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FEKETE-SZEGÖ INEQUALITIES ASSOCIATED WITH CERTAIN GENERALIZED CLASS OF NON-BAZILEVIC FUNCTIONS

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Abstract

For function $f(z)$ of the form:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ..., \quad z \in U$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We determine Fekete-Szegö inequalities for functions belonging to the generalized class $\Delta(\alpha, \beta, \lambda, \theta)$ of non-Bazilevic functions in the open unit disk .

Keywords and Phrases: Analytic, Univalent, Starlike, Convex, Spiral-like function.
MSC2010:30C45

1 Introduction and Definitions

A function $g(z)$ is said to be analytic at a point z_0 in a chosen domain if its derivative exists in that domain. Since these functions are analytic, there exists in their domain Taylor series expansion of the form

$$g(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{1.0.1}$$

where the coefficient $b_k = \frac{g^k(0)}{k!}$ and is obtainable from the famous Cauchy integral formula

$$g^k(z) = \frac{k!}{2\pi i} \int_\Gamma \frac{g(\xi)}{\xi - z} d\xi.$$

Here, we can say that $g(z)$ is normalized such that: $g(z)$ takes the value zero at the origin ($i.e, g(0) = 0$) and its first derivative takes the value 1 at the origin ($g'(0) = 1$).
With this in mind, we can write that

$$\frac{g(z) - b_0}{b_1} = z + \frac{b_2}{b_1} z^2 + \frac{b_3}{b_1} z^3 + ... \ .$$

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Let us define

$$f(z) = \frac{g(z) - b_0}{b_1}. \tag{1.0.2}$$

Then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.0.3}$$

where $a_k = \frac{b_k}{b_1}$, $k \geq 2$.
This function in (1.0.3) is analytic and so let A denotes the class of all such functions which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Here, we recall that a single value function $f(z)$ is said to be univalent if it never takes on the same value twice ($i.e,\ if\ f(z_1) = f(z_2) \Rightarrow z_1 = z_2\ or\ z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$) for all $z_1, z_2 \in U$. Furthermore, it is equally possible to show graphically that $f(z)$ is one-to-one (injective) if and only if $f'(z) \neq 0$ for all $z \in U$. On the other hand, $f(z)$ is injective if and only if it does not turn in its domain, if it does, then in some neighbourhood of its turning point, it must assign the same value twice.
A set Ω is said to be starlike with respect to a fixed point w_0 in it, if the line segment joining w_0 to every other points $w \in \Omega$ lies entirely in the set Ω or if every other points $w \in \Omega$ is visible from the fixed point w_0. If a function $f(z)$ maps U onto a star domain with respect to w_0, then $f(z)$ is said to be starlike with respect to w_0. In particular, if w_0 is the origin, then we say that $f(z)$ is a starlike function in U. Likewise, a set Ω is said to be convex if the line segment joining any two points of Ω lies entirely in Ω. If a function $f(z)$ maps U onto a convex domain, then we say that $f(z)$ is a convex function (see Hamzat [1]).
Now, suppose that S denotes the class of all functions in A which are univalent in the disk U. Then $f(z)$ belonging to S is said to be starlike of order α if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{1.0.4}$$

for some α $(0 \leq \alpha < 1)$. Let $S^*(\alpha)$ denote the class of all functions in S which are starlike of order α. In like manner, a function $f(z)$ belonging to S is said to be convex of order α if and only if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \qquad (z \in U) \tag{1.0.5}$$

for some α $(0 \leq \alpha < 1)$. Let $S^*(\alpha)$ denote the class of all functions in S which are starlike of order α.
Here, we note that $f(z) \in K(\alpha)$ if and only if

$$S^*(\alpha) \subseteq S^*(0) \equiv S^*,$$

$$K(\alpha) \subseteq K(0) \equiv K,$$

and

$$K(\alpha) \subset S^*(\alpha) \subset S,$$

for some α $(0 \leq \alpha < 1)$.
These two classes $S^*(\alpha)$ and $K(\alpha)$ were first initiated by Robertson [2] and were subsequently studied by several authors (MacGregor [3], Pinchuck [4], Schild [5] and Srivastava and Owa [6] among others).
However, in the present investigation, the following definitions shall be considered.

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Definition 1.1. Let $f(z)$ be defined by (1.0.3), then we say that $f(z)$ belong to $\Delta(\alpha,\beta,\lambda,\theta)$ if and only if

$$\Re\left\{\frac{\beta e^{i\theta}\left[z\Big(f'(z)\Big)^{\lambda}\right]+(1-\beta)\left[zf'(z)+z^2\Big(\big[f'(z)\big]^{\lambda}\Big)'\right]}{\beta f(z)+(1-\beta)zf'(z)}\right\}>\alpha \tag{1.0.6}$$

for $0\leq\alpha<1$, $\lambda\geq 1$, $0\leq\beta\leq 1$, $0\leq\theta<\frac{\pi}{2}$ and $z\in U$. Here, we shall give the following remarks (see Hamzat [7]).

Remarks:

(i.) Let $f(z)\in\Delta(\alpha,1,\lambda,0)$, then

$$\Re\left\{\frac{z\Big(f'(z)\Big)^{\lambda}}{f(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.7}$$

This is the class of analytic function known as $\lambda-$*pseudo*$-$*starlike* class of order α.

(ii.) Let $f(z)\in\Delta(\alpha,0,\lambda,0)$, then

$$\Re\left\{\frac{z\Big(\big[f'(z)\big]^{\lambda}\Big)'}{f'(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.8}$$

This is the class of analytic function called $\lambda-$*pseudo*$-$*convex* class of order α.

(iii.) If $f(z)\in\Delta(\alpha,1,\lambda,\theta)$, then

$$\Re\left\{e^{i\theta}\frac{z\Big(f'(z)\Big)^{\lambda}}{f(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.9}$$

This is the class of analytic function known as $\lambda-$*pseudo*$-$*spiralike* class of order α.

(iv.) Let $f(z)\in\Delta(\alpha,0,1,0)$, then

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.10}$$

This is the class of analytic function known as *convex* class of order α.

(v.) If $f(z)\in\Delta(\alpha,1,1,\theta)$, then

$$\Re\left\{e^{i\theta}\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.11}$$

This is the class of analytic function known as *spiralike* class of order α.

(vi.) Let $f(z)\in\Delta(\alpha,1,1,0)$, then

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha,\quad z\in U. \tag{1.0.12}$$

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This is the class of analytic function known as *starlike* class of order α.
(vii.) If $f(z) \in (\alpha, 1, 2, \theta)$, then

$$\Re\left\{e^{i\theta} f'(z)\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U, \tag{1.0.13}$$

which is the product of combination of geometric expression for *bounded turning* and *spiralike* function of order α.
(viii.) If $f(z) \in (\alpha, 1, 2, 0)$, then

$$\Re\left\{f'(z)\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U, \tag{1.0.14}$$

which is the product of combination of geometric expression for *bounded turning* and *starlike* function of order α.
Now, suppose that ψ is an analytic function with positive real part in the unit disk U and has the Taylor's series expansion of the form

$$\psi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \tag{1.0.15}$$

with $B_n > 0,\ n \in \mathbb{N}$.
Definition 1.2. A function $f \in A$ of the form (1.0.3) is said to be in the class $\Delta(\alpha, \beta, \lambda, \theta; \psi)$ if it satisfies the condition that

$$\frac{\beta\Big[e^{i\theta} z\big(f'(z)\big)^{\lambda} - \alpha f(z)\Big] + (1-\beta)z\Big[(1-\alpha)f'(z) + z\Big(\big[f'(z)\big]^{\lambda}\Big)'\Big]}{\Big[1 + \beta(e^{i\theta} - 1) - \alpha\Big]\Big[\beta f(z) + (1-\beta)zf'(z)\Big]} \prec \psi(z). \tag{1.0.16}$$

Here, some well-known Lemmas would be considered.
Lemma 1.1. If a function $q \in P$ (class of Caratheodory functions) is given by

$$q(z) = 1 + c_1 z + c_2 z^2 + ... = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in U),$$

then

$$|c_k| \leq 2 \qquad\qquad (k \in \mathbb{N}),$$

where

$$q(0) = 1 \qquad and \qquad \Re e\,(q(z)) > 0 \qquad (z \in U).$$

Lemma 1.2. If $q_1(z) = 1 + c_1 z + c_2 z^2 + ...$ is an analytic function with positive real part and σ is a complex number, then

$$|c_2 - \sigma c_1^2| \leq 2 \max\{1;\ |2\sigma - 1|\}.$$

The result is sharp for the functions given by

$$q(z) = \frac{1+z^2}{1-z^2} \ \ and \ \ q(z) = \frac{1+z}{1-z}.$$

Lemma 1.3. If $q(z) = 1 + c_1 z + c_2 z^2 + ... \in P$ is an analytic function with positive real part and σ is a complex number, then

$$|c_2 - \sigma c_1^2| \leq \begin{cases} 2 - 4\sigma, & if \quad \sigma \leq 0; \\ 2, & if \quad 0 \leq \sigma \leq 1; \\ 4\sigma - 2, & if \quad \sigma \geq 1. \end{cases}$$

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For $\sigma < 0$ or $\sigma > 1$, the equality holds true if and only if

$$q_1(z) = \frac{1+z}{1-z}$$

or one of its rotations.
If $0 < \sigma < 1$, then the equality holds true if and only if

$$q_1(z) = \frac{1+z^2}{1-z^2}$$

or one of its rotations.
Also, for $\sigma = 0$, the equality holds true if and only if

$$q_1(z) = \Big(\frac{1}{2}+\frac{1}{2}t\Big)\frac{1+z}{1-z} + \Big(\frac{1}{2}-\frac{1}{2}t\Big)\frac{1-z}{1+z}, \ \ (0 \le t \le 1),$$

or one of its rotations.
Further, if $\sigma = 1$, the equality holds true if and only if

$$\frac{1}{q_1(z)} = \Big(\frac{1}{2}+\frac{1}{2}t\Big)\frac{1+z}{1-z} + \Big(\frac{1}{2}-\frac{1}{2}t\Big)\frac{1-z}{1+z}, \ \ (0 \le t \le 1),$$

or one of its rotations.
The above upper bound is sharp and it can be improved as follows when $0 < \sigma < 1$:

$$\Big|c_2 - \sigma c_1^2\Big| + \sigma\big|c_1\big|^2 \le 2, \quad \big(0 < \sigma < \frac{1}{2}\big)$$

and

$$\Big|c_2 - \sigma c_1^2\Big| + (1-\sigma)\big|c_1\big|^2 \le 2, \quad \big(0 < \sigma < \frac{1}{2}\big).$$

For more details on the Lemmas above, interested reader can refer Ma and Minda [8], Shanmugham *et al.* [9] and Aouf *et al.* [10] among others.

2 Preliminary Result

Here, except otherwise stated, we shall assume throughout this paper that $0 \le \alpha < 1, \lambda \ge 1, 0 \le \beta \le 1$ and $|\theta| < \frac{\pi}{2}$.
Theorem 2.1. Let $f(z)$ be defined by (1.0.3). Also let $\psi(z) = 1+B_1z+B_2z^2+B_3z^3+...$, $\psi(z) \in A$ and $\psi'(z) > 0$. If $f(z) \in \Delta(\alpha,\beta,\lambda,\theta;\psi)$, then

$$|a_2| \le \frac{|Y|B_1}{\Omega_1} \tag{2.0.1}$$

and

$$|a_3| \le \frac{|Y|B_1}{\Omega_2} \cdot \max\left\{1, \ \left|\frac{B_2}{B_1} + \frac{YB_1}{\Omega_1^2}\Big[(2-\beta)\Omega_1 - 2\lambda(\lambda-1)\Big(2+\beta(e^{i\theta}-2)\Big)\Big]\right|\right\}, \tag{2.0.2}$$

where,

$$Y = 1-\alpha+\beta(e^{i\theta}-1),$$
$$\Omega_1 = 2\lambda\beta\big(e^{i\theta}-1\big) - (2-\beta)\big(e^{i\theta}-1\big) + 2\lambda - \beta$$

and

$$\Omega_2 = 3\lambda\beta\big(e^{i\theta}-2\big) - (2-\beta)\big(e^{i\theta}-1\big) + 6\lambda - \beta.$$

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Proof. Since $f(z) \in \Delta(\alpha, \beta, \lambda, \theta; \psi)$, there exists a Schwarz function $w(z)$ such that $w(0) = 0$ and $|w(z)| < 1$ in U, then

$$\frac{\beta\Big[e^{i\theta}z\big(f'(z)\big)^{\lambda} - \alpha f(z)\Big] + (1-\beta)z\Big[(1-\alpha)f'(z) + z\Big([f'(z)]^{\lambda}\Big)'\Big]}{\Big[1+\beta(e^{i\theta}-1)-\alpha\Big]\Big[\beta f(z) + (1-\beta)zf'(z)\Big]} = \psi\big(w(z)\big). \tag{2.0.3}$$

If the function $q_1(z)$ is an analytic function with positive real part and $q_1(0) = 1$, then we define $q_1(z)$ such that

$$q_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + ... \quad . \tag{2.0.4}$$

Since $w(z)$ is a Schwarz function, observe that $\Re\{q_1(z)\}$ is positive with $q_1(0) = 1$. We define

$$q(z) = \frac{\beta\Big[e^{i\theta}z\big(f'(z)\big)^{\lambda} - \alpha f(z)\Big] + (1-\beta)z\Big[(1-\alpha)f'(z) + z\Big([f'(z)]^{\lambda}\Big)'\Big]}{\Big[1+\beta(e^{i\theta}-1)-\alpha\Big]\Big[\beta f(z) + (1-\beta)zf'(z)\Big]} = \psi\big(w(z)\big)$$

$$= 1 + d_1 z + d_2 z^2 + ... \qquad z \in U. \tag{2.0.5}$$

From (2.0.4), we have that

$$\omega(z) = \Big(\frac{q_1(z)-1}{q_1(z)+1}\Big) = \frac{1}{2}\Big[c_1 z + \Big(c_2 - \frac{c_1^2}{2}\Big)z^2 + \Big(c_3 + \frac{c_1^3}{4} - c_1c_2\Big)z^3 + ...\Big]. \tag{2.0.6}$$

Since

$$q(z) = \psi\Big(\frac{q_1(z)-1}{q_1(z)+1}\Big) = 1 + d_1 z + d_2 z^2 + ... ,$$

and a simple computation yields

$$\psi\Big(\frac{q_1(z)-1}{q_1(z)+1}\Big) = 1 + \frac{1}{2}B_1c_1 z + \Big[\frac{1}{2}B_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}B_2c_1^2\Big]z^2 + ... \quad , \tag{2.0.7}$$

where

$$d_1 = \frac{1}{2}B_1c_1, \;\; d_2 = \frac{1}{2}B_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}B_2c_1^2, \; ... \; .$$

Therefore, in view of (2.0.3), we can say that

$$\frac{\big[1-\alpha+\beta(e^{i\theta}-1)\big]z + \big[2\lambda\beta(e^{i\theta}-1) + 2(\lambda-\alpha-\beta+1) + \alpha\beta\big]a_2z^2 + Lz^3 + ...}{\big[1-\alpha+\beta(e^{i\theta}-1)\big]\big[z + (2-\beta)a_2z^2 + (3-2\beta)a_3z^3 + ...\big]}$$

$$= 1 + \frac{1}{2}B_1c_1 z + \Big[\frac{1}{2}B_1\Big(c_2 - \frac{c_1^2}{2}\Big) + \frac{1}{4}B_2c_1^2\Big]z^2 + ... \tag{2.0.8}$$

where

$$L = \big(3\lambda\beta(e^{i\theta}-2) + 3(2\lambda-\alpha-\beta+1)\big)a_3 + 2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)a_2^2.$$

From (2.0.8), we obtain

$$a_2 = \frac{YB_1c_1}{2\Omega_1} \tag{2.0.9}$$

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and

$$a_3 = \frac{YB_1}{2\Omega_2}\left\{c_2 - \frac{YB_1^2\left[\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]-\left(B_2-B_1\right)\Omega_1^2}{2B_1\Omega_1^2}c_1^2\right\}, \tag{2.0.10}$$

where

$$Y = 1-\alpha+\beta(e^{i\theta}-1),$$
$$\Omega_1 = 2\lambda\beta\left(e^{i\theta}-1\right)-(2-\beta)\left(e^{i\theta}-1\right)+2\lambda-\beta$$

and

$$\Omega_2 = 3\lambda\beta\left(e^{i\theta}-2\right)-(2-\beta)\left(e^{i\theta}-1\right)+6\lambda-\beta.$$

Using Lemma 1.1 and Lemma 1.2 respectively, in (2.0.9) and (2.0.10), we obtain the inequalities in (2.0.1) and (2.0.2). This obviously completes the proof of Theorem 2.1.
Next, we present the main results.

3 Main Results

Theorem 3.1. Let $\psi = 1+B_1z+B_2z^2+B_3z^3+\ldots$, $\left(B_n>0, n\in\mathbb{N}\right)$, $\psi(z)\in A$ and $\psi'(z)>0$. Also let

$$\rho_1 = \frac{\Omega_1^2}{YB_1\Omega_2}\left\{\frac{B_2}{B_1}-\frac{YB_1}{\Omega_1^2}\left[2\lambda(\lambda-1)\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]-1\right\}.$$

and

$$\rho_2 = \frac{\Omega_1^2}{YB_1\Omega_2}\left\{1+\frac{B_2}{B_1}-\frac{YB_1}{\Omega_1^2}\left[2\lambda(\lambda-1)\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]\right\},$$

where all the parameters involved are as earlier defined.
Suppose that $f(z)$ is of the form (1.0.3) and belongs to the class $\Delta(\alpha,\beta,\lambda,\theta;\psi)$, then for real μ we obtain the following results:
(i) If $\mu\le\rho_1$, then

$$\left|a_3-\mu a_2^2\right| \le \frac{|Y|B_1}{\Omega_2}\left\{\frac{B_2}{B_1}-\frac{|Y|B_1}{\Omega_1^2}\left[\mu\Omega_2+2\lambda(\lambda-1)\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]\right\}.$$

(ii) If $\rho_1\le\mu\le\rho_2$, then

$$\left|a_3-\mu a_2^2\right| \le \frac{|Y|B_1}{\Omega_2}.$$

(iii) If $\mu\ge\rho_2$, then

$$\left|a_3-\mu a_2^2\right| \le \frac{|Y|B_1}{\Omega_2}\left\{\frac{|Y|B_1}{\Omega_1^2}\left[\mu\Omega_2+2\lambda(\lambda-1)\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]-\frac{B_2}{B_1}\right\},$$

where Y,Ω_1 and Ω_2 are as earlier defined.
Proof. Suppose that $f(z)\in\Delta(\alpha,\beta,\lambda,\theta;\psi)$. Also let $q(z)$ be given by (2.0.5) and $q_1(z)$ be of the form (2.0.4), then a_2 and a_3 are given as in (2.0.9) and (2.0.10) of Theorem 2.1 respectively. Observe that

$$a_3-\mu a_2^2 = \frac{YB_1}{2\Omega_2}\left\{c_2-\sigma c_1^2\right\}, \tag{3.0.1}$$

where

$$\sigma = \frac{YB_1^2\left[2\lambda(\lambda-1)\left(2+\beta(e^{i\theta}-2)\right)-(2-\beta)\Omega_1\right]-\left(B_2-B_1\right)\Omega_1^2+\mu YB_1^2\Omega_2}{2B_1\Omega_1^2}.$$

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Now, we considered the following cases .
Case 1.a
If $\mu \leq \rho_1$, then $\sigma \leq 0$. Applying Lemma 1.3 to (3.0.1), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|Y|B_1}{\Omega_2}\left\{\frac{B_2}{B_1} - \frac{|Y|B_1}{\Omega_1^2}\Big[\mu\Omega_2 + 2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big) - (2-\beta)\Omega_1\Big]\right\}$$

which obviously satisfies the item (i) of Theorem 3.1.
Case 1.b
If $\mu = \rho_1$, then $\sigma = 0$. Therefore equality holds true if and only if

$$q_1(z) = \Big(\frac{1+t}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-t}{2}\Big)\frac{1-z}{1+z}, \quad \Big(0 \leq t \leq 1; z \in U\Big).$$

Case 2.a
If $\rho_1 \leq \mu \leq \rho_2$, observe that

$$\max\left\{\frac{|Y|B_1^2\Big[2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big) - (2-\beta)\Omega_1\Big] - \big(B_2 - B_1\big)\Omega_1^2 + \mu|Y|B_1^2\Omega_2}{2B_1\Omega_1^2}\right\} \leq 1,$$

then applying Lemma 1.3 to (3.0.1), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|Y|B_1}{\Omega_2}$$

which satisfies the item (ii) of Theorem 3.1.
Case 2.b
If $\rho_1 < \mu < \rho_1$, then we obtain

$$q_1(z) = \Big(\frac{1+t}{2}\Big)\frac{1+z^2}{1-z^2}.$$

Case 3.a
If $\mu \geq \rho_2$, then $\sigma \geq 1$. Therefore, applying Lemma 1.3 to as before, we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|Y|B_1}{\Omega_2}\left\{\frac{|Y|B_1}{\Omega_1^2}\Big[\mu\Omega_2 + 2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big) - (2-\beta)\Omega_1\Big] - \frac{B_2}{B_1}\right\}$$

which ultimately satisfies the item (iii) of Theorem 3.1.
Case 3.b
If $\mu = \rho_2$, then $\sigma = 1$. Then, equality holds if and only if

$$\frac{1}{q_1(z)} = \Big(\frac{1+t}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-t}{2}\Big)\frac{1-z}{1+z}, \quad \Big(0 \leq t \leq 1; z \in U\Big).$$

This evidently completes the proof of Theorem 3.1.
Theorem 3.2. Let $\psi = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ...,$ $\big(B_n > 0, n \in \mathbb{N}\big)$, $\psi(z) \in A$ and $\psi'(z) > 0$. Also let $f(z)$ be given by (1.0.3) and belongs to the class $\Delta(\alpha, \beta, \lambda, \theta; \psi)$ with $\rho_1 \leq \mu \leq \rho_2$, then in view of Lemma 1.3, Theorem 3.1 can be improved. So let

$$\rho_3 = \frac{\Omega_1^2}{YB_1\Omega_2}\left\{\frac{B_2}{B_1} - \frac{YB_1}{\Omega_1^2}\Big[2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big) - (2-\beta)\Omega_1\Big]\right\}.$$

(i) If $\rho_1 \leq \mu \leq \rho_3$, then we have

$$|a_3 - \mu a_2^2| + \frac{\Omega_1^2}{|Y|B_1\Omega_2}\left\{1 - \frac{B_2}{B_1} + \frac{|Y|B_1}{\Omega_1^2}\Big[2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big) - (2-\beta)\Omega_1\Big] + \frac{|Y|B_1\Omega_2}{\Omega_1^2\mu}\right\}\left|a_2\right|^2 \leq \frac{|Y|B_1}{\Omega_2},$$

(ii) If $\rho_3 \leq \mu \leq \rho_2$, then we have

$$\left|a_3-\mu a_2^2\right|+\frac{\Omega_1^2}{|Y|B_1\Omega_2}\left\{1+\frac{B_2}{B_1}-\frac{|Y|B_1}{\Omega_1^2}\left[2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-(2-\beta)\Omega_1\right]+\frac{|Y|B_1\Omega_2}{\Omega_1^2\mu}\right\}\left|a_2\right|^2 \leq \frac{|Y|B_1}{\Omega_2},$$

where Y, Ω_1 and Ω_2 are as earlier defined.
Proof. For all values of $\rho_1 \leq \mu \leq \rho_3$, we obtain

$$\left|a_3-\mu a_2^2\right|+\big(\mu-\rho_1\big)\left|a_2\right|^2=\frac{|Y|B_1}{2\Omega_2}\left|c_2-\sigma c_1^2\right|$$
$$+\left\{\mu-\frac{\Omega_1^2}{|Y|B_1\Omega_2}\left[\frac{B_2}{B_1}-\frac{|Y|B_1}{\Omega_1^2}\Big(2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-(2-\beta)\Omega_1\Big)-1\right]\right\}.\frac{|Y|^2B_1^2}{4\Omega_1^2}\left|c_1\right|^2$$
$$=\frac{|Y|B_1}{2\Omega_2}\left\{\left|c_2-\sigma c_1^2\right|+\sigma\left|c_1\right|^2\right\}. \tag{3.0.2}$$

Applying Lemma 1.3 to equality (3.0.2), we obtain

$$\left|a_3-\mu a_2^2\right|+\big(\mu-\rho_1\big)\left|a_2\right|^2 \leq \frac{|Y|B_1}{2\Omega_2}$$

where Y and Ω_2 are as earlier defined. This obviously satisfies the item (i) of Theorem 3.2. Finally, for the values of $\mu : \rho_3 \leq \mu \leq \rho_2$, we have that

$$\left|a_3-\mu a_2^2\right|+\big(\rho_2-\mu\big)\left|a_2\right|^2=\frac{|Y|B_1}{2\Omega_2}\left|c_2-\sigma c_1^2\right|$$
$$+\left\{\frac{\Omega_1^2}{|Y|B_1\Omega_2}\left[1+\frac{B_2}{B_1}-\frac{|Y|B_1}{\Omega_1^2}\Big(2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-(2-\beta)\Omega_1\Big)\right]\right\}.\frac{|Y|^2B_1^2}{4\Omega_1^2}\left|c_1\right|^2$$
$$=\frac{|Y|B_1}{2\Omega_2}\left\{\left|c_2-\sigma c_1^2\right|+\big(1-\sigma\big)\left|c_1\right|^2\right\}. \tag{3.0.3}$$

Applying Lemma 1.3 to equality (3.0.3), we obtain

$$\left|a_3-\mu a_2^2\right|+\big(\rho_2-\mu\big)\left|a_2\right|^2 \leq \frac{|Y|B_1}{2\Omega_2}$$

where Y and Ω_2 are as earlier defined. This obviously satisfies the item (ii) of Theorem 3.2 and this evidently ends the proof.
Theorem 3.3. Let $f(z)$ be defined by (1.0.3). Also let $\psi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + ...$, $\big(B_n > 0, n \in \mathbb{N}\big)$, $\psi(z) \in A$ and $\psi'(z) > 0$. If $f(z) \in \Delta(\alpha, \beta, \lambda, \theta; \psi)$, then for real μ

$$\left|a_3-\mu a_2^2\right| \leq \frac{|Y|B_1}{\Omega_2}.\max\left\{1,\left|\frac{B_2}{B_1}+\frac{|Y|B_1}{\Omega_1^2}\Big((2-\beta)\Omega_1-2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-\mu\Omega_2\Big)\right|\right\}.$$

The result is sharp.
Proof. In view of (2.0.9) and (2.0.10), we have that

$$a_3-\mu a_2^2=\frac{YB_1}{2\Omega_2}\Big[c_2-\sigma c_1^2\Big],$$

where

$$\sigma=\frac{YB_1^2\Big[2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-(2-\beta)\Omega_1\Big]-\big(B_2-B_1\big)\Omega_1^2+\mu YB_1^2\Omega_2}{2B_1\Omega_1^2}.$$

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Using Lemma 1.2, then we have

$$\left|a_3-\mu a_2^2\right| \leq \frac{|Y|B_1}{\Omega_2}\cdot \max\left\{1, \left|\frac{B_2}{B_1}+\frac{YB_1}{\Omega_1^2}\Big((2-\beta)\Omega_1-2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)\Big)-\frac{YB_1\Omega_2}{\Omega_1^2}\mu\right|\right\}$$

and this evidently completes the proof of Theorem 3.3.
Let $\psi(z)=\frac{1+Az}{1-Bz}$, $\big(-1\leq B<A\leq 1\big)$ or equivalently, $B_1=A-B$ and $B_2=-B(A-B)$ in Theorem 3.3, then we obtain the following corollary:
Corollary 3.4. Let $f(z)\in\Delta(\alpha,\beta,\lambda,\theta;A,B)$. Then

$$\left|a_3-\mu a_2^2\right|$$

$$\leq \frac{|Y|(A-B)}{\Omega_2}\cdot \max\left\{1, \left|\frac{Y(A-B)}{\Omega_1^2}\Big((2-\beta)\Omega_1-2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-\mu\Omega_2\Big)-B\right|\right\}.$$

The result is sharp. Also, setting $\psi(z)=\frac{1+(1-2\gamma)z}{1-z}$, $\big(-1\leq B<A\leq 1\big)$ in Theorem 3.3, then we obtain the following corollary:
Corollary 3.5. Let $f(z)\in\Delta(\alpha,\beta,\lambda,\theta;(1-2\gamma),-1)$. Then

$$\left|a_3-\mu a_2^2\right|$$

$$\leq \frac{2(1-\gamma)|Y|}{\Omega_2}\cdot \max\left\{1, \left|\frac{2(1-\gamma)Y}{\Omega_1^2}\Big((2-\beta)\Omega_1-2\lambda(\lambda-1)\big(2+\beta(e^{i\theta}-2)\big)-\mu\Omega_2\Big)+1\right|\right\}.$$

The result is sharp. Also, setting
Corollary 3.6. Let $f(z)\in\Delta(\alpha,1,\lambda,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{2(1-\alpha)}{3\lambda-1}\cdot \max\left\{1, \left|\frac{2(1-\alpha)}{(2\lambda-1)^2}\Big(-2\lambda^2+4\lambda-1-\mu(3\lambda-1)\Big)+1\right|\right\}.$$

Corollary 3.7. Let $f(z)\in\Delta(\alpha,0,\lambda,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{(1-\alpha)}{3\lambda}\cdot \max\left\{1, \left|\frac{(1-\alpha)}{\lambda}\Big(2(2-\lambda)-3\mu\Big)+1\right|\right\}.$$

Corollary 3.8. Let $f(z)\in\Delta(\alpha,1,15,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq (1-\alpha)\cdot \max\left\{1, \left|2(1-\alpha)\Big(2-3\mu\Big)+1\right|\right\}.$$

Corollary 3.9. Let $f(z)\in\Delta(\alpha,0,1,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{(1-\alpha)}{3}\cdot \max\left\{1, \left|(1-\alpha)\Big(2-3\mu\Big)+1\right|\right\}.$$

Corollary 3.10. Let $f(z)\in\Delta(0,1,\lambda,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{2}{3\lambda-1}\cdot \max\left\{1, \left|\frac{2}{(2\lambda-1)^2}\Big(-2\lambda^2+4\lambda-1-\mu(3\lambda-1)\Big)+1\right|\right\}.$$

Corollary 3.11. Let $f(z)\in\Delta(0,0,\lambda,0;1,-1)$. Then

$$\left|a_3-\mu a_2^2\right| \leq \frac{(1}{3\lambda}\cdot \max\left\{1, \left|\frac{1}{\lambda}\Big(2(2-\lambda)-3\mu\Big)+1\right|\right\}.$$

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Corollary 3.12. Let $f(z) \in \Delta(0,1,1,0;1,-1)$. Then

$$|a_3 - \mu a_2^2| \leq \max\left\{1, \left|\Big(2(1-2\mu)\Big)+1\right|\right\}.$$

Corollary 3.13. Let $f(z) \in \Delta(0,0,1,0;1,-1)$. Then

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}.\max\left\{1, \left|3(1-\mu)\right|\right\}.$$

For recent works on Fekete-Szegö problems, refer to Hamzat [7], Ali *et al.* [11], Alsoboh and Darus [12] and Arikan *et al.* [13], Hamzat and Makinde [14] and Jahangiri *et al* [15] among others.

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Competing financial interests

The author declares no competing financial interests".

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