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Maximum Likelihood Estimation and Applications of the Weibull-Rayleigh Distribution

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Abstract

Recently, the Weibull-Rayleigh Distribution (WRD) has been derived as a new class of the Weibull-X family of distributions and expressions for some of its distributional properties defined and studied. Further to these, this article employs the method of maximum likelihood to find estimators for parameters of the Weibull-Rayleigh distribution. The method of maximum likelihood yielded a closed form estimator for the scale parameter but cannot produce a closed form estimator for the shape parameter of WRD. Some properties of the estimator of the scale parameter are discussed. Two numerical data sets on insurance claims and marriage survival time are used to illustrate the applicability of the method of maximum likelihood of the Weibull-Rayleigh Distribution.

Keywords: Weibull-Rayleigh distribution, Weibull distribution, Rayleigh distribution, Weibull-*X* family, Maximum Likelihood method.
MSC 2010: 47N30; 60A10; 62G07; 46F20; 60E20

1 Introduction

The Weibull distribution is a widely used distribution in survival, reliability, finance and climatology. [1]Tadikamalla (1978) illustrates the effectiveness and adequacy the Weibull distribution to approximate the lead time demand in inventory control problems. In interpreting environmental pollution and biological sciences data, the Weibull distribution has been found useful ([2]Mikolaj, 1972; [3]Perry, 1998; [4]Cordeiro *et al.*, 2008). Attempts at generalizing and extending the Weibull distribution can be found in the works of [5]Mudholkar and Srivastava (1993), [6]Famoye *et al.* (2005), [7]Alzaatreh *et al.* (2013a), [8]Adeleke *et al.* (2013) and [9]Akarawak *et al.* (2014). The Rayleigh distribution and its extension are also well investigated and used in survival and environmental studies ([10]Voda, 2007).

Motivated by the current development in the generalization of the Weibull distribution provided by [11]Alzaatreh et al. (2013b) and the need for its continuous extension and generalization to more complex situations, [12]Akarawak et al. (2013) considered certain results characterizing the generalization of the Weibull and Rayleigh distributions through their probability density and distribution functions. Consequently, a new member of the Weibull-X family of distributions as introduced by [11]Alzaatreh et al. (2013b) was proposed. This continuous probability distribution is the two-parameter Weibull-Rayleigh Distribution

(WRD). Expressions for some distributional properties such as the survival function, hazard function, moments and moment generating function were derived and studied ([12]Akarawak et al., 2013). Simulation studies results revealed that the newly derived distribution is unimodal, peaked and right-skewed.

Further extension and generalization of the Weibull distribution need investigation owing to the central role this distribution plays in modelling and survival studies. An important aspect of this continuous distribution requiring investigation is parameter estimation. This article employs the method of maximum likelihood to derive estimators for parameters of the Weibull-Rayleigh distribution. R project will be used for the implementations.

The remaining sections of the article are organized as follows. In Section 2, a review of the Weibull-Rayleigh distribution and some of its properties are presented. In Section 3, parameter estimation for the parameters of the Weibull-Rayleigh Distribution is considered. Applications of the distribution are done in Section 4; while Section 5 concludes the article.

2 Review of the Weibull-Rayleigh Distribution (WRD)

2.1 The pdf and cdf of the Weibull-Rayleigh Distribution

[12]Akarawak et al. (2013) defined and studied a new continuous probability distribution as a class of the Weibull-X family of distributions introduced by [11]Alzaatreh et al. (2013b). According to the authors, the pdf g(x) and cdf F(x) of the two-parameter Weibull-Rayleigh Distribution (WRD) are, respectively, given by

$$g(x)=\frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a \exp\left\{-\left(\frac{x^2}{k}\right)^a\right\}; \quad x>0, a, k>0\,; \tag{2.1}$$

$$F(x)=1-e^{-\left(\frac{x^2}{k}\right)^a}, \tag{2.2}$$

Where a is the shape parameter and k is the scale parameter. As shown in [12]Akarawak et al. (2013), if a random variable X is WR-distributed, then X^2 is Rayleigh distributed. Furthermore, the authors has shown that the Rayleigh distribution is a special case of WRD for a = 1; while the Weibull distribution is a special case of WRD with parameters m= 2a and $\lambda = \sqrt{k}$.

From (2.1) if a = 1 and k = $2\sigma^2$, then we have

$$g(x)=\frac{x}{\sigma^2}\exp\left\{-\left(\frac{x^2}{2\sigma^2}\right)\right\}; \quad x>0,, \sigma>0 \text{ is a Rayleigh distribution}$$

Also, from (2.1) if a = m/2 and k = λ^2, then we have

$$g(x)=\frac{m}{\lambda}\left\{\frac{x}{\lambda}\right\}^{m-1} \exp\left\{-\left(\frac{x}{\lambda}\right)^m\right\}; \quad x>0, m>0 \text{ is a Weibull distribution.}$$

While the procedure for deriving the pdf and cdf in (2.1) and (2.2) are omitted, we state here that the two-parameter Weibull-Rayleigh distribution as defined by [12]Akarawak et al. (2013) is quite different from the three-parameter Weibull Rayleigh distribution of [13]Merovci and Elbatal (2015) and this paper is not intended to bring out the difference between the two.

2.2 Moments of Weibull-Rayleigh Distributions

[12]Akarawak et al. (2013) studied WRD analytically and by simulation and obtained the moments and other properties of the distribution. The moments and moment generating function are presented in this section.

Moments

The rth moment of WRD is:

$$E(X^r) = k^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2a}+1\right); \text{ for all integer } r > 0.$$

Therefore, the first and second moments and variance of the distribution are given by:

$$u_1' = E(X) = k^{\frac{1}{2}}\Gamma\left(\frac{1}{2a}+1\right), \quad u_2' = E(X^2) = k\Gamma\left(\frac{1}{a}+1\right), \tag{2.3}$$

and,
$$Var(X) = k\left\{\Gamma\left(\frac{1}{a}+1\right) - \left[\Gamma\left(\frac{1}{2a}+1\right)\right]^2\right\}, \tag{2.4}$$

$$= k\left\{\Gamma\left(\frac{1}{a}+1\right) - \Gamma\left(\frac{1}{2a}+1\right)^2\right\}. \tag{2.5}$$

$$= \frac{k}{a}\left\{\Gamma\left(\frac{1}{a}\right) - \frac{1}{4a}\Gamma\left(\frac{1}{2a}\right)^2\right\}. \tag{2.6}$$

The standard deviation of X is given by:

$$\sigma = Std(X) = \sqrt{Var(X)}$$

$$= k^{\frac{1}{2}}\left[\Gamma\left(\frac{1}{a}+1\right) - \Gamma\left(\frac{1}{2a}+1\right)^2\right]^{\frac{1}{2}} = \left(\frac{k}{a}\right)^{\frac{1}{2}}\left[\Gamma\left(\frac{1}{a}\right) - \frac{1}{4a}\Gamma\left(\frac{1}{2a}\right)^2\right]^{\frac{1}{2}} \tag{2.7}$$

Coefficient of variation of X is given by:

$$CV = \frac{Std(X)}{E(X)} = \frac{\left[\Gamma\left(\frac{1}{a}+1\right) - \Gamma\left(\frac{1}{2a}+1\right)^2\right]^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2a}+1\right)}, \tag{2.8}$$

$$= \frac{2\left[\Gamma\left(\frac{1}{a}\right) - \frac{1}{4a}\Gamma\left(\frac{1}{2a}\right)^2\right]^{\frac{1}{2}}}{a^{\frac{1}{2}}\Gamma\left(\frac{1}{2a}\right)}. \tag{2.9}$$

To access skewness and kurtosis, μ_3 and were obtained as μ_3:

$$\mu_3 = k^{\frac{3}{2}}\left[\Gamma\left(\frac{3}{2a}+1\right) - 3\Gamma\left(\frac{1}{2a}+1\right)\Gamma\left(\frac{1}{a}+1\right) + 2\Gamma\left(\frac{1}{2a}+1\right)^3\right], \tag{2.10}$$

$$\mu_4 = k^2\left[\Gamma\left(\frac{2}{a}+1\right) - 4\Gamma\left(\frac{1}{2a}+1\right)\Gamma\left(\frac{3}{2a}+1\right) + 6\Gamma\left(\frac{1}{a}+1\right)\Gamma\left(\frac{1}{2a}+1\right)^2 - 3\Gamma\left(\frac{1}{2a}+1\right)^4\right], \tag{2.11}$$

Then, $$Skewness = \frac{\mu_3}{\sigma^3} = \frac{\frac{3a^2}{2}\Gamma\left(\frac{3}{2a}\right) - \frac{3a}{2}\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{1}{2a}\right) + \frac{1}{4}\Gamma\left(\frac{1}{2a}\right)^3}{\left[\Gamma\left(\frac{1}{a}\right) - \frac{1}{4a}\Gamma\left(\frac{1}{2a}\right)^2\right]^{\frac{3}{2}}} \tag{2.12}$$

$$Kurtosis = \frac{\mu_4}{\sigma^4} = \left\{\frac{2a\Gamma\left(\frac{2}{a}\right) - 3\Gamma\left(\frac{1}{2a}\right)\Gamma\left(\frac{3}{2a}\right) + \frac{3}{2a}\Gamma\left(\frac{1}{a}\right)\Gamma\left(\frac{1}{2a}\right)^2 - \frac{3}{16a^2}\Gamma\left(\frac{1}{2a}\right)^4}{\left[\Gamma\left(\frac{1}{a}\right) - \frac{1}{4a}\Gamma\left(\frac{1}{2a}\right)^2\right]^2}\right\} \tag{2.13}$$

Moment Generating Function

The moment generating function of the Weibull-Rayleigh distributed random variable X is given by:

$$M_X(t) = \sum_{r=0}^{\infty} k^{\left(\frac{r}{2}\right)}\Gamma\left(\frac{r}{2a}+1\right)\frac{t^r}{r!}. \tag{2.14}$$

3 Parameter Estimation of WRD

3.1 The Method of Maximum Likelihood (ML) Estimation

For estimating an unknown parameter θ, the likelihood principle can be used to obtain the maximum likelihood estimator (MLE) $\hat{\theta}$ ([14]Bai and Fu, 1987). The definition of maximum likelihood estimator is presented below.

Definition 3.1: Likelihood Function ([15]Mood et al., 1974)

The likelihood function of n random variables X_1, X_2, ..., X_n is defined to be the joint density of the n random variables, say $f_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n;\theta)$, which is considered to be a function of θ. In particular, if $X_1,X_2,\cdots,X_n$ is random sample from the density $f(x;\theta)$, then the likelihood function is given by $L(\theta) = f(x_1;\theta)f(x_1;\theta)\cdots f(x_1;\theta)$.

Definition 3.2: Maximum Likelihood Estimator (MLE)

Let $L(\theta) = L(\theta;x_1,\cdots,x_n) = f(x_1,\theta)f(x_2,\theta)\cdots f(x_n,\theta) = \prod_{i=1}^{n} f(x_i;\theta)$ be the likelihood function of the random variables $X_1,X_2,\cdots,X_n$ and $\hat{\theta} = h(X_1,\cdots,X_n)$ a function of the random variables. If the value of $\hat{\theta}$ given by $h(x_1,\cdots,x_n)$ maximizes $L(\theta)$, then $\hat{\theta} = h(X_1,\cdots,X_n)$ is a maximum likelihood estimator of θ. Under certain regularity conditions, the maximum likelihood estimator of θ is obtained by solving the likelihood equation:

$$\frac{dL(\theta)}{d\theta}=0, \tag{3.1}$$

such that $\frac{\partial^2 L}{\partial\hat{\theta}^2}<0$.

If the likelihood function contains k parameters, that is, if

$$L(\theta_1,\cdots,\theta_k)=\prod_{i=1}^{n} f(x_i,\theta_1,\cdots,\theta_k), \tag{3.2}$$

then, the maximum likelihood estimators are the random variables $\hat{\theta}_1=h_1(X_1,X_2,\cdots,X_n),\hat{\theta}_2=h_2(X_1,X_2,\cdots,X_n),\cdots,\hat{\theta}_k=h_k(X_1,X_2,\cdots,X_n)$, whose values maximizes $L(\theta_1,\cdots,\theta_k)$. If the regularity conditions are satisfied, the point where the likelihood is a maximum is a solution set of the k equations:

$$\frac{\partial L(\theta_1,\theta_2,\cdots,\theta_k)}{\partial\theta_1}=0$$
$$\frac{\partial L(\theta_1,\theta_2,\cdots,\theta_k)}{\partial\theta_2}=0$$
$$\vdots$$
$$\frac{\partial L(\theta_1,\theta_2,\cdots,\theta_k)}{\partial\theta_k}=0,$$

Such that, $\frac{\partial^2 L}{\partial\hat{\theta}_i^2}<0;\ i=1,\cdots,k$.

For ease of computation, the logarithm of the likelihood function is often used as it has the same maximum point as [16]Wackerley et al. (1996) remarked that the method of maximum likelihood often leads to minimum-variance unbiased estimators (MVUE) and that despite its computational complications, MLEs are always preferred because they have optimal asymptotic properties. For works on applications of MLE see [17]Golding (1993), [18]Comets and Gidas (1991), [19]Gupta and Szekely (1994), [20]Huang and Yu (2008) and [21]Dent and Hildreth (1977).

3.2 Derivation of the ML Estimators for the Parameters of the Weibull-Rayleigh Distribution

In this section, the estimators for parameters *a* and *k* of the Weibull-Rayleigh distribution are derived using the method of maximum likelihood.

Let a random sample be taken from WRD with probability density function (pdf) given as:

$$g(x)=\frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a\exp\left\{-\left(\frac{x^2}{k}\right)^a\right\};\quad x\geq 0,a,k>0$$

The likelihood function is given by:

$$L(a,k;x)=(2a)^n k^{-na}\prod_{i=1}^{n} x_i^{2a-1}\exp\left[-\left(\frac{x_i^2}{k}\right)^a\right] \tag{3.3}$$

$$=(2a)^n k^{-na}\left[\prod_{i=1}^{n} x_i^{2a-1}\right]\exp\left[-\sum\left(\frac{x_i^2}{k}\right)^a\right] \tag{3.4}$$

And the log-likelihood function is given by:

$$\log L = n \ln 2a - na \ln k + (2a-1)\sum \ln x_i - \frac{\sum x_i^{2^a}}{k^a} \tag{3.5}$$

Taking partial derivatives with respect to the parameters:

$$\frac{\partial \ln L}{\partial a} = \frac{n}{a} - n \ln k + 2\sum \ln x_i - \sum_{i=1}^{n}\left[\frac{x_i^{2a}}{k^a}\ln\left(\frac{x_i^2}{k}\right)\right] \tag{3.6}$$

$$\frac{\partial \ln L}{\partial k} = -\frac{na}{k} + \frac{a\sum\left(x_i^{2a}\right)}{k^{a+1}} \tag{3.7}$$

Equating (3.6) and (3.7) to zero and replacing the parameters by their estimates gives the following:

$$\frac{n}{\hat{a}} - n \ln \hat{k} + 2\sum \ln x_i - \sum_{i=1}^{n}\frac{\left(x_i^2\right)^a}{k^a}\ln\left(\frac{x_i^2}{k}\right) = 0 \tag{3.8}$$

$$-\frac{n\hat{a}}{\hat{k}} + \frac{\hat{a}\sum x_i^{2a}}{\hat{k}^{\hat{a}+1}} = 0 \tag{3.9}$$

Then solving (3.9) gives

$$-n\hat{k}^{\hat{a}} + \sum x_i^{2\hat{a}} = 0, \tag{3.10}$$

$$\Rightarrow \hat{k}^{\hat{a}} = \frac{\sum x_i^{2\hat{a}}}{n} \tag{3.11}$$

$$\Rightarrow \hat{k} = \left(\frac{\sum x_i^{2\hat{a}}}{n}\right)^{\frac{1}{\hat{a}}} \tag{3.12}$$

From (3.8), we have

$$\frac{n}{\hat{a}} - n \ln \hat{k} + 2\sum \ln x_i - \sum_{i=1}^{n}\frac{\left(x_i^2\right)^{\hat{a}}}{k^{\hat{a}}}\ln\left(\frac{x_i^2}{\hat{k}}\right) = 0$$

Multiplying through by $\hat{a}\hat{k}^{\hat{a}}$:

$$\hat{k}^{\hat{a}} n - n\hat{a}\hat{k}^{\hat{a}} \ln \hat{k} + 2\hat{a}\hat{k}^{\hat{a}}\sum \ln x_i - \hat{a}\sum x_i^{2\hat{a}} \ln\left(\frac{x_i^2}{\hat{k}}\right) = 0 \tag{3.13}$$

$$\Rightarrow \hat{k}^{\hat{a}} n - n\hat{a}\hat{k}^{\hat{a}} \ln \hat{k} + 2\hat{a}\hat{k}^{\hat{a}}\sum \ln x_i - \hat{a}\sum x_i^{2\hat{a}} \ln x_i^2 + \hat{a}\sum x_i^{2\hat{a}} \ln \hat{k} = 0 \tag{3.14}$$

From (3.10), $\sum x_i^{2\hat{a}} = n\hat{k}^{\hat{a}}$, and substituting into (3.14) gives:

$$\sum x_i^{2\hat{a}} - \hat{a}\sum x_i^{2\hat{a}} \ln \hat{k} + 2\hat{a}\hat{k}^{\hat{a}}\sum \ln x_i - \hat{a}\sum x_i^{2\hat{a}} \ln x_i^2 + \hat{a}\sum x_i^{2\hat{a}} \ln \hat{k} = 0$$

$$\Rightarrow \sum x_i^{2\hat{a}} + 2\hat{a}\frac{\sum x_i^{2\hat{a}}}{n}\sum \ln x_i - \hat{a}\sum x_i^{2\hat{a}} \ln x_i^2 = 0$$

$$\Rightarrow n\sum x_i^{2\hat{a}} + 2\hat{a}\sum x_i^{2\hat{a}}\sum \ln x_i - \hat{a}n\sum x_i^{2\hat{a}} \ln x_i^2 = 0. \tag{3.15}$$

The likelihood equation in (3.15) cannot be solved to obtain $\hat{a}$ in closed form; however, it will be solved numerically. Once $\hat{a}$ is determined, $\hat{k}$ can be obtained using (3.12).

3.3 Expectation and Variance of the Estimator $\hat{k}$

The properties of the estimator $\hat{k}$ of the scale parameter of the Weibull-Rayleigh distribution could not be obtained easily due to its complex form. However, an attempt is made in this section to obtain the expectation and variance of the estimators using the following approximation to expected value and variance obtained by a Taylor series about μ.

Approximation to Expected value and Variance: ([22]Benaroya et al., 2005)

Let f(X) be any function of the random variable X. The expected value and variance of f(X) can be approximated by:

$$E[f(X)] \approx f(\mu) + \frac{\sigma_X^2}{2} f''(\mu). \tag{3.16}$$

$$Var[f(X)] \approx \sigma_X^2 [f'(\mu)]^2. \tag{3.17}$$

These approximations are expected to be accurate provided f(X) is a decreasing function that does not have a closed form for its estimate and it is consistent.

Expectation of $\hat{k}$

$$\hat{k} = \left(\frac{\sum x_i^{2\hat{a}}}{n}\right)^{\frac{1}{\hat{a}}} = n^{-\frac{1}{\hat{a}}}\left(\sum x_i^{2\hat{a}}\right)^{\frac{1}{\hat{a}}}$$, is a function of a random variable X.

$$\Rightarrow f(\mu) = \left(\sum \mu^{2\hat{a}}\right)^{\frac{1}{\hat{a}}} = \left(n\mu^{2\hat{a}}\right)^{\frac{1}{\hat{a}}} = n^{\frac{1}{\hat{a}}}\mu^2,$$

and, $f'(\mu) = 2\mu n^{\frac{1}{\hat{a}}}$; $f''(\mu) = 2n^{\frac{1}{\hat{a}}}$.

$$E(\hat{k}) = n^{-\frac{1}{\hat{a}}} E\left[\left(\sum x_i^{2\hat{a}}\right)^{\frac{1}{\hat{a}}}\right],$$

$$f(X) = \left(\sum x_i^{2\hat{a}}\right)^{\frac{1}{\hat{a}}}, \tag{3.18}$$

By (3.16),

$$E\left[\left(\sum x_i^{2\hat{a}}\right)^{\frac{1}{\hat{a}}}\right] \approx n^{\frac{1}{\hat{a}}}\mu + \frac{\sigma_X^2}{2}\left(2n^{\frac{1}{\hat{a}}}\right).$$

$$E[f(X)] \approx n^{\frac{1}{\hat{a}}}\mu^2 + \sigma_X^2 n^{\frac{1}{\hat{a}}} = n^{\frac{1}{\hat{a}}}\left(\mu^2 + \sigma_X^2\right), \tag{3.19}$$

Substituting for μ^2 and σ_X^2 gives,

$$E[f(X)] \approx n^{\frac{1}{\hat{a}}}\left\{\left[k^{\frac{1}{2}}\Gamma\left(\frac{1}{2\hat{a}}+1\right)\right]^2 + k\left\{\Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right\}\right\}, \tag{3.20}$$

$$\approx n^{\frac{1}{\hat{a}}}\left[k\Gamma\left(\frac{1}{2\hat{a}}+1\right)^2 + k\left\{\Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right\}\right],$$

$$\approx kn^{\frac{1}{\hat{a}}}\left[\Gamma\left(\frac{1}{2\hat{a}}+1\right)^2 + \Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right],$$

$$\approx kn^{\frac{1}{\hat{a}}}\left[\Gamma\left(\frac{1}{\hat{a}}+1\right)\right], \tag{3.21}$$

Hence, $E(\hat{k}) \approx n^{-\frac{1}{\hat{a}}}.n^{\frac{1}{\hat{a}}}\left[\mu^2 + \sigma_X^2\right] = \left[\mu^2 + \sigma_X^2\right],$

$$\Rightarrow E(\hat{k}) \approx \hat{k}\left[\Gamma\left(\frac{1}{\hat{a}}+1\right)\right], \tag{3.22}$$

(3.22) implies that the estimator $\hat{k}$ is a biased estimator for k.

Variance of $\hat{k}$:

$$\hat{k} = \left(\frac{\sum x_i^{2\hat{a}}}{n}\right)^{\frac{1}{\hat{a}}} = n^{-\frac{1}{\hat{a}}}\left(\sum x^{2\hat{a}}\right)^{\frac{1}{\hat{a}}},$$

From (3.17), (3.18), (3.19) and (3.20):

$$Var\left[f(X)\right] \approx Var(X)[h'(\mu)]^2,$$

$$\approx \hat{k}\left\{\Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right\}[2n^{\frac{1}{\hat{a}}}\mu]^2$$

Since, $Var(\hat{k}) = Var\left[n^{-\frac{1}{\hat{a}}}\left(f(X)\right)\right] = n^{-\frac{2}{\hat{a}}}Var\left[f(X)\right],$ (3.23)

$$\therefore Var(\hat{k}) \approx 4\mu^2\hat{k}\left\{\Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right\} \tag{3.24}$$

$$\therefore Var(\hat{k}) \approx 4\hat{k}^2\Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\left\{\Gamma\left(\frac{1}{\hat{a}}+1\right) - \Gamma\left(\frac{1}{2\hat{a}}+1\right)^2\right\}. \tag{3.25}$$

3.4 Unbiased Estimators for the parameter k

Here attempt is made to obtain an unbiased estimator for the scale parameter k using the concept of sufficient statistics and Rao-Blackwell theorem.

Unbiased Estimator for k:

Theorem 3.1:

A possible unbiased estimator for the parameter k is given by:

$$\breve{k} = \frac{\sum\left(X_i^2\right)}{n\Gamma\left(\frac{1}{a}+1\right)}. \tag{3.26}$$

Proof:

$$E(\breve{k}) = E\left(\frac{\sum (X_i^2)}{n\left[\Gamma\left(\frac{1}{a}+1\right)\right]}\right) \tag{3.27}$$

The expectation of a sum is the sum of the expectation. So, (3.27) becomes

$$E(\breve{k}) = \frac{\sum E(X_i^2)}{n\left[\Gamma\left(\frac{1}{a}+1\right)\right]}. \tag{3.28}$$

Recall from (2.3) that $u_2' = E(X^2) = k\Gamma\left(\frac{1}{a}+1\right)$, so that (3.28) becomes

$$E(\breve{k}) = \frac{\sum k\left[\Gamma\left(\frac{1}{a}+1\right)\right]}{n\left[\Gamma\left(\frac{1}{a}+1\right)\right]} = \frac{nk\left[\Gamma\left(\frac{1}{a}+1\right)\right]}{n\left[\Gamma\left(\frac{1}{a}+1\right)\right]} = k \tag{3.29}$$

Thus, $E(\breve{k}) =$ k.

Theorem 3.2 (Minimum Variance Unbiased Estimator for k):

The estimator $\breve{k} = \dfrac{\sum\left(X_i^2\right)}{n\Gamma\left(\frac{1}{\hat{a}}+1\right)}$ is a minimum variance unbiased estimator (MVUE) of the parameter k.

Proof:

Joint Sufficient Statistics for *a* and *k*:

Let $X_1, X_2, \cdots, X_n$ be a random sample from WRD with pdf given by, $g(x) = \frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a \exp\left\{-\left(\frac{x^2}{k}\right)^a\right\}, x > 0, a, k > 0$. Then, the likelihood function is given by:

$$L(a,k;x_i) = (2a)^n k^{-na} \prod_{i=1}^{n} x_i^{2a-1} \exp\left[-\frac{1}{k^a}\sum_{i=1}^{n} x_i^{2a}\right] = \left(\frac{2a}{k^a}\right)^n \prod_{i=1}^{n} x_i^{2a-1} \exp\left[-\sum_{i=1}^{n}\left(\frac{x_i^2}{k}\right)^a\right], \tag{3.30}$$

$$L(x;a,k) = f\left(S(x);a,k\right).h(x_1, x_2, \ldots, x_n),$$

(3.30) satisfy the factorization criterion, where $h(x_i) = 1$ does not depend on a and k and

$$f(S(x);a,k) = \left(\frac{2a}{k^a}\right)^n \left(\prod_{i=1}^{n}(x_i)^{2a-1}\right)\exp\left\{-\frac{1}{k^a}\sum_{i=1}^{n}\left(x_i^2\right)^a\right\}, \tag{3.31}$$

Therefore, x_i and x_i^2 are joint sufficient statistics for a and k. Hence, x_i and x_i^2 best summarize all information about the parameters a and k.

Joint Complete Sufficient Statistics of Weibull-Rayleigh Distribution Parameters

Let $x_1, x_2, \ldots, x_n, n > m$ denote a random sample from a Weibull-Rayleigh distribution that depends on parameters a and k has a pdf $g(x) = \frac{2a}{x}\left(\frac{x}{k}\right)^a \exp\left\{-\left(\frac{x^2}{k}\right)^a\right\}$.

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Exponential Class Form

A distribution is said to belong to exponential class of distributions if its pdf can be expressed in the form:

$$Exp[P(\theta_1,\theta_2)K(x)+S(x)+q(\theta_1,\theta_2)]$$

$$R(x_1,x_2,...,x_m)\exp[P(\theta_1,\theta_2,...,\theta_m)x_i+q(\theta_1,\theta_2,...,\theta_m)] \tag{3.32}$$

To show that the pdf of WRD is in form of (3.32), we do the following:

$$\ln g(x;a,k)=\ln(\frac{2a}{x}\left\{\frac{x^2}{k}\right\}^a \exp\left\{-\left(\frac{x^2}{k}\right)^a\right\})=\ln 2a-\ln x+a\ln\left(\frac{x^2}{k}\right)-\left(\frac{x^2}{k}\right)^a$$

$$=\ln 2a-\ln x+a\ln x^2-a\ln k-\left(\frac{x^2}{k}\right)^a$$

$$e^{\ln g(x;a,k)}=e^{\left[\ln 2a-\ln x+a\ln x^2-a\ln k-\left(\frac{x^2}{k}\right)^a\right]}$$

$$=e^{-\ln x}e^{\left[\ln 2a+a\ln x^2-a\ln k-\left(\frac{x^2}{k}\right)^a\right]} \tag{3.33}$$

Comparing equation (3.32) and (3.33)

$$R(x_1,x_2,...,x_m)=e^{-\ln x}$$

$$q(\theta_1,\theta_2)=\ln 2a-a\ln k$$

$$P(\theta_1,\theta_2)x=a\ln x^2-\left(\frac{x^2}{k}\right)^a$$

Since the pdf can be written as a family of an exponential class, then it is complete. We proceed to finding the joint complete sufficient statistics as follows:

$$e^{\ln g(x;a,k)} = e^{\left[n\ln 2a - \sum_{i=1}^{n} \ln x_i + a\sum_{i=1}^{n} \ln x_i^{2} - na\ln k - \left(\frac{\sum_{i=1}^{n} x_i^{2}}{k} \right)^{a} \right]}$$

$$= e^{-\sum_{i=1}^{1} \ln x_i} e^{\left[n\ln 2a + a\sum_{i=1}^{n} \ln x_i^{2} - na\ln k - \left(\frac{\sum_{i=1}^{n} x_i^{2}}{k} \right)^{a} \right]}$$

$$= e^{-\ln \prod_{i=1}^{n} x_i} e^{\left[n\ln 2a - n\ln k + a\ln \prod_{i=1}^{n} x_i^{2} - \left(\frac{\sum_{i=1}^{n} x_i^{2}}{k} \right)^{a} \right]}$$

$$= e^{-\ln \prod_{i=1}^{n} x_i} e^{\left[n(\ln 2a - \ln k) + a\ln \prod_{i=1}^{n} x_i^{2} - \left(\frac{\sum_{i=1}^{n} x_i^{2}}{k} \right)^{a} \right]} \quad (3.34)$$

Comparing equation (3.32) and (3.34), we have

$$R(x_1, x_2, \ldots, x_m) = e^{-\ln \prod_{i=1}^{n} x_i}$$

$$nq(\theta_1, \theta_2, \ldots, \theta_n) = n(\ln 2a - a\ln k)$$

$$\sum_{i=1}^{n} P(\theta_1, \theta_2) x_i = a\ln \prod_{i=1}^{n} x_i^{2} - \left(\frac{\sum_{i=1}^{n} x_i^{2}}{k} \right)^{a} = a\ln \prod_{i=1}^{n} x_i^{2} - \frac{1}{k^{a}} \left(\sum_{i=1}^{n} x_i^{2} \right)^{a}$$

The joint complete sufficient statistics for the parameters of Weibull-Rayleigh distributionn

are $\sum_{i=1}^{n} x_i^{2}$ and $\prod_{i=1}^{n} x_i^{2}$ for k and a.

Therefore, by the Rao-Blackwell theorem, since the unbiased estimator of k, $\breve{k}$ is a function of the complete sufficient statistics $\sum X_i^2$, it is a minimum variance unbiased estimator.

Variance of $\breve{k}$:

$$\breve{k} = \frac{\sum X_i^2}{n\Gamma\left(\frac{1}{a}+1\right)},$$

$$Var(\breve{k}) = \frac{Var\left(\sum X_i^2\right)}{n^2\Gamma\left(\frac{1}{a}+1\right)^2},$$

$$= \frac{nVar\left(X^2\right)}{n^2\Gamma\left(\frac{1}{a}+1\right)^2},$$

But $Var(X^2) = E(X^4) - \left[E(X^2)\right]^2$,

$$\Rightarrow Var(\breve{k}) = \frac{k^2\left[\Gamma\left(\frac{2}{a}+1\right) - \Gamma\left(\frac{1}{a}+1\right)^2\right]}{n\Gamma\left(\frac{1}{a}+1\right)^2}, \tag{3.35}$$

Clearly, the variance of $\breve{k}$ in (3.35) is smaller than the variance of $\hat{k}$ given in (3.25).

4 Applications and Results

In this section, the Weibull-Rayleigh distribution is applied to two data sets on vehicle insurance claims and marriage survival. The performance criterion used is the Kolmogorov-Smirnov (K-S) statistics and smaller value of the K-S statistics is desirable.

4.1 Application to Insurance Claims Data

Data Description and Exploration

The distribution was applied to a real life data on 1721 claims on motor vehicle from an insurance company. A characteristic feature of the claims data is that many of the insurees were paid very little claims. Table 1 shows the summary of statistics for the claims data. The histogram of the vehicle claims data generated in SPSS 18.0 is also presented in Figure 1.

Statistics	N	Mean	Variance	Std Dev.	Skewness	Std Error of Skewness	Kurtosis	Std Error of Kurtosis
Values	1721	48663.5	2.355 × 1011	485286.93	25.387	0.059	786.557	0.118

Table 1: Summary of Statistics for claims data

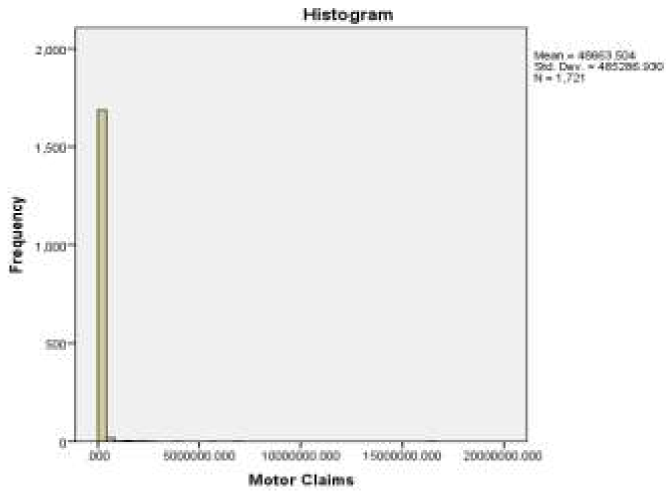

Figure 1: Histogram of Claims Data

It is seen from both the descriptive statistics and histogram that the distribution of the data is highly peaked and skewed to the right.

Data Analysis and Results

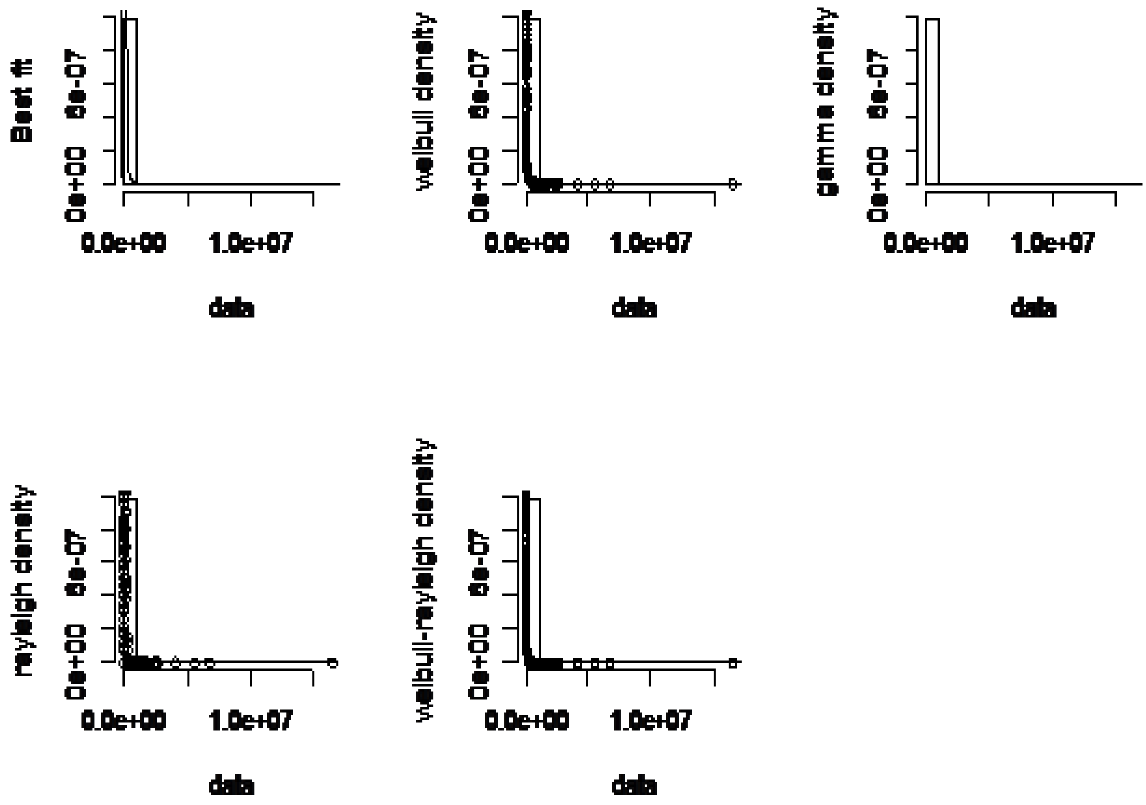

Figure 2: Density Plots for the Claims Data

Distribution	Estimates of the Parameters	K-S Statistics
Weibull	$\alpha = 0.1307$, $\beta = 4.8836$	0.4550
Rayleigh	$\sigma = 74380.0788$	0.8460
Gamma	Shape = 0.0233, Rate = 527.788	0.9995
WRD	$a = 0.0653$, $k = 23.4452$	0.4546

Table 2: Estimates of Parameters and K-S Statistics for Claims Data

Discussion of Result

The results presented in Figure 2 above show the density plots of the data generated in R, respectively for the best fit, Weibull distribution, Gamma distribution, Rayleigh distribution and WRD. The estimates and fits of the data from the various distributions are presented in Table 2. The K-S statistics in the results show that WRD distribution is the best with smallest K-S statistic, followed by Weibull, Rayleigh and Gamma. However, the difference between WRD and Weibull is extremely small and insignificant. So, WRD does not perform differently from Weibull distribution. WRD is, therefore competitive in fitting the claims data. Or, since the difference between Weibull and WRD is little and as already shown analytically, WRD can be used as an alternative distribution to the Weibull distribution.

4.2 Application to Marriage Survival Data

Data Description and Exploration

The marriage survival data represents time to divorce of marriage from when it was contracted. The sample size for the study is 300. A questionnaire was used to collect the primary data for the study. Table 3 gives the summary of statistics for the survival data while Figure 3 gives the histogram of the data generated in SPSS 18.0.

Statistics	N	Mean	Variance	Std Dev.	Skewness	Std Error of Skewness	Kurtosis	Std Error of Kurtosis
Values	300	12.233	57.731	7.598	0.925	0.141	0.724	0.281

Table 3: Summary of Statistics for survival data

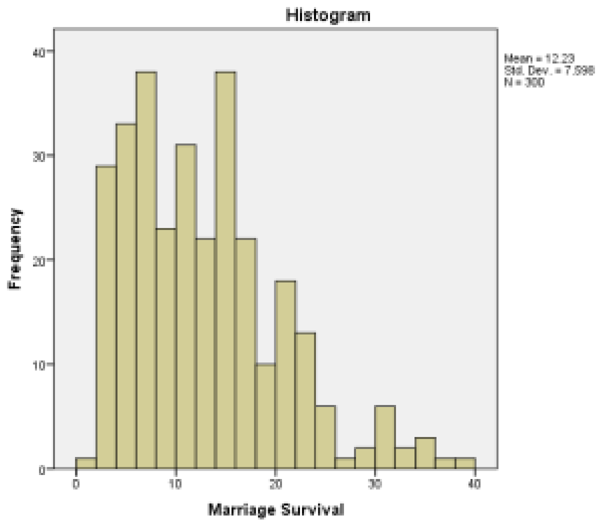

Figure 3: Histogram of Claims Data

Data Analysis and Results

Distribution	Estimates of the Parameters	K-S Statistics
Weibull	α = 1.6859, β = 13.7432	0.0623
Rayleigh	σ = 14.3949	0.1261
Gamma	Shape = 2.4508, Rate = 4.9915	0.7955
WRD	a = 0.8425, k = 188.3980	0.0617

Table 4: Estimates of Parameters and Goodness of fits

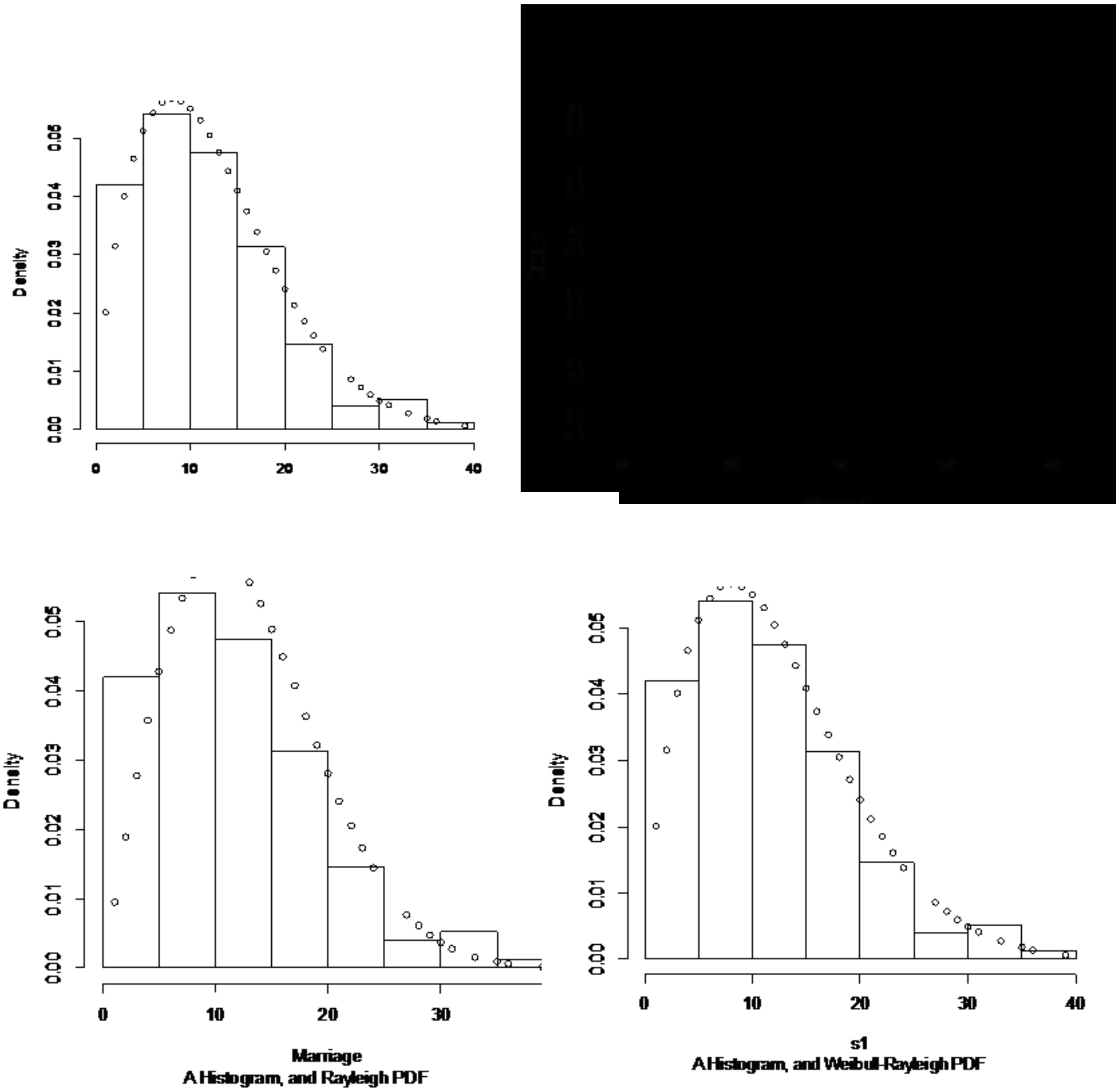

Figure 4: Density Plots for Marriage Survival Data

Discussion of Result

The estimates of the parameters and K-S statistics for the survival data are given in Table 4 for the Weibull, Rayleigh, Gamma distributions as well as WRD. Again, the K-S statistics show that WRD fits the survival data best followed by the Weibull, Rayleigh and Gamma distributions. However, as stated before, the difference between WRD and Weibull is extremely small and insignificant. WRD, therefore can be an alternative distribution to Weibull distribution and its performance shows a little improvement on Weibull distribution. WRD is more of a Weibull distribution than a Rayleigh distribution.

5 Conclusion

In this research work, parameter estimation and applications of the two-parameter Weibull-Rayleigh Distribution (WRD) have been considered. The method of likelihood estimation has been used to estimate the shape and scale parameters of WRD. The maximum likelihood method only yields closed form estimator for the scale parameter. The expected value and variance of the estimator of the scale parameter has been derived. Furthermore, minimum variance unbiased estimator has been obtained for the scale parameter.
The distribution is then applied to two data sets on vehicle insurance claims and marriage survival in Lagos State. The results show that WRD produce good fits and performs competitively well in fitting both data types compared to Weibull, Rayleigh and Gamma distributions.

We therefore recommended that the newly derived Weibull-Rayleigh distribution be used as an alternative to some existing distributions in modelling data on survival, reliability and financial studies.

Competing financial interests

The author(s) declare no competing financial interests.

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