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New Multi-valued Contractions with Applications in Dynamic Programming

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Abstract

In this paper, two well-celebrated results in metric fixed point theory due to Jaggi and Dass-Gupta are revisited. To this end, the concepts of Jaggi and Dass-Gupta type bilateral multi-valued contractions are introduced and suitable conditions for existence of fixed points for such mappings are established. A nontrivial example is provided to support the hypotheses of our main results. Moreover, a few consequences which dwell upon the generality of the ideas presented herein are pointed out and discussed. Finally, one of our theorems is applied to investigate sufficient conditions for existence of solutions of nonlinear functional equations arising in dynamic programming and in optimization theory.

Keywords and Phrases: Fixed point, Multi-Valued Contraction, Jagg-type bilateral Contraction, Dass-Gupta-type bilateral Contraction, Functional equation.
MSC2010: 47H10, 54H25

1 Introduction and Preliminaries

Banach fixed point theorem (see [5]) is one of the earliest, simple and versatile classical results for single-valued mappings in fixed point theory with metric space structure. More than a handful of literature embrace applications and modifications of this principle from different perspectives, for example, by weakening the hypotheses, employing different mappings and various forms of quasi and pseudo-metric spaces. In this context, the work of Rhoades [20] is useful for recalling a few salient Banach-type contractive definitions. Two obvious intersecting properties of most generalizations of the Banach fixed point theorem is that their proofs are similar and the contractive conditions consist of linear combinations of the distances between two distinct points and their images. The first-two most embraced extensions of Banach contraction principle involving rational inequalities were presented by Dass-Gupta [9] and Jaggi [10].

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Theorem 1.1 *[10] Let (Ω, μ) be a complete metric space and $\Psi : \Omega \longrightarrow \Omega$ be a continuous mapping. Further, Let Ψ satisfies the condition:*

$$\mu(\Lambda\varsigma, \Lambda\jmath) \leq \frac{\alpha\mu(\varsigma, \Lambda\varsigma)\mu(\jmath, \Lambda\jmath)}{\mu(\varsigma, \jmath)} + \eta\mu(\varsigma, \jmath),$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$ and for some $\alpha, \eta \in [0, 1)$ with $\alpha + \eta < 1$. Then Ψ has a unique fixed point in Ω.

Following [9, 10], a lot of related results (see, e.g. [17, 18]) have been established. Moreover, the theory of set-valued mappings plays major roles in several branches of mathematics because of its applications in areas such as control theory, game theory, biomathematics, qualitative physics, viability theory, and so on. In particular, the idea of multivalued mappings in fixed point theory was initiated by von Neumann in 1935 in the study of game theory. On the other hand, the notion multivalued mappings in metric fixed point theory was brought up by Nadler [16] who used the concept of Hausdorff metric to obtain a generalization of Banach fixed point theorem. Meanwhile, a number of generalizations in diverse frames of Nadler's fixed point theorem have also been investigated by several authors; see, for example, [1–4, 11, 15, 19, 21] and references therein.

On the other hand, in 1976, Caristi [7] presented a fixed point theorem whose statement and proof are significantly different from the Banach fixed point theorem. The main result of [7, Theorem 2.1] is as follows.

Theorem 1.2 *[7] Let (Ω, μ) be a complete metric space, A be a closed subset of Ω. Suppose that $\Lambda : A \longrightarrow A$ is an arbitrary function and $\Psi : A :\longrightarrow \Omega$ is continuous. If there exists a real number $\eta < 0$ such that*

$$\mu(\Lambda(\varsigma), \Psi\Lambda(\varsigma)) \leq \mu(\varsigma, \Lambda\varsigma) + \eta\mu(\varsigma, \Lambda(\varsigma)),$$

for all $\varsigma \in A$, then Λ has a fixed point in A.

Not long ago, a new type of contraction called bilateral contraction was introduced by Chi-Ming et al [8]. The presented idea combined two well-known results in fixed point theory due to Caristi [7] and Jaggi [10].

Definition 1.3 *[8] Let (Ω, μ) be a metric space. A mapping $\Psi : \Omega \longrightarrow \Omega$ is called a Jaggi-type bilateral contraction if there exists a function $\vartheta : \Omega \longrightarrow [0, \infty)$ such that*

$$\mu(\varsigma, \Psi\varsigma) > 0 \text{ implies } \mu(\Psi\varsigma, \Psi\jmath) \leq (\vartheta(\varsigma) - \vartheta(\Psi\varsigma))R_\Psi(\varsigma, \jmath),$$

for all $\varsigma, \jmath \in \Omega$, where

$$R_\Psi(\varsigma, \jmath) = \max\left\{\mu(\varsigma, \jmath), \frac{\mu(\varsigma, \Psi\varsigma)\mu(\jmath, \Psi\jmath)}{\mu(\varsigma, \jmath)}\right\}.$$

Definition 1.4 *[8] Let (Ω, μ) be a metric space. A mapping $\Psi : \Omega \longrightarrow \Omega$ is called a Dass-Gupta-type bilateral contraction if there exists a function $\vartheta : \Omega \longrightarrow [0, \infty)$ such that*

$$\mu(\varsigma, \Psi\varsigma) > 0 \text{ implies } \mu(\Psi\varsigma, \Psi\jmath) \leq (\vartheta(\varsigma) - \vartheta(\Psi\varsigma))Q_\Psi(\varsigma, \jmath),$$

for all $\varsigma, \jmath \in \Omega$, where

$$Q_\Psi(\varsigma, \jmath) = \max\left\{\mu(\varsigma, \jmath), \frac{1 + \mu(\varsigma, \Psi\varsigma)\mu(\jmath, \Psi\jmath)}{1 + \mu(\varsigma, \jmath)}\right\}.$$

The following are the two main results in [8].

IJMSO

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Theorem 1.5 *[8, Theorem 1] Let (Ω, μ) be a complete metric space. If the mapping $\Psi : \Omega \longrightarrow \Omega$ is continuous and forms a Jaggi-type bilateral contraction on Ω, then there exists at least one $u \in \Omega$ such that $\Psi u = u$.*

Theorem 1.6 *[8, Theorem 2] Let (Ω, μ) be a complete metric space. If the mapping $\Psi : \Omega \longrightarrow \Omega$ is continuous and forms a Dass-Gupta-type bilateral contraction on Ω, then there exists at least one $u \in \Omega$ such that $\Psi u = u$.*

Hereafter, we denote by $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ and $\mathcal{K}(\Omega)$ the sets of natural numbers, non-negative reals, real numbers and the family nonempty compact subsets of Ω, respectively.

Definition 1.7 *Let (Ω, μ) be a metric space. Then, a subset A of Ω is called:*

(i) compact if and only if for every sequence of elements of A, there exists a subsequence that converges to an element of A.

(ii) closed if and only if for every sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ of elements of A that converges to an element ς, we have $\varsigma \in A$.

For $A, B \in \mathcal{K}(\Omega)$, the function $\aleph : \mathcal{K}(\Omega) \times \mathcal{K}(\Omega) \longrightarrow \mathbb{R}_+$, defined by

$$\aleph(A, B) = \begin{cases} \max\left\{\sup_{\varsigma\in A} \mu(\varsigma, B), \sup_{\varsigma\in B} \mu(\varsigma, A)\right\}, & \text{if it exists} \\ \infty, & \text{otherwise} \end{cases}$$

is called the generalized Hausdorff-Pompeiu metric on $\mathcal{K}(\Omega)$ induced by the metric μ, where

$$\mu(\varsigma, A) = \inf_{\jmath\in A} \mu(\varsigma, \jmath).$$

Definition 1.8 *Let Ω be a nonempty set. A mapping $\Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ is called a multivalued mapping. A point $u \in \Omega$ is called a fixed point of a multivalued mapping Ψ if $u \in \Psi u$.*

Definition 1.9 *Let (Ω, μ) be a metric space. A mapping $\Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ is called a multi-valued contraction, if there exists $\lambda \in (0, 1)$ such that for all $\varsigma, \jmath \in \Omega$,*

$$\aleph(\Psi\varsigma, \Psi\jmath) \leq \lambda\mu(\varsigma, \jmath).$$

The following result due to Nadler [16] is the first metric fixed point theorem for multi-valued contraction.

Theorem 1.10 *Every multi-valued contraction on a complete metric space has at least one fixed point.*

Lemma 1.11 *[16] Let (Ω, μ) be a metric space and $A, B \in \mathcal{K}(\Omega)$. If $a \in A$, then there exists $b \in B$ such that $\mu(a, b) \leq \aleph(A, B)$.*

The aim of this work is to extend and unify various renowned metric fixed point results due to Caristi [7], Chen etal. [8], Jaggi [10], and related articles in the framework of multi-valued mappings. In particular, we noticed that the assumptions of [8, Theorems 1 and 2] do not guarantee the uniqueness of fixed of the concerned single-valued mapping, thereby making the ideas therein more appropriate for fixed theorems of multi-valued mappings. To this end, the concept of Jaggi-type multi-valued bilateral contraction is initiated herein to properly address the aforementioned gap. Moreover, encouraged by the celebrated fixed theorems of Dass-Gupta and Caristi [9], we inaugurate the idea of Dass-Gupta-type bilateral multi-valued contraction and analyze the existence of fixed for such contraction under suitable hypotheses. A few consequences which dwell upon the generality of our main results are noted and discussed. A nontrivial example is provided to authenticate the assertions of our main theorem. From application point of view, one of our results is employed to investigate sufficient conditions for the existence of solutions of nonlinear functional equations arising in dynamic programming and optimization theory.

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2 Main Results

We start this section by introducing the concept of Jaggi-type bilateral multivalued contraction.

Definition 2.1 *Let (Ω, μ) be a metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multivalued mappings. Then, the pair (Υ, Ψ) is said to form a Jaggi-type bilateral multivalued contraction, if there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Psi\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma, \Psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\, M_{(\Upsilon,\Psi)}(\varsigma, \jmath), \tag{2.1}$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, where

$$M_{(\Upsilon,\Psi)}(\varsigma, \jmath) = \max\left\{\mu(\varsigma, \jmath), \frac{\mu(\varsigma, \Upsilon\varsigma)\mu(\jmath, \Psi\jmath)}{1 + \mu(\varsigma, \jmath)}\right\}.$$

Theorem 2.2 *Let (Ω, μ) be a complete metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multivalued mappings. If the pair (Υ, Ψ) forms a Jaggi-type bilateral multivalued contraction, then there exists $u \in \Omega$ such that $u \in \Upsilon u \cap \Psi u$.*

Proof 2.1 *Let $\varsigma_0 \in \Omega$ be arbitrary. Then, by hypothesis, $\Upsilon\varsigma_0 \in \mathcal{K}(\Omega)$. Therefore, there exists $\varsigma_1 \in \Upsilon\varsigma_0$ such that $\mu(\varsigma_0, \varsigma_1) = \mu(\varsigma_0, \Upsilon\varsigma_0)$. For this $\varsigma_1 \in \Omega$, we have $\Psi\varsigma_1 \in \mathcal{K}(\Omega)$. Since $\Upsilon\varsigma_0, \Psi\varsigma_1 \in \mathcal{K}(\Omega)$ and $\varsigma_1 \in \Upsilon\varsigma_0$, there is a point $\varsigma_2 \in \Psi\varsigma_1$ such that $\mu(\varsigma_1, \varsigma_2) = \mu(\varsigma_1, \Psi\varsigma_1)$. Take $\varsigma_2 \in \Omega$, then $\Upsilon\varsigma_2 \in \mathcal{K}(\Omega)$. So we can find $\varsigma_3 \in \Upsilon\varsigma_2$ such that $\mu(\varsigma_2, \varsigma_3) = \mu(\varsigma_2, \Upsilon\varsigma_2)$. Similarly, $\Psi\varsigma_3$ is a nonempty compact subset of Ω. Thus, there exists $\varsigma_4 \in \Psi\varsigma_3$ with $\mu(\varsigma_3, \varsigma_4) = \mu(\varsigma_3, \Psi\varsigma_3)$. By continuing in this fashion, we build a sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ of elements of Ω, with*

$$\varsigma_{2k+1} \in \Upsilon\varsigma_{2k} \text{ and } \varsigma_{2k+2} \in \Psi\varsigma_{2k+1}, \ k \in \mathbb{N}.$$

Notice that if there exists some $k^* \in \mathbb{N}$ such that $\varsigma_{k^*+1} = \varsigma_{k^*} \in \Upsilon\varsigma_{k^*} \cap \Psi\varsigma_{k^*}$, then we are done with the proof. So, presume that $\varsigma_k \neq \varsigma_{k+1}$ for all $k \in \mathbb{N}$. It follows that

$$\mu(\varsigma_{2k}, \varsigma_{2k+1}) = \mu(\varsigma_{2k}, \Upsilon\varsigma_{2k}) > 0$$

and

$$\mu(\varsigma_{2k+1}, \varsigma_{2k+2}) = \mu(\varsigma_{2k+1}, \Psi x_{2k+1}) > 0.$$

Hence, by Lemma 1.11, and the contractive condition (2.1), we have

$$\begin{aligned}
\mu(\varsigma_{2k}, \varsigma_{2k+1}) &\leq \aleph(\Upsilon\varsigma_{2k-1}, \Psi\varsigma_{2k}) \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k}))\, M_{(\Upsilon,\Psi)}(\varsigma_{2k-1}, \varsigma_{2k}) \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k})) \max\left\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \frac{\mu(\varsigma_{2k-1}, \Upsilon\varsigma_{2k-1})\mu(\varsigma_{2k}, \Psi\varsigma_{2k})}{1 + \mu(\varsigma_{2k-1}, \varsigma_{2k})}\right\} \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k})) \max\left\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \frac{\mu(\varsigma_{2k-1}, \varsigma_{2k})\mu(\varsigma_{2k}, \varsigma_{2k+1})}{1 + \mu(\varsigma_{2k-1}, \varsigma_{2k})}\right\} \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k})) \max\left\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \mu(\varsigma_{2k}, \varsigma_{2k+1})\right\}.
\end{aligned} \tag{2.2}$$

Now, we consider the following cases:

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Case 1: Assume that $\max\{\mu(\varsigma_{2k-1},\varsigma_{2k}),\mu(\varsigma_{2k},\varsigma_{2k+2})\}=\mu(\varsigma_{2k-1},\varsigma_{2k})$*. Then, on account of* (2.2)*, we have*

$$\mu(\varsigma_{2k},\varsigma_{2k+1})\leq\left(\Lambda(\varsigma_{2k-1})-\Lambda(\varsigma_{2k})\right)\mu(\varsigma_{2k-1},\varsigma_{2k}).$$

In other words,

$$0<\frac{\mu(\varsigma_{2k},\varsigma_{2k+1})}{\mu(\varsigma_{2k-1},\varsigma_{2k})}\leq\left(\Lambda(\varsigma_{2k-1})-\Lambda(\varsigma_{2k})\right). \tag{2.3}$$

In this case, by repeating the above steps, for all $n\in\mathbb{N}$*, we have*

$$0<\frac{\mu(\varsigma_n,\varsigma_{n+1})}{\mu(\varsigma_{n-1},\varsigma_n)}\leq\Lambda(\varsigma_{n-1})-\Lambda(\varsigma_n). \tag{2.4}$$

(2.4) *implies that* $\Lambda(\varsigma_n)\leq\Lambda(\varsigma_{n-1})$*, for all* $n\in\mathbb{N}$*. Hence, the sequence* $\{\Lambda(\varsigma_n)\}_{n\in\mathbb{N}}$ *is decreasing and positive, and thus converges to some* $p\geq 0$*. Further, notice that*

$$\begin{aligned}0<\sum_{i=1}^{n}\frac{\mu(\varsigma_i,\varsigma_{i+1})}{\mu(\varsigma_{i-1},\varsigma_i)}\leq&\sum_{i=1}^{n}\left(\Lambda(\varsigma_{i-1})-\Lambda(\varsigma_i)\right)\\=&\left(\Lambda(\varsigma_0)-\Lambda(\varsigma_1)\right)+\left(\Lambda(\varsigma_1)-\Lambda(\varsigma_2)\right)+\cdots+\left(\Lambda(\varsigma_{n-1})-\Lambda(\varsigma_n)\right)\\=&\Lambda(\varsigma_0)-\Lambda(\varsigma_n)\longrightarrow\Lambda(\varsigma_0)-p<\infty\text{ as }n\longrightarrow\infty.\end{aligned} \tag{2.5}$$

It follows that

$$0<\frac{\mu(\varsigma_n,\varsigma_{n+1})}{\mu(\varsigma_{n-1},\varsigma_n)}<\infty,\ \text{for all } n\in\mathbb{N}.$$

Consequently,

$$\lim_{n\longrightarrow\infty}\frac{\mu(\varsigma_n,\varsigma_{n+1})}{\mu(\varsigma_{n-1},\varsigma_n)}=0. \tag{2.6}$$

By (2.6)*, for* $\eta\in(0,1)$*, there exists* $m\in\mathbb{N}$ *such that for all* $n\geq m$*,*

$$\frac{\mu(\varsigma_n,\varsigma_{n+1})}{\mu(\varsigma_{n-1},\varsigma_n)}\leq\eta.$$

Equivalently,

$$\mu(\varsigma_n,\varsigma_{n+1})\leq\eta\mu(\varsigma_{n-1},\varsigma_n). \tag{2.7}$$

Case 2: Assume that $\max\{\mu(\varsigma_{2k-1},\varsigma_{2k}),\mu(\varsigma_{2k},\varsigma_{2k+1})\}=\mu(\varsigma_{2k},\varsigma_{2k+1})$*. Then, using* (2.2)*, we get*

$$\mu(\varsigma_{2k},\varsigma_{2k+1})\leq\left(\Lambda(\varsigma_{2k-1})-\Lambda(\varsigma_{2k})\right)\mu(\varsigma_{2k},\varsigma_{2k+1}). \tag{2.8}$$

Taking $2k=n\in\mathbb{N}$ *in* (2.8)*, yields*

$$\mu(\varsigma_n,\varsigma_{n+1})\leq\left(\Lambda(\varsigma_{n-1}-\Lambda(\varsigma_n))\right)\mu(\varsigma_n,\varsigma_{n+1}). \tag{2.9}$$

As $n\longrightarrow\infty$ *in* (2.9)*,*

$$1\leq\left(\Lambda(\varsigma_{n-1})-\Lambda(\varsigma_n)\right)\longrightarrow 0,$$

gives a contradiction. Combining the results from Case 1 and Case 2, and using (2.7)*, we conclude that the sequence* $\{\mu(\varsigma_n,\varsigma_{n+1})\}_{n\in\mathbb{N}}$ *is decreasing and bounded below, and hence converges to its infimum, say* $\rho\geq 0$*. Since* $\eta<1$*, then clearly,* $\eta=0$*. Now, from* (2.7)*, for each* $i,j\in\mathbb{N}$ *with* $i>j$*, we get*

$$\mu(\varsigma_i,\varsigma_j)\leq\sum_{t=i}^{j-1}\mu(\varsigma_t,\varsigma_{t+1})\leq\frac{\eta^i}{1-\eta}\mu(\varsigma_0,\varsigma_1).$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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Hence, $\lim_{i\longrightarrow\infty}\sup\{\mu(\varsigma_i,\varsigma_j) : i > j\} = 0$*. This proves that the sequence* $\{\varsigma_n\}_{n\in\mathbb{N}}$ *is Cauchy. By completeness of* Ω*, there exists* $u \in \Omega$ *such that* $\varsigma_n \longrightarrow u$ *as* $n \longrightarrow \infty$*. To show that* $u \in \Upsilon u$*, we apply Lemma 1.11 as follows:*

$$\begin{aligned}\mu(u,\Upsilon u) &\leq \mu(u,\varsigma_{2n}) + \mu(\varsigma_{2n},\Upsilon u)\\ &\leq \mu(u,\varsigma_{2n}) + \aleph(\Psi\varsigma_{2n-1},\Upsilon u)\\ &\leq \mu(u,\varsigma_{2n}) + (\Lambda(u)-\Lambda(\varsigma_{2n-1}))\max\left\{\mu(u,\varsigma_{2n-1}),\frac{\mu(u,\Upsilon u)\mu(\varsigma_{2n-1},\Psi\varsigma_{2n-1})}{1+\mu(u,\varsigma_{2n-1})}\right\} \quad (2.10)\\ &\leq \mu(u,\varsigma_{2n}) + (\Lambda(u)-\Lambda(\varsigma_{2n-1}))\max\left\{\mu(u,\varsigma_{2n}),\frac{\mu(u,\Upsilon u)\mu(\varsigma_{2n-1},\varsigma_{2n})}{1+\mu(u,\varsigma_{2n-1})}\right\}.\end{aligned}$$

Letting $n \longrightarrow \infty$ *in (2.10), and using the continuity of* Λ*, we have*

$$\begin{aligned}\mu(u,\Upsilon u) &\leq \mu(u,u) + (\Lambda(u)-\Lambda(u))\max\left\{\mu(u,u),\frac{\mu(u,\Upsilon u)\mu(u,u)}{1+\mu(u,u)}\right\}\\ &\leq 0.\end{aligned}$$

This implies that $u \in \Upsilon u$*. On similar steps, by using*

$$\mu(u,\Psi u) \leq \mu(u,\varsigma_{2n}) + \mu(\varsigma_{2n},\Psi u),$$

we can show that $u \in \Psi u$*. Consequently,* $u \in \Upsilon u \cap \Psi u$*.*

Corollary 2.3 *Let* (Ω,μ) *be a complete metric space and* $\Upsilon : \Omega \longrightarrow \mathcal{K}(\Omega)$ *be a multivalued mapping. If there exists a continuous function* $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ *such that*

$$\mu(\varsigma,\Upsilon\varsigma) > 0 \text{ and } \mu(\jmath,\Upsilon\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma,\Upsilon\jmath) \leq (\Lambda(\varsigma)-\Lambda(\jmath))M_{(\Upsilon,\Upsilon)},$$

for all $\varsigma,\jmath \in \Omega$*,* $\varsigma \neq \jmath$*, where*

$$M_{(\Upsilon,\Upsilon)} = \max\left\{\mu(\varsigma,\jmath),\frac{\mu(\varsigma,\Upsilon\varsigma)\mu(\jmath,\Upsilon\jmath)}{1+\mu(\varsigma,\jmath)}\right\},$$

then, there exists $u \in \Omega$ *such that* $u \in \Upsilon u$*.*

Proof 2.2 *Put* $\Upsilon = \Psi$ *in Theorem 2.2.*

Corollary 2.4 *Let* (Ω,μ) *be a complete metric space and* $\Upsilon,\Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ *be any two multivalued mappings. Assume that there exists a continuous function* $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ *such that*

$$\mu(\varsigma,\Upsilon\varsigma) > 0 \text{ and } \mu(\jmath,\Psi\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma,\Psi\jmath) \leq (\Lambda(\varsigma)-\Lambda(\jmath))\left[r_1\mu(\varsigma,\jmath) + r_2\frac{\mu(\varsigma,\Upsilon\varsigma)\mu(\jmath,\Psi\jmath)}{1+\mu(\varsigma,\jmath)}\right],$$

for all $\varsigma,\jmath \in \Omega$*,* $\varsigma \neq \jmath$*, where* $\sum_{i=1}^{2} r_i = 1$*, then, there exists* $u \in \Omega$ *such that* $u \in \Upsilon u \cap \Psi u$*.*

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Corollary 2.5 *Let (Ω, μ) be a complete metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multivalued mappings. If there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Psi\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma, \Psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\left(\frac{\mu(\varsigma, \Upsilon\varsigma)\mu(\jmath, \Psi\jmath)}{1 + \mu(\varsigma, \jmath)}\right),$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, then, there exists $u \in \Omega$ such that $u \in \Upsilon u \cap \Psi u$.

Remark 2.6 *By setting $\Upsilon = \Psi$ in corollaries 2.4 and 2.5, we can derive independent corollaries analogous to Corollary 2.3.*

Motivated by the results of Dass-Gupta [9], we launch the notion of Dass-Gupta type bilateral multivalued contraction as follows:

Definition 2.7 *Let (Ω, μ) be a metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multivalued mappings. Then, the pair (Υ, Ψ) is said to form a Dass-Gupta- type bilateral multivalued contraction, if there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Psi\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma, \Psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\, W_{(\Upsilon,\Psi)}, \tag{2.11}$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, where

$$W_{(\Upsilon,\Psi)} = \max\left\{\mu(\varsigma, \jmath), \frac{(1 + \mu(\varsigma, \Upsilon\varsigma))\mu(\jmath, \Psi\jmath)}{1 + \mu(\varsigma, \jmath)}\right\}.$$

Theorem 2.8 *Let (Ω, μ) be a complete metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multivalued mappings. Assume that the pair (Υ, Ψ) forms a Dass-Gupta -type bilateral multi-valued contraction, then there exists $u \in \Omega$ such that $u \in \Upsilon u \cap \Psi u$.*

Proof 2.3 *Following the proof of Theorem 2.2, we construct a sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ of points of Ω such that*

$$\varsigma_{2k+1} \in \Upsilon\varsigma_{2k} \text{ and } \varsigma_{2k+2} \in \Psi\varsigma_{2k+1},\ k \in \mathbb{N}.$$

Taking

$$\mu(\varsigma_{2k}, \varsigma_{2k+1}) = \mu(\varsigma_{2k}, \Upsilon\varsigma_{2k}) > 0$$

and

$$\mu(\varsigma_{2k+1}, \varsigma_{2k+2}) = \mu(\varsigma_{2k+1}, \Psi\varsigma_{2k+1}) > 0,$$

Lemma 1.11 and the contractive condition 2.11 guarantee that

$$\begin{aligned}
\mu(\varsigma_{2k}, \varsigma_{2k+1}) &\leq \aleph(\Upsilon\varsigma_{2k-1}, \Psi\varsigma_{2k}) \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k}))\, W_{(\Upsilon,\Psi)}(\varsigma_{2k-1}, \varsigma_{2k}) \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k}))\max\left\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \frac{(1 + \mu(\varsigma_{2k-1}, \Upsilon\varsigma_{2k-1}))\mu(\varsigma_{2k}, \Psi\varsigma_{2k})}{1 + \mu(\varsigma_{2k-1}, \varsigma_{2k})}\right\} \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k}))\max\left\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \frac{(1 + \mu(\varsigma_{2k-1}, \varsigma_{2k}))\mu(\varsigma_{2k}, \varsigma_{2k+1})}{(1 + \mu(\varsigma_{2k-1}, \varsigma_{2k}))}\right\} \\
&\leq (\Lambda(\varsigma_{2k-1}) - \Lambda(\varsigma_{2k}))\max\{\mu(\varsigma_{2k-1}, \varsigma_{2k}), \mu(\varsigma_{2k}, \varsigma_{2k+1})\}.
\end{aligned} \tag{2.12}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 924 - 938.
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From (2.12), we can follow all the steps in Case 1 and Case 2 of Theorem 2.2. After then, we deduce that $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Using the completeness of Ω, there exists $u \in \Omega$ such that $\varsigma_n \longrightarrow u$ as $n \longrightarrow \infty$. To prove that $u \in \Psi u$, we appeal to Lemma 1.11 as follows:

$$\begin{aligned} \mu(u, \Psi u) &\leq \mu(u, \varsigma_{2n}) + \mu(\varsigma_{2n}, \Psi u) \\ &\leq \mu(u, \varsigma_{2n}) + \aleph(\Upsilon\varsigma_{2n-1}, \Psi u) \\ &\leq \mu(u, \varsigma_{2n}) + (\Lambda(\varsigma_{2n-1}) - \Lambda(u)) \\ &\times \max\left\{\mu(u, \varsigma_{2n-1}), \frac{(1+\mu(\varsigma_{2n-1}, \Upsilon\varsigma_{2n-1}))\mu(u, \Psi u)}{(1+\mu(\varsigma_{2n-1}, u))}\right\} \\ &\leq \mu(u, \varsigma_{2n}) + (\Lambda(\varsigma_{2n-1}) - \Lambda(u)) \\ &\times \max\left\{\mu(u, \varsigma_{2n-1}), \frac{(1+\mu(\varsigma_{2n-1}, \varsigma_{2n}))\mu(u, \Psi u)}{1+\mu(\varsigma_{2n-1}, u)}\right\}. \end{aligned} \tag{2.13}$$

Taking limit in (2.13) as $n \longrightarrow \infty$, and using the continuity of Λ, we have

$$\mu(u, \Psi u) \leq 0\,(\mu(u, \Psi u)) = 0.$$

Hence, $u \in \Psi u$. On similar steps, we can show that $u \in \Upsilon u$. Therefore, $u \in \Upsilon u \cap \Psi u$.

Corollary 2.9 *Let (Ω, μ) be a complete metric space and $\Upsilon : \Omega \longrightarrow \mathcal{K}(\Omega)$ be a multi-valued mapping. Assume that there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Upsilon\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma, \Upsilon\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))W_{(\Upsilon,\Upsilon)},$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, where

$$M_{(\Upsilon,\Upsilon)} = \max\left\{\mu(\varsigma, \jmath), \frac{(1+\mu(\varsigma, \Upsilon\varsigma))\mu(\jmath, \Upsilon\jmath)}{1+\mu(\varsigma, \jmath)}\right\},$$

then there exists $u \in \Omega$ such that $u \in \Upsilon u$.

Proof 2.4 *Take $\Upsilon = \Psi$ in Theorem 2.8.*

Corollary 2.10 *Let (Ω, μ) be a complete metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multi-valued mappings. Assume that there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Psi\jmath) > 0$$

imply

$$\aleph(\Upsilon\varsigma, \Psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\left[r_1\mu(\varsigma, \jmath) + r_2\frac{(1+\mu(\varsigma, \Upsilon\varsigma))\mu(\jmath, \Psi\jmath)}{1+\mu(\varsigma, \jmath)}\right],$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, where $\sum_{i=1}^{2} r_i = 1$, then there exists $u \in \Omega$ such that $u \in \Upsilon u \cap \Psi u$.

Corollary 2.11 *Let (Ω, μ) be a complete metric space and $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ be any two multi-valued mappings. Assume that there exists a continuous function $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ such that*

$$\mu(\varsigma, \Upsilon\varsigma) > 0 \text{ and } \mu(\jmath, \Psi\jmath) > 0$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 924 - 938.
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imply

$$\aleph(\Upsilon\varsigma, \Psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\left(\frac{(1+\mu(\varsigma, \Upsilon\varsigma))\mu(\jmath, \Psi\jmath)}{1+\mu(\varsigma, \jmath)}\right),$$

for all $\varsigma, \jmath \in \Omega$, $\varsigma \neq \jmath$, *then, there exists* $u \in \Omega$ *such that* $u \in \Upsilon u \cap \Psi u$.

Remark 2.12 *By setting* $\Upsilon = \Psi$ *in corollaries 2.10 and 2.11, we can derive independent corollaries similar to Corollary 2.9.*

In what follows, we provide an example to support the hypotheses of Theorem 2.2.

Example 2.13 *Let* $\Omega = \{(0,0), (3,3), (3,4), (5,5)\}$ *be endowed with the taxicab metric* $\mu : \Omega^2 \longrightarrow \mathbb{R}$ *defined by*

$$\mu((\varsigma_1, \varsigma_2), (\jmath_1, \jmath_2)) = |\varsigma_1 - \jmath_1| + |\varsigma_2 - \jmath_2|$$

for all $\varsigma_1, \varsigma_2, \jmath_1, \jmath_2 \in \Omega$. *Obviously,* (Ω, μ) *is a complete metric space. Consider two multivalued mappings* $\Upsilon, \Psi :\longrightarrow \mathcal{K}(\Omega)$ *defined by*

$$\Upsilon\varsigma = \{(3,4), (5,5)\} \text{ for all } \varsigma \in \Omega$$

and

$$\Psi\varsigma = \{(0,0), (3,4)\} \text{ for all } \varsigma \in \Omega.$$

Also, defined the mapping $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ *as follows:*

$$\Lambda(\varsigma) = \begin{cases} 12, & \text{if } \varsigma = (0,0) \\ 10, & \text{if } \varsigma = (3,3) \\ 7, & \text{if } \varsigma \in \{(3,4), (5,5)\}. \end{cases}$$

Now, for all $\varsigma \in \Omega$, *we have*

$$\begin{aligned} \mu((0,0), \Upsilon\varsigma) &= \inf\{\mu((0,0), a) : a \in \Upsilon\varsigma\} \\ &= \inf\{7, 10\} = 7. \end{aligned}$$

Similarly, $\mu((3,3), \Upsilon\varsigma) = 1$, $\mu((3,4), \Upsilon\varsigma) = 0$ *and* $\mu((5,5), \Upsilon\varsigma) = 0$.
For the mapping Ψ, *we have*

$$\mu((0,0), \Psi\varsigma) = \inf\{\mu((0,0), b) : b \in \Psi\varsigma\} = 0.$$

On same steps, $\mu((3,3), \Psi\varsigma) = 1$, $\mu((3,4), \Psi\varsigma) = 0$ *and* $\mu((5,5), \Psi\varsigma) = 3$.
Now, considering the points for which $\mu(\varsigma, \Upsilon\varsigma) > 0$ *and* $\mu(\jmath, \Psi\jmath) > 0$, *that is, for all* $\varsigma \in \{(0,0), (3,3)\}$ *and* $\jmath \in \{(3,3), (5,5)\}$, *we have*

$$\begin{aligned} \aleph(\Upsilon\varsigma, \Psi\jmath) = \aleph(\Upsilon(0,0), \Psi(3,3)) &= \max\{\sup_{\varsigma \in \Upsilon(0,0)} \mu(\varsigma, \Psi(3,3)), \sup_{\jmath \in \Psi(3,3)} \mu(\Upsilon(0,0), \jmath)\} \\ &= \max\{3, 7\} = 7, \end{aligned}$$

$$\mu(\varsigma, \jmath) = \mu((0,0), (3,3)) = 6.$$

$$\mu(\varsigma, \Upsilon\varsigma) = \mu((0,0), \Upsilon(0,0)) = 7.$$

$$\mu(\jmath, \Psi\jmath) = \mu((3,3), \Psi(3,3)) = 3.$$

IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
VOL. 6, NO. 2, PP. 924 - 938.
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Therefore,

$$\begin{aligned} M_{(\Upsilon,\Psi)}(\varsigma,\jmath) &= M_{(\Upsilon,\Psi)}((0,0),(3,3)) \\ &= \max\left\{\mu(\varsigma,\jmath), \frac{\mu(\varsigma,\Upsilon\varsigma)\mu(\jmath,\Psi\jmath)}{1+\mu(\varsigma,\jmath)}\right\} \\ &= \max\{6,3\} = 6, \end{aligned}$$

and

$$\Lambda(\varsigma) - \Lambda(\jmath) = \Lambda((0,0)) - \Lambda((3,3)) = 2.$$

Consequently, $\mu(\varsigma,\Upsilon\varsigma) > 0$ *and* $\mu(\jmath,\Psi\jmath) > 0$ *imply*

$$\begin{aligned} \aleph(\Upsilon\varsigma,\Psi\jmath) &= 7 \leq 2(6) \\ &= (\Lambda(\varsigma) - \Lambda(\jmath))M_{(\Upsilon,\Psi)}(\varsigma,\jmath). \end{aligned}$$

Similarly, for $\varsigma = (3,3)$ *and* $\jmath = (5,5)$*, we have*

$$\aleph(\Upsilon\varsigma,\Psi\jmath) = \aleph(\Upsilon(3,3),\Psi(5,5)) = 7.$$

$$\mu(\varsigma,\jmath) = \mu((3,3),(5,5)) = 4.$$

$$\mu(\varsigma,\Upsilon\varsigma) = \mu((3,3),\Upsilon(3,3)) = 1.$$

$$\mu(\jmath,\Psi\jmath) = \mu((5,5),\Psi(5,5)) = 3.$$

Hence,

$$\begin{aligned} M_{(\Upsilon,\Psi)}(\varsigma,\jmath) &= M_{(\Upsilon,\Psi)}((3,3),(5,5)) \\ &= \max\left\{\mu(\varsigma,\jmath), \frac{\mu(\varsigma,\Upsilon\varsigma)\mu(\jmath,\Psi\jmath)}{1+\mu(\varsigma,\jmath)}\right\} \\ &= \max\left\{4, \frac{3}{5}\right\} = 4, \end{aligned}$$

and

$$\Lambda(\varsigma) - \Lambda(\jmath) = \Lambda((3,3)) - \Lambda((5,5)) = 3.$$

It follows that $\mu(\varsigma,\Upsilon\varsigma) > 0$ *and* $\mu(\jmath,\Psi\jmath) > 0$ *imply*

$$\begin{aligned} \aleph(\Upsilon\varsigma,\Psi\jmath) &= 7 \leq 3(4) \\ &= (\Lambda(\varsigma) - \Lambda(\jmath))M_{(\Upsilon,\Psi)}(\varsigma,\jmath). \end{aligned}$$

Thus, all the conditions of Theorem 2.2 are satisfied. Consequently, we can see that there exists $(3,4) \in \Omega$ *such that* $(3,4) \in \Upsilon(3,4) \cap \Psi(3,4)$.

3 Consequences in Single-valued Mappings

In this section, we deduce a few single-valued counterparts of the results from Section 2.

Theorem 3.1 *Let* (Ω,μ) *be a complete metric space and* $\vartheta,\psi : \Omega \longrightarrow \Omega$ *be any two single-valued mappings. If there exists a continuous function* $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ *such that*

$$\mu(\varsigma,\vartheta\varsigma) > 0 \text{ and } \mu(\jmath,\psi\jmath) > 0$$

imply

$$\mu(\vartheta\varsigma,\psi\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\Omega_{(\vartheta,\psi)}(\varsigma,\jmath)$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 924 - 938.
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for all $\varsigma, \jmath \in \Omega,\ \varsigma \neq \jmath$, *where*

$$\Omega_{(\vartheta,\psi)}(\varsigma,\jmath) = \max\left\{\mu(\varsigma,\jmath), \frac{\mu(\varsigma,\vartheta\varsigma)\mu(\jmath,\psi\jmath)}{1+\mu(\varsigma,\jmath)}\right\}.$$

Then, ϑ *and* ψ *have a common fixed point in* Ω*, that is, there exists* $u \in \Omega$ *such that* $u = \vartheta(u) = \psi(u)$.

Proof 3.1 *Consider two multivalued mappings* $\Upsilon, \Psi : \Omega \longrightarrow \mathcal{K}(\Omega)$ *defined as*

$$\Upsilon\varsigma = \{\vartheta\varsigma\} \text{ and } \Psi\varsigma = \{\psi\varsigma\}$$

for all $\varsigma \in \Omega$*. Clearly,* $\{\vartheta\varsigma\}, \{\psi\varsigma\} \in \mathcal{K}(\Omega)$ *for each* $\varsigma \in \Omega$*. Notice that in this case,* $\aleph(\Upsilon\varsigma, \Psi\jmath) = \mu(\varsigma,\jmath)$ *for all* $\varsigma,\jmath \in \Omega$*. Consequently, Theorem 2.2 can be applied to find* $u \in \Omega$ *such that* $u \in \Upsilon u = \{\vartheta u\}$ *and* $u \in \Psi u = \{\psi u\}$*, which further implies that* $u = \vartheta(u) = \psi(u)$.

Corollary 3.2 *Let* (Ω, μ) *be a complete metric space and* $\vartheta : \Omega \longrightarrow \Omega$ *be a single-valued mapping. If there exists a continuous function* $\Lambda : \Omega \longrightarrow \mathbb{R}^+$ *such that*

$$\mu(\varsigma, \vartheta\varsigma) > 0$$

implies

$$\mu(\vartheta\varsigma, \vartheta\jmath) \leq (\Lambda(\varsigma) - \Lambda(\jmath))\Omega_{(\vartheta,\vartheta)}(\varsigma,\jmath)$$

for all $\varsigma, \jmath \in \Omega,\ \varsigma \neq \jmath$, *where*

$$\Omega_{(\vartheta)}(\varsigma,\jmath) = \max\left\{\mu(\varsigma,\jmath), \frac{\mu(\varsigma,\vartheta\varsigma)\mu(\jmath,\vartheta\jmath)}{1+\mu(\varsigma,\jmath)}\right\}.$$

Then, there exists $u \in \Omega$ *such that* $\vartheta(u) = u$.

Remark 3.3

(i) Note that Corollary 3.2 and the result of Chen et al. [8, Theorem 1] are independent. In the latter, the mapping Ψ is required to be continuous and the function Λ is taken arbitrarily. In our case, (Corollary 3.2), the conditions are reversed. Similar observation concerns Theorem 2.8 and the result of Chen et al. [8, Theorem 2].

(ii) Following the ideas of Theorem 3.1 and Corollary 3.2, we can establish similar results using Theorem 2.8 and its associated consequences.

4 Applications in Dynamic Programming

Mathematical optimization is one of the areas in which the techniques of fixed point theory are generously used. It is a known fact that dynamic programming provides important tools for mathematical optimization and computer programming. There are more than a handful of literature dealing with the study of functional equations arising in dynamic programming. The interested reader may consult [6, 12–14, 19] and references therein for detail analysis on this topic.

Hereafter, let M and Q be Banach spaces and $F \subseteq M$, $G \subseteq Q$, where F and G are the state and decision spaces, respectively. Recall that a state space is the set of all feasible state and a decision space is the resultant network formed by the nodes of feasible states and all the feasible decisions. Let $B(F)$ be the set of all bounded real-valued functions defined on F. It is well-known that $B(F)$ equipped with the metric μ, given as

$$\mu(p,q) = \sup_{\varsigma \in F} |p(\varsigma) - q(\varsigma)|,$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 924 - 938.
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for all $p, q \in B(F)$ is a complete metric space. The classical form of functional equation arising in dynamic programming is given as

$$h(\varsigma) = opt_{\jmath} \aleph(\varsigma, \jmath, h(\kappa(\varsigma, \jmath)))$$

where $\varsigma \in F$ and $\jmath \in G$ are the state and decision vectors, respectively, the *opt* denotes either sup, inf, max or min, $\kappa : F \times G \longrightarrow F$ is the transformation of the process and $h(\varsigma)$ represents the optimal return function with initial state ς.

Our aim in this section is to apply Theorem 3.1 to study the optimal decision in the given state space using dynamic programming in connection with the problem of solving the functional equations, given as

$$g(\varsigma) = \sup_{\varsigma \in F} \{\pi(\varsigma, \jmath) + \Delta(\varsigma, \jmath, g(\kappa(\varsigma, \jmath))) : \jmath \in G\} \tag{4.1}$$

and

$$h(\varsigma) = \sup_{\varsigma \in F} \{\pi(\varsigma, \jmath) + \Theta(\varsigma, \jmath, h(\kappa(\varsigma, \jmath))) : \jmath \in G\} \tag{4.2}$$

where $\pi : F \times G \longrightarrow \mathbb{R}$ and $\Delta, \Theta : F \times G \times \mathbb{R} \longrightarrow \mathbb{R}$ are bounded functions.
Consider two mappings $L, \Phi : B(F) \longrightarrow B(F)$ defined by

$$L(p(\varsigma)) = \sup_{\varsigma \in F} \{\pi(\varsigma, \jmath) + \Delta(\varsigma, \jmath, p(\kappa(\varsigma, \jmath))) : \jmath \in G\}. \tag{4.3}$$

$$\Phi(p(\varsigma)) = \sup_{\varsigma \in F} \{\pi(\varsigma, \jmath) + \Theta(\varsigma, \jmath, p(\kappa(\varsigma, \jmath))) : \jmath \in G\}. \tag{4.4}$$

Notice that fixed points of the mappings $L(p(\varsigma))$ and $\Phi(p(\varsigma))$ are solutions of problems (4.1) and (4.2), respectively. Now, we investigate the existence of a common solution of Equations (4.1) and (4.2) under the following hypotheses.

Theorem 4.1 *Consider Equations* (4.3) *and* (4.4)*. Assume that the following conditions are satisfied:*

(i) the functions Δ, Θ *and* π *are bounded and continuous;*

(ii) for all $(\varsigma, \jmath) \in F \times G$*,* $p, q \in B(F)$ *and* $t \in F$*, there exits a continuous function* $\varrho : B(F) \longrightarrow \mathbb{R}_+$ *such that*

$$|p - L(p)| > 0 \text{ and } |q - M(q)| > 0$$

imply

$$|\Delta(\varsigma, \jmath, p(t)) - \Theta(\varsigma, \jmath, q(t))| \leq (\varrho(p) - \varrho(q))|p - q|, \tag{4.5}$$

where $(\varrho(p) - \varrho(q)) = \sup_{\varsigma \in F}(\varrho(p(\varsigma)) - \varrho(q(\varsigma)))$.

Then, the functional equations (4.1) *and* (4.2) *have a common bounded solution in* $B(F)$.

Proof 4.1 *Let* ω *be an arbitrary positive real number and* $p_1, p_2 \in B(F)$ *with* $p_1 \neq p_2$*. For* $\varsigma \in F$*, choose* $\jmath_1, \jmath_2 \in G$ *such that*

$$L(p_1(\varsigma)) < \pi(\varsigma, \jmath_1) + \Delta(\varsigma, \jmath_1, p_1(\kappa(\varsigma, \jmath_1))) + \omega, \tag{4.6}$$

$$\Phi(p_2(\varsigma)) < \pi(\varsigma, \jmath_2) + \Theta(\varsigma, \jmath_2, p_2(\kappa(\varsigma, \jmath_2))) + \omega. \tag{4.7}$$

And, from Equations (4.3), (4.4)*, we have*

$$L(p_1(\varsigma)) \geq \pi(\varsigma, \jmath_2) + \Delta(\varsigma, \jmath_2, p_1(\kappa(\varsigma, \jmath_2))), \tag{4.8}$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 6, No. 2, pp. 924 - 938.
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$$\Phi(p_2(\varsigma)) \geq \pi(\varsigma, \jmath_1) + \Theta(\varsigma, \jmath_1, p_2(\kappa(\varsigma, \jmath_2))). \tag{4.9}$$

Then, Equations (4.6) and (4.9) together with (4.5) give

$$\begin{aligned} & L(p_1(\varsigma)) - \Phi(p_2(\varsigma)) \\ & \leq \ \Delta(\varsigma, \jmath_1, p_1(\kappa(\varsigma, \jmath_1))) - \Theta(\varsigma, \jmath_1, p_2(\kappa(\varsigma, \jmath_1))) + \omega \\ & \leq \ |\Delta(\varsigma, \jmath_1, p_1(\kappa(\varsigma, \jmath_1))) - \Theta(\varsigma, \jmath_1, p_2(\kappa(\varsigma, \jmath_1)))| + \omega \\ & \leq (\varrho(p_1) - \varrho(p_2))|p_1(\varsigma) - p_2(\varsigma)| + \omega. \end{aligned} \tag{4.10}$$

Similarly, from Equations (4.7), (4.8) and (4.5), we have

$$\begin{aligned} & \Phi(p_2(\varsigma)) - L(p_1(\varsigma)) \\ & \leq \ \Theta(\varsigma, \jmath_2, p_2(\kappa(\varsigma, \jmath_2))) - \Delta(\varsigma, \jmath_2, p_1(\kappa(\varsigma, \jmath_2))) + \omega \\ & \leq \ |\Theta(\varsigma, \jmath_2, p_2(\kappa(\varsigma, \jmath_2))) - \Delta(\varsigma, \jmath_2, p_1(\kappa(\varsigma, \jmath_2)))| + \omega \\ & = |\Delta(\varsigma, \jmath_2, p_1(\kappa(\varsigma, \jmath_2))) - \Theta(\varsigma, \jmath_2, p_2(\kappa(\varsigma, \jmath_2)))| + \omega \\ & \leq \ (\varrho(p_1) - \varrho(p_2))|p_1(\varsigma) - p_2(\varsigma)| + \omega. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$|L(p_1(\varsigma)) - \Phi(p_2(\varsigma))| \leq (\varrho(p_1) - \varrho(p_2))|p_1(\varsigma) - p_2(\varsigma)| + \omega. \tag{4.12}$$

Taking supremum over all $\varsigma \in F$ in (4.12), and noting that $\omega > 0$ is arbitrary, we obtain

$$\begin{aligned} \mu(L(p_1), \Phi(p_2)) & \leq (\varrho(p_1) - \varrho(p_2))\mu(p_1, p_2) \\ & \leq \ (\varrho(p_1) - \varrho(p_2)) \max\left\{\mu(p_1, p_2), \frac{\mu(p_1, L(p_1))\mu(p_2, M(p_2))}{1 + \mu(p_1, p_2)}\right\}. \end{aligned}$$

Consequently, all the assertions of Theorem 3.1 are satisfied. It follows that the mappings L and Φ have a common fixed point in $B(F)$, which corresponds to the common bounded solution of problems (4.1) and (4.2).

5 Conclusion

In this work, we have established two concepts, namely, Jaggi-type bilateral multi-valued contraction and Dass-Gupta -type bilateral multi-valued contraction. The ideas are multi-valued generalizations of a recently introduced concept of bilateral contractions for single-valued mappings due to Chen et al. [8]. Moreover, our results also extended the well-known metric fixed point theorems of Caristi [7], Dass-Gupta and Caristi [9] and Jaggi [10], and related articles in the framework of multi-valued mappings. The presented results herein were motivated by the fact that the hypotheses of [8, Theorems 1 and 2] do not guarantee the uniqueness of fixed points of the concerned single-valued mappings, thereby making the notions therein more suitable for fixed point theorems of multi-valued mappings. It is noteworthy that the results of this paper can be improved upon when discussed in the setting of some generalized metric spaces such as b-metric space, F-metric space, G-metric space, modular metric space, and some other quasi or pseudo metric spaces. Also, from application perspective, the multi-valued contractions constructed in this work can be applied to establish some existence theorems of differential inclusions of various types.

Competing Interests

Authors declare that they have no competing interests.

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