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A 3-step fourth derivatives method for numerical integration of third order ordinary differential equations

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Abstract

A 3-Step Hybrid Block Method (S3HBM) with three mid-step grid points based on Linear Multistep Method is presented in this work for direct approximation of solution of third-order Initial and Boundary Value Problems (IVPs and BVPs). Multiple Finite Difference formulas are derived using the collocation technique. These formulas are unified in a block formulation to form a numerical integrator that solves general third-order ordinary differential equations. Basic properties of the derived method are discussed. The superiority of this method over existing methods is established numerically on different test problems, to show its better performance in terms od accuracy.

Keywords: Third Order Ordinary Differential Equation, Initial Value Problem, Boundary Value Problems, Block Method, Finite Differences, Linear Multistep Hybrid Method.
MSC2010:65A05

1 Introduction

In this work, third-order problems of the type:

$$y''' = f(x, y, y', y''), \quad a \leq x \leq b \tag{1.1}$$

with initial conditions

$$y(a) = \alpha_0, y'(a) = \alpha_1, y''(a) = \alpha_2 \tag{1.2}$$

or boundary conditions

$$\begin{aligned} y(a) = \alpha_0, y'(a) = \alpha_1, y(b) = \beta_0 \\ y(a) = \alpha_0, y'(a) = \alpha_1, y'(b) = \beta_1 \end{aligned} \tag{1.3}$$

are considered, where α_i $i = 0, 1, 2$, β_0, β_1, a, b are real constants. $x \in [a, b]$. We assume that the function f is continuous in $[a, b] \times \mathbb{R}^3$. Thus we as well assume the existence and uniqueness of the solution of (1.1) with (1.2) and (1.1) with any of (1.3) as discussed in [1,2]. It is also assumed that (1.1) is well posed and the numerical solution is the interest. Accurate numerical methods for solving third-order initial and boundary value problems are available in literature. To mention but few are: Non-polynomial splines [3], Quartic Splines [4], Collocation method [5–7]. Block methods

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credited to [7] and many others. Emphasis is on collocation technique which is employed in this work. This method is found to be flexible and more efficient in that; (i) it approximates the solution of (1.1) at several intra-points; (ii) its block formulations consists of several linear multistep methods viz-a-viz the main and additional methods required for direct solution of (1.1), such that it overcomes the overlapping of pieces of solutions; (iii) it does not require any starting value from other methods i.e it is self starting.

In this work a three-step continuous hybrid block method (S3HBM) with three intra-step points is developed via collocation approach, for the direct approximation of the solution of (1.1) with (1.2) or any of the conditions in (1.3).

2 Derivation of the Methods

Here, the derivation of a continuous implicit three mid intra-step hybrid block method is described, for the solution of (1.1) over the integration interval $[a, b]$,

$$\pi_N \equiv \{a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b\}$$

with h the constant step-size, $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, N$.
Method similar to this work can be seen in [9], where the three off-step points in the interval $0 < x_{n+r} < x_{n+s} < x_{n+t} < 1$ are given such that $(r, s, t) = (\frac{3}{8}, \frac{5}{8}, \frac{7}{8})$. Optimized two-step block method with three hybrid points for solving third order initial value problems can be found in [8]. In [9], the off-step points r, s, g are such that $0 < r, s, g < 2$. In the case of this paper, the off-step points are mid points in the interval $0 < s < 1 < u < 2 < v < 3$, where $s = \frac{1}{2}$, $u = \frac{3}{2}$ and $v = \frac{5}{2}$.

Consider the approximation $p(x)$ of $y(x)$ given by the polynomial

$$y(x) \simeq p(x) = \sum_{i=0}^{11} \rho_j x^j \,. \tag{2.1}$$

whose third derivatives yields

$$y'''(x) \simeq p'''(x) = \sum_{j=3}^{11} \rho_j j(j-1)(j-2)x^{j-3} \tag{2.2}$$

and on differentiation further

$$y^{(iv)}(x) \simeq p^{(iv)}(x) = \sum_{j=4}^{11} \rho_j j(j-1)(j-2)(j-3)x^{j-4} \,. \tag{2.3}$$

where ρ_j are real unknown coefficients to be determined. Interpolating (2.1) at the points x_{n+i}, $i = 0, 1, 2$, collocating (2.2) at the points $x_{n+\frac{i}{2}} =$, $i = 0, 1, \ldots, 6$ and finally collocating (2.3) at the points x_{n+i}, $i = 0, 3$, we obtain a system of 12 equations with 12 unknowns (the ρ_j, $i = 0, 1, \ldots, 11$). This system can be written in matrix form as

$$\begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^{11} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^{11} \\
1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & \cdots & x_{n+2}^{11} \\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & \cdots & 990x_n^8 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{2}} & 60x_{n+\frac{1}{2}}^2 & 120x_{n+\frac{1}{2}}^3 & \cdots & 990x_{n+\frac{1}{2}}^8 \\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & \cdots & 990x_{n+1}^8 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{2}} & 60x_{n+\frac{3}{2}}^2 & 120x_{n+\frac{3}{2}}^3 & \cdots & 990x_{n+\frac{3}{2}}^8 \\
0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & \cdots & 990x_{n+2}^8 \\
0 & 0 & 0 & 6 & 24x_{n+\frac{5}{2}} & 60x_{n+\frac{5}{2}}^2 & 120x_{n+\frac{5}{2}}^3 & \cdots & 990x_{n+\frac{5}{2}}^8 \\
0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & \cdots & 990x_{n+3}^8 \\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & \cdots & 7920x_n^7 \\
0 & 0 & 0 & 0 & 24 & 120x_{n+3} & 360x_{n+3}^2 & \cdots & 7920x_{n+3}^7
\end{pmatrix}
\begin{pmatrix}
\rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \\ \rho_9 \\ \rho_{10} \\ \rho_{11}
\end{pmatrix}
=
\begin{pmatrix}
y_0 \\ y_1 \\ y_2 \\ h^3 f_0 \\ h^3 f_{\frac{1}{2}} \\ h^3 f_1 \\ h^3 f_{\frac{3}{2}} \\ h^3 f_2 \\ h^3 f_{\frac{5}{2}} \\ h^3 f_3 \\ h^4 g_0 \\ h^4 g_3
\end{pmatrix}$$

where $y^{(j)}_{n+i} \simeq y^{(j)}(x_{n+i})$, $f_{n+i} \simeq f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})$, and $g_{n+i} \simeq \frac{df(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})}{dx_{n+i}}$.

Solving the above system using a computer algebraic system (cas) in Mathematica we obtain the values of the coefficients ρ_i, $i = 0, 1, \ldots, 11$, (not included here). Substituting these values of ρ_i's into the polynomial (2.1), the formula is obtained as:

$$p(x) = \sum_{i=0}^{2} \alpha_i(x) y_{n+i} + h^3 \left(\sum_{i=0}^{3} \beta_i(x) f_i + \sum_{i=1}^{3} \hat{\beta}_i(x) f_{\frac{2i-1}{2}} \right) + h^4 \left(\gamma_0(x) g_0 + \gamma_3(x) g_3 \right) \tag{2.4}$$

where the α, β, $\hat{\beta}$ and γ are continuous coefficients (which are large expressions and are not included here, but can be easily obtained with the help of a cas).

Evaluating $p(x)$ in (2.4) at the points $x = x_{n+\frac{1}{2}}, x_{n+\frac{3}{2}}, x_{n+\frac{5}{2}}$ and x_{n+3} and after some simplifications, we obtain the following methods:

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} = {} & \frac{3y_n}{8} + \frac{3y_{n+1}}{4} - \frac{y_{n+2}}{8} + h^3 \left(\frac{131381 f_n}{580608000} + \frac{89281 f_{n+\frac{1}{2}}}{4032000} + \frac{29047 f_{n+1}}{860160} + \frac{1963 f_{n+\frac{3}{2}}}{362880} + \frac{3491 f_{n+2}}{2580480} \right. \\
& \left. - \frac{261 f_{n+\frac{5}{2}}}{448000} + \frac{35137 f_{n+3}}{193536000} \right) - h^4 \left(\frac{3 g_n}{102400} + \frac{89 g_{n+3}}{2764800} \right) \\
y_{n+\frac{3}{2}} = {} & -\frac{y_n}{8} + \frac{3y_{n+1}}{4} + \frac{3y_{n+2}}{8} - h^3 \left(\frac{536243 f_n}{1741824000} + \frac{9367 f_{n+\frac{1}{2}}}{1344000} + \frac{9169 f_{n+1}}{286720} + \frac{401 f_{n+\frac{3}{2}}}{17010} - \frac{563 f_{n+2}}{860160} \right. \\
& \left. + \frac{617 f_{n+\frac{5}{2}}}{1344000} - \frac{235507 f_{n+3}}{1741824000} \right) - h^4 \left(\frac{257 g_n}{8294400} + \frac{193 g_{n+3}}{8294400} \right) \\
y_{n+\frac{5}{2}} = {} & \frac{3y_n}{8} - \frac{5y_{n+1}}{4} + \frac{15y_{n+2}}{8} + h^3 \left(\frac{7813 f_n}{23224320} + \frac{2049 f_{n+\frac{1}{2}}}{89600} + \frac{47801 f_{n+1}}{516096} + \frac{1097 f_{n+\frac{3}{2}}}{8064} + \frac{10357 f_{n+2}}{172032} \right. \\
& \left. + \frac{23 f_{n+\frac{5}{2}}}{161280} + \frac{33871 f_{n+3}}{116121600} \right) - h^4 \left(\frac{g_n}{110592} + \frac{29 g_{n+3}}{552960} \right) \\
y_{n+3} = {} & y_n - 3y_{n+1} + 3y_{n+2} + h^3 \left(\frac{4699 f_n}{3402000} + \frac{52 f_{n+\frac{1}{2}}}{875} + \frac{421 f_{n+1}}{1680} + \frac{3208 f_{n+\frac{3}{2}}}{8505} + \frac{421 f_{n+2}}{1680} + \frac{52 f_{n+\frac{5}{2}}}{875} \right. \\
& \left. + \frac{4699 f_{n+3}}{3402000} \right) + h^4 \left(\frac{g_n}{16200} - \frac{g_{n+3}}{16200} \right)
\end{aligned}\right\} \tag{2.5}$$

Then, evaluating $p'(x)$ in (2.4), at the points $x = x_{\frac{i}{2}}$, $i = 1, 2, \ldots, 6$, we obtain the following formulas for approximating the first derivatives:

$$\left.\begin{aligned}
hy'_n = {} & -\frac{3y_n}{2} + 2y_{n+1} - \frac{y_{n+2}}{2} + h^3 \left(\frac{221533 f_n}{8981280} + \frac{1883 f_{n+\frac{1}{2}}}{12375} + \frac{13997 f_{n+1}}{110880} + \frac{8378 f_{n+\frac{3}{2}}}{280665} + \frac{3 f_{n+2}}{12320} \right. \\
& \left. + \frac{f_{n+\frac{5}{2}}}{3465} - \frac{3611 f_{n+3}}{32076000} \right) + h^4 \left(\frac{431 g_n}{213840} + \frac{23 g_{n+3}}{1069200} \right) \\
hy'_{n+\frac{1}{2}} = {} & -y_n + y_{n+1} - h^3 \left(\frac{48527 f_n}{14784000} + \frac{2359649 f_{n+\frac{1}{2}}}{66528000} + \frac{1257 f_{n+1}}{394240} - \frac{2869 f_{n+\frac{3}{2}}}{11975040} + \frac{493 f_{n+2}}{5322240} \right. \\
& \left. + \frac{2147 f_{n+\frac{5}{2}}}{22176000} + \frac{89239 f_{n+3}}{2395008000} \right) - h^4 \left(\frac{1813 g_n}{5702400} + \frac{g_{n+3}}{140800} \right) \\
hy'_{n+1} = {} & -\frac{y_n}{2} + \frac{y_{n+2}}{2} - h^3 \left(\frac{6389 f_n}{37422000} + \frac{2791 f_{n+\frac{1}{2}}}{86625} + \frac{212 f_{n+1}}{2079} + \frac{2986 f_{n+\frac{3}{2}}}{93555} + \frac{5 f_{n+2}}{11088} \right. \\
& \left. - \frac{31 f_{n+\frac{5}{2}}}{259875} + \frac{29 f_{n+3}}{519750} \right) + h^4 \left(\frac{g_n}{19800} + \frac{g_{n+3}}{89100} \right) \\
hy'_{n+\frac{3}{2}} = {} & -y_{n+1} + y_{n+2} - h^3 \left(\frac{1079861 f_n}{14370048000} - \frac{2819 f_{n+\frac{1}{2}}}{7392000} + \frac{21779 f_{n+1}}{7096320} + \frac{1298369 f_{n+\frac{3}{2}}}{35925120} + \frac{21779 f_{n+2}}{7096320} \right. \\
& \left. - \frac{2819 f_{n+\frac{5}{2}}}{7392000} + \frac{1079861 f_{n+3}}{14370048000} \right) - h^4 \left(\frac{779 g_n}{68428800} - \frac{779 g_{n+3}}{68428800} \right) \\
hy'_{n+2} = {} & \frac{y_n}{2} - 2y_{n+1} + \frac{3y_{n+2}}{2} + h^3 \left(\frac{47513 f_n}{74844000} + \frac{7753 f_{n+\frac{1}{2}}}{259875} + \frac{13843 f_{n+1}}{110880} + \frac{698 f_{n+\frac{3}{2}}}{4455} + \frac{7759 f_{n+2}}{332640} \right. \\
& \left. - \frac{31 f_{n+\frac{5}{2}}}{12375} + \frac{38911 f_{n+3}}{74844000} \right) + h^4 \left(\frac{7 g_n}{356400} - \frac{29 g_{n+3}}{356400} \right) \\
hy'_{n+\frac{5}{2}} = {} & y_n - 3y_{n+1} + 2y_{n+2} + h^3 \left(\frac{226489 f_n}{159667200} + \frac{187963 f_{n+\frac{1}{2}}}{3168000} + \frac{190603 f_{n+1}}{760320} + \frac{4519733 f_{n+\frac{3}{2}}}{11975040} + \frac{292613 f_{n+2}}{1182720} \right. \\
& \left. + \frac{318803 f_{n+\frac{5}{2}}}{13305600} - \frac{2276639 f_{n+3}}{1197504000} \right) + h^4 \left(\frac{157 g_n}{2280960} + \frac{487 g_{n+3}}{1900800} \right) \\
hy'_{n+3} = {} & \frac{3y_n}{2} - 4y_{n+1} + \frac{5y_{n+2}}{2} + h^3 \left(\frac{109981 f_n}{56133000} + \frac{7747 f_{n+\frac{1}{2}}}{86625} + \frac{331}{880} f_{n+1} + \frac{23882 f_{n+\frac{3}{2}}}{40095} + \frac{13919 f_{n+2}}{27720} \right. \\
& \left. + \frac{20903 f_{n+\frac{5}{2}}}{86625} + \frac{3001763 f_{n+3}}{112266000} \right) + h^4 \left(\frac{19 g_n}{267300} - \frac{1127 g_{n+3}}{534600} \right)
\end{aligned}\right\} \tag{2.6}$$

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Similarly, evaluating $p''(x)$ in (2.4), at the points $x = x_{\frac{i}{2}}$, $i = 1, 2, \ldots, 6$, we obtain the formulas for approximating the second derivatives:

$$\left.\begin{aligned}
h^2 y''_n =\;& y_n - 2y_{n+1} + y_{n+2} - h^3\left(\frac{1547893 f_n}{6804000} + \frac{11429 f_{n+\frac{1}{2}}}{23625} + \frac{6229 f_{n+1}}{30240} + \frac{118 f_{n+\frac{3}{2}}}{1215} - \frac{137 f_{n+2}}{6048}\right.\\
&\left. + \frac{289 f_{n+\frac{5}{2}}}{23625} - \frac{8959 f_{n+3}}{2268000}\right) - h^4\left(\frac{169 g_n}{10800} + \frac{23 g_{n+3}}{32400}\right)\\
h^2 y''_{n+\frac{1}{2}} =\;& y_n - 2y_{n+1} + y_{n+2} + h^3\left(\frac{26872157 f_n}{1306368000} - \frac{82841 f_{n+\frac{1}{2}}}{504000} - \frac{624473 f_{n+1}}{1935360} - \frac{398 f_{n+\frac{3}{2}}}{25515} - \frac{16909 f_{n+2}}{645120}\right.\\
&\left. + \frac{18187 f_{n+\frac{5}{2}}}{1512000} - \frac{701179 f_{n+3}}{186624000}\right) + h^4\left(\frac{17843 g_n}{6220800} + \frac{4147 g_{n+3}}{6220800}\right)\\
h^2 y''_{n+1} =\;& y_n - 2y_{n+1} + y_{n+2} - h^3\left(\frac{20401 f_n}{4082400} - \frac{409 f_{n+\frac{1}{2}}}{4725} - \frac{5 f_{n+1}}{288} + \frac{2938 f_{n+\frac{3}{2}}}{25515} - \frac{719 f_{n+2}}{30240}\right.\\
&\left. + \frac{17 f_{n+\frac{5}{2}}}{1575} - \frac{13229 f_{n+3}}{4082400}\right) - h^4\left(\frac{19 g_n}{19440} + \frac{11 g_{n+3}}{19440}\right)\\
h^2 y''_{n+\frac{3}{2}} =\;& y_n - 2y_{n+1} + y_{n+2} + h^3\left(\frac{2086271 f_n}{435456000} + \frac{71173 f_{n+\frac{1}{2}}}{1512000} + \frac{557063 f_{n+1}}{1935360} + \frac{1604 f_{n+\frac{3}{2}}}{8505} - \frac{72071 f_{n+2}}{1935360}\right.\\
&\left. + \frac{2669 f_{n+\frac{5}{2}}}{216000} - \frac{494933 f_{n+3}}{145152000}\right) + h^4\left(\frac{443 g_n}{691200} + \frac{1201 g_{n+3}}{2073600}\right)\\
h^2 y''_{n+2} =\;& y_n - 2y_{n+1} + y_{n+2} - h^3\left(\frac{37951 f_n}{20412000} - \frac{79 f_{n+\frac{1}{2}}}{1125} - \frac{6859 f_{n+1}}{30240} - \frac{12562 f_{n+\frac{3}{2}}}{25515} - \frac{2351 f_{n+2}}{10080}\right.\\
&\left. + \frac{641 f_{n+\frac{5}{2}}}{23625} - \frac{130199 f_{n+3}}{20412000}\right) - h^4\left(\frac{49 g_n h^2}{97200} + \frac{101 g_{n+3} h^2}{97200}\right)\\
h^2 y''_{n+\frac{5}{2}} =\;& y_n - 2y_{n+1} + y_{n+2} + h^3\left(\frac{6712669 f_n}{1306368000} + \frac{71669 f_{n+\frac{1}{2}}}{1512000} + \frac{178573 f_{n+1}}{645120} + \frac{10022 f_{n+\frac{3}{2}}}{25515} + \frac{31699 f_{n+2}}{55296}\right.\\
&\left. + \frac{112793 f_{n+\frac{5}{2}}}{504000} - \frac{25067741 f_{n+3}}{1306368000}\right) + h^4\left(\frac{4531 g_n}{6220800} + \frac{17459 g_{n+3}}{6220800}\right)\\
h^2 y''_{n+3} =\;& y_n - 2y_{n+1} + y_{n+2} - h^3\left(\frac{2497 f_n}{972000} - \frac{1693 f_{n+\frac{1}{2}}}{23625} - \frac{6893 f_{n+1}}{30240} - \frac{4034 f_{n+\frac{3}{2}}}{8505} - \frac{13807 f_{n+2}}{30240}\right.\\
&\left. - \frac{12833 f_{n+\frac{5}{2}}}{23625} - \frac{519097 f_{n+3}}{2268000}\right) - h^4\left(\frac{7 g_n}{10800} - \frac{509 g_{n+3}}{32400}\right)
\end{aligned}\right\} \tag{2.7}$$

It is noteworthy that methods in (2.5)-(2.7) form the required unique block method applied sequentially for solving (1.1)-(1.2)) and simultaneously (for solving (1.1)-(1.3).

3 Analysis of the Method

3.1 Local truncation error and order

Consider the linear difference operators associated with the formulas in (2.5), which could be written as

$$\begin{aligned}
\mathcal{L}[z(x);\; h] \equiv\,& h^j z^{(j)}_{n+\frac{i}{2}} - \left[\sum_{k=0}^{2} \alpha_k z_{n+k} + h^3\left(\sum_{k=0}^{3} \beta_k z'''_{n+k} + \sum_{k=1}^{3} \hat{\beta}_k z'''_{n+\frac{2k-1}{2}}\right)\right.\\
&\left. + h^4\left(\gamma_0 z^{(4)}_n + \gamma_3 z^{(4)}_{n+3}\right)\right]
\end{aligned} \tag{3.1}$$

for $i = 0(1)6$, $j = 0, 1, 2$.

The local truncation error of each of the formulas in (2.5)-(2.7) is the amount by which the exact solution of the ODE fails to satisfy the corresponding difference operator. Consider the exact solution $y(x)$ in (3.1), expanding in Taylor series around x the following is arrived at

$$\mathcal{L}[y(x);\; h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_q h^q y^{(q)}(x) + O(h^{(q+1)})\,. \tag{3.2}$$

where the C_i are constants. Suppose the first $p + 2$ constants are such that

$$C_0 = C_1 = C_2 = \cdots = C_{p+2} = 0$$

and $C_{p+3} \neq 0$, then

$$\mathcal{L}[y(x);\ h] = C_{p+3}h^{p+3}y^{(p+3)}(x) + O(h^{p+4}) \tag{3.3}$$

Here, p is called the order of the method and C_{p+3} is the principal error constant. As such, the method is said to be consistent of order p (see [13]).

Setting $x = x_n$, the local truncation errors for formulas in (2.5) are given by

$$\begin{aligned}
\mathcal{L}_{\frac{1}{2}}[y(x_n);\ h] &= -\frac{140911y^{(12)}(x_n)h^{12}}{11771943321600} + O(h)^{13};\\
\mathcal{L}_{\frac{3}{2}}[y(x_n);\ h] &= -\frac{297y^{(12)}(x_n)h^{12}}{1468006400} + (h)^{13};\\
\mathcal{L}_{\frac{5}{2}}[y(x_n);\ h] &= -\frac{274375y^{(12)}(x_n)h^{12}}{470877732864} + (h)^{13};\\
\mathcal{L}_{3}[y(x_n);\ h] &= -\frac{27y^{(12)}(x_n)h^{12}}{31539200} + (h)^{13}
\end{aligned} \tag{3.4}$$

For the formulas in (2.6)-(2.7) the local truncation errors may be obtained similarly. From the (3.3), the order of the block method is $p = 9$.

3.2 Zero-stability and convergence

A numerical method is zero-stable if the solutions remain bounded as $h \to 0$. Following the procedure in [12], to show the zero-stability the block method (2.5)-(2.7) may be rewritten in a form such that $y^{(k)}_{n+\frac{j}{2}}$, for each $k = 0, 1, 2$, $j = 1(1)6$ are on the left hand side. Thus, as $h \to 0$ the method in matrix form becomes

$$A_0 Y_\mu = A_1 Y_{\mu-1} \tag{3.5}$$

where

$$Y_\mu = \left(Y^0_\mu, Y^1_\mu, Y^2_\mu\right)^T, \quad Y_{\mu-1} = \left(Y^0_{\mu-1}, Y^1_{\mu-1}, Y^2_{\mu-1}\right)^T$$

$$\begin{aligned}
Y^0_\mu &= (y_{\frac{1}{2}}, y_1, y_{\frac{3}{2}}, y_2, y_{\frac{5}{2}}, y_3)\\
&\vdots\\
Y^2_\mu &= (y''_{\frac{1}{2}}, y''_1, y''_{\frac{3}{2}}, y''_2, y''_{\frac{5}{2}}, y''_3)\\
Y^0_{\mu-1} &= (y_{\frac{1}{2}-1}, y_0, y_{\frac{3}{2}-1}, y_1, y_{\frac{5}{2}-1}, y_2)\\
&\vdots\\
Y^3_{\mu-1} &= (y''_{\frac{1}{2}-1}, y''_0, y''_{\frac{3}{2}-1}, y''_1, y''_{\frac{5}{2}-1}, y'''_2)
\end{aligned}$$

A_0 is the identity matrix of order 18, $A_0 = I_{18}$, and A_1 is a 18×18 matrix given by

$$\begin{pmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{pmatrix}$$

with the A_{11}, A_{22} and A_{33} being 6×6 matrices respectively, given by

$$A_{ii} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad i = 1, 2, 3$$

The characteristic polynomial of each of the matrix A_{11}, A_{22} and A_{33} is given as $|A_{mm} - \lambda I_6| = 0$, for $m = 1, 2, 3$, that is, $\lambda^5(\lambda - 1) = 0$. The roots of the characteristic polynomial are $\lambda_\tau = 0$, for $\tau = 1, \ldots, 5$ and $\lambda_6 = 1$. Consequently, the method is zero-stable, since the roots of the characteristic polynomial are all zero except one, whose modulus is one (see [13, 15]). For convergence, we state the following theorem.

Theorem 3.1. *Henrici [?]. A linear multistep method is said to be convergent if it is consistent (with order $p \geq 1$) and it is zero-stable.*

By the above analysis, the method has order $p = 9$, and is zero-stable. Then, by Theorem 3.1, the method is convergent.

3.3 Computational procedure

3.3.1 For IVPs of the form (1.1)-(1.2)

The block method has been implemented using *Mathematica*, enhanced by the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as summarized in the following code:

Algorithm

a, b (integration interval), N (number of steps), y_{00}, y_{10}, y_{20} (initial values), $f, \frac{df}{dx}$ *sol*, discrete approximate solution of the IVP (1.1)-(1.2)) Let $n = 0, h = \frac{b-a}{N}$ Let $x_n = a, y_n = y_{00}, y'_n = y_{10}, y''_n = y_{20}$. Let sol$=\{(x_n, y_n)\}$.
Solve (2.5)-(2.7) to get $y_{n+i}, y'_{n+i}, y''_{n+i}$, $i = 1(1)3$
Let $sol = sol \cup \{(x_{n+i}, y_{n+i})\}_{i=1(1)3}$.
Let $x_n = x_n + 3h, y_n = y_{n+3}, y'_n = y'_{n+3}, y''_n = y''_{n+3}$
Let $n = n + 3$
$n = N$ go to 12 go to 3
End

3.3.2 For BVPs of the form (1.1)-(1.3)

Assume the boundary conditions (1.3) are known:

$$y(a) = \alpha_0, \;\; y(b) = \beta_0, \;\; y'(a) = \alpha_1$$

the vector of unknowns y is given by

$$y = (y_1, \ldots, y_{N-1}, y'_1, \ldots, y'_N, y''_0, \ldots, y''_N)^T$$

This makes a total of $(N-1)+(N)+(N+1) = 3N$ unknowns. Consider the formulas in (2.5), for $n = 0(3), N-3$, there are $4N/3$ formulas. Also, consider (2.6), for $n = 0(3), N-3$, there are $7N/3$ total formulas therein. Finally, consider (2.7), $n = 0(3), N-3$, there are also $7N/3$ total formulas as well. In total for (2.5)-(2.7) there are a total of $4N/3 + 7N/3 + 7N/3 = 3N$. Hence we have a system with 3N equations and 3N unknowns, whose solution provides a set of approximate values of the BVP in (1.1)-(1.3). Now, the formulae (2.5)-(2.7) form a block and solved simultaneously using codes written in Mathematica, enhanced by the feature *NSolve[]* for linear problems and the feature *FindRoot[]* for nonlinear problems solved by Newton?s method. The following is the algorithm

Algorithm

ENTER PARTITION a, b, NUMBER OF STEPS N, BOUNDARY VALUES y_a, y'_a, y_b, $f, \frac{df}{dx}$ *sol*, discrete approximate solution of the BVP (1.1)-(1.3)) Set $n = 0, h = \frac{b-a}{N}$ Let sol$=\{(x_n, y_n)\}$.
Generate block system Solve (2.5)-(2.7) to get unknown values
Let $sol = sol \cup \{(x_{n+i}, y_{n+i})\}_{i=1(1)3}$.
Let $n = n + 3$
End

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4 Numerical Examples

Numerical examples are presented to show the accuracy of the method proposed (S3HBM). We state here that we have not been able to find any method as ours which of order 9 in literature. As such, we have tried to make fair comparison with some of the methods in literature to show the efficiency of our method in terms of errors and Maximum errors (ME) obtained where applicable. We also need to reiterate that the number of sub-interval used in each problem is a multiple of 3 so as to obtain results at the right hand boundary interval. Hence, we have extended the right hand boundary of some of the problems during computation. In all, the problems considered are mainly to show the efficiency in terms of the errors obtained using the S3HBM.

Problem 1. Consider the BVP discussed in [9]

$$\begin{array}{l} y''' - xy = (x^3 - 2x^2 - 5x - 3)e^x, \quad x \in [0,1] \\ y(0) = y'(0) = 1, \quad y'(1) = -e \end{array} \tag{4.1}$$

whose exact solution is $y(x) = xe^x(1+x)$.

Table 1: Comparison of Maximum Errors (ME) for Problem 1

h	ME in [9]	h	ME in S3HBM	N	ME in [5]	N	ME in S3HBM
$\frac{1}{16}$	1.3647×10^{-10}	$\frac{1}{15}$	7.7229×10^{-17}	7	4.12×10^{-12}	6	4.3392×10^{-15}
$\frac{1}{32}$	4.2086×10^{-12}	$\frac{1}{30}$	4.4805×10^{-22}	14	1.56×10^{-14}	12	4.2653×10^{-18}
$\frac{1}{64}$	1.2939×10^{-13}	$\frac{1}{60}$	4.3765×10^{-25}	28	6.08×10^{-17}	24	4.1719×10^{-21}
				56	2.37×10^{-19}	48	4.0758×10^{-24}
				112	9.27×10^{-22}	96	3.9807×10^{-27}

Table 1 compares the S3HBM and those of [5] and [9]. This problem has a misprint in [9] but the correct version is in [5]. The method in [5] is of order 8 and that of [9] has order 5. In both cases, the S3HBM performs better for different step-sizes (h) and for different number of subintervals (N), which shows a better performance over both methods.

Problem 2. Consider the BVP discussed in [9].

$$\begin{array}{l} y''' + y = (x-4)\sin(x) + (1-x)\cos(x), \quad x \in [0,1] \\ y(0) = 0, \quad y'(0) = -1, \quad y'(1) = \sin 1 \end{array} \tag{4.2}$$

whose exact solution is $y(x) = (x-1)\sin(x)$.

Table 2: Comparison of maximum Errors (ME) for Problem 2

h	ME in [9]	h	ME in S3HBM
$\frac{1}{16}$	1.03179×10^{-11}	$\frac{1}{15}$	1.0507×10^{-18}
$\frac{1}{32}$	3.24907×10^{-13}	$\frac{1}{30}$	9.1161×10^{-19}
$\frac{1}{64}$	1.02789×10^{-14}	$\frac{1}{60}$	9.1145×10^{-20}

Table 2 shows the Maximum Error obtained using different step-sizes and compared with the method in [9] having order 5 and the S3HBM in this work. It clearly shows that the method presented is more superior to the cited literature.

Problem 3. Consider the nonlinear BVP discussed in [5].

$$\begin{array}{l} y''' + 2e^{-3y} = 4(1+x)^{-3}, \quad x \in [0,1] \\ y(0) = 0, \quad y'(0) = 1, \quad y'(1) = 0.5 \end{array} \tag{4.3}$$

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whose exact solution is $y(x) = \ln(1 + x)$.

Table 3: Comparison of maximum Errors (ME) for Problem 3

N	ME in S3HBM	N	ME in [11]
6	9.046×10^{-10}	7	5.24×10^{-9}
12	2.374×10^{-12}	14	2.39×10^{-11}
24	3.189×10^{-15}	28	9.50×10^{-14}
48	3.407×10^{-18}	56	3.62×10^{-16}
96	3.406×10^{-21}	112	2.27×10^{-17}

From Table 3 different values of N has been used to obtain the maximum error for Problem 3. There is a missprint in the boundary condition of this problem in [5] but has been corrected in this work. As stated earlier, The N used in this work for comparison is a multiple of 3, hence a closer value to the N used in [5] for this particular problem was used so as to make comparison. It should also be stated here that the order for FDM in [5] is $p = 8$. A fair comparison in the use of N is thus considered.

Problem 4. Consider the Sandwich problem discussed in [5] governed by a linear third order differential equation with the boundary conditions at three different points.

$$\begin{aligned} &y''' - l^2 y' + a = 0, \quad x \in [0,1] \\ &y'(0) = 0, \quad y'(1) = 0, \quad y\left(\tfrac{1}{2}\right) = 0.5 \end{aligned} \tag{4.4}$$

whose exact solution is $y(x) = \dfrac{z\left(l(2x-1) - 2\sinh(lx) + 2\cosh(lx)\tanh\left(\frac{l}{2}\right)\right)}{2l^3}$.
where $y(x)$ is shear deformation of the Sandwich beams, l and a are physical parameters depending on the elasticity of the layers.

Table 4: Comparison of maximum Errors (ME) for Problem 4

$l = 5$				$l = 10$			
N	ME in S3HBM	N	ME in [5]	N	ME in S3HBM	N	ME in [5]
12	8.008×10^{-14}	14	5.78×10^{-12}	12	7.068×10^{-12}	14	2.26×10^{-10}
24	9.179×10^{-17}	28	2.16×10^{-14}	24	1.109×10^{-14}	28	7.91×10^{-13}
48	9.346×10^{-20}	56	4.94×10^{-16}	48	1.251×10^{-17}	56	2.96×10^{-15}
96	9.223×10^{-23}	112	1.07×10^{-16}	96	1.269×10^{-20}	112	1.73×10^{-17}

Table 4 shows the Maximum Error (ME) obtained by S3HBM compared to those obtained in [5] for different values of the parameter l. The Number of steps employed is a multiple of 3 but still less than that which was used in method or order 8 in [5]. The S3HBM performed better compare to the Finite Difference Method (FDM) in [5].

Problem 5. Consider the IVP discussed in [8]

$$\begin{aligned} &y''' = 3\sin(x), \quad x \in [0,1] \\ &y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 2 \end{aligned} \tag{4.5}$$

whose exact solution is $y(x) = 3\cos(x) + \frac{x^2}{2}$.

The order of the method in [8] is 8. We solved Problem 5 for $x \in [0, 3]$. This is because in other to reach to the end of the integration interval (end point of integration), N has to be a multiple of 3. In this case, $N = 30, 300$ is used so as to have $h = 0.1, 0.01$ respectively for the sake of comparison. Hence, we recorded the solution for $x = 0.1(0.1)1$ only in Table 5. This shows the performance of the S3HBM for this problem.

Table 5: Comparison of errors for Problem 5

	$h = 0.1$		$h = 0.01$	
x	Error in S3HBM	Error in [8]	Error in S3HBM	Error in [8]
0.1	2.6030×10^{-19}	4.1078×10^{-15}	2.8491×10^{-29}	4.4409×10^{-16}
0.2	1.1080×10^{-18}	1.6875×10^{-14}	1.1377×10^{-28}	1.2212×10^{-15}
0.3	2.5430×10^{-18}	5.0848×10^{-14}	2.5508×10^{-28}	2.4425×10^{-15}
0.4	4.4779×10^{-18}	1.1779×10^{-13}	4.5109×10^{-28}	3.7748×10^{-15}
0.5	6.9530×10^{-18}	2.4081×10^{-13}	7.0009×10^{-28}	5.5511×10^{-15}
0.6	9.9679×10^{-18}	4.3709×10^{-13}	9.9985×10^{-28}	8.4377×10^{-15}
0.7	1.3422×10^{-17}	7.3708×10^{-13}	1.3476×10^{-27}	1.1324×10^{-14}
0.8	1.7326×10^{-17}	1.1662×10^{-12}	1.7404×10^{-27}	1.4544×10^{-14}
0.9	2.1680×10^{-17}	1.7587×10^{-12}	2.1746×10^{-27}	1.8985×10^{-14}
1.0	2.6379×10^{-17}	2.5466×10^{-12}	2.6464×10^{-27}	2.3870×10^{-14}

Problem 6. Consider the IVP discussed in [11]

$$\begin{aligned} &y''' + 4y' = x, \quad x \in [0,1] \\ &y(0) = y'(0) = 0, \quad y''(0) = 1 \end{aligned} \tag{4.6}$$

whose exact solution is $y(x) = \frac{3}{16}(1 \cos(2x)) + \frac{x^2}{8}$.

Table 6: Comparison of maximum errors for Problem 6

S3HBM			Method in [11]		
b	TS	ME	b	TS	ME
5	30	2.14×10^{-12}	5	46	1.20×10^{-10}
	45	3.79×10^{-14}		56	3.69×10^{-11}
	60	1.32×10^{-15}		88	2.44×10^{-12}
10	60	4.81×10^{-12}	10	61	5.54×10^{-09}
	75	5.24×10^{-13}		91	5.04×10^{-10}
	90	8.52×10^{-14}		136	4.53×10^{-11}
15	75	4.69×10^{-11}	15	76	2.67×10^{-08}
	90	7.70×10^{-12}		110	2.91×10^{-09}
	105	1.66×10^{-12}		180	1.52×10^{-10}
20	90	1.75×10^{-10}	20	91	5.29×10^{-08}
	105	3.88×10^{-11}		129	6.54×10^{-09}
	120	1.03×10^{-11}		204	4.19×10^{-10}

Table 6 shows the Maximum Error obtained for different values of N (Total nummber of sub intervals TS) and the different end point of integration (b). Comparing with an order 5 method in [11], the S3HBM demonstrates its efficiency by using less number of subinternals to achieve better results.

Problem 7. Consider the IVP discussed in [11]

$$\begin{aligned} &y''' - \frac{3}{8y^5} = 0, \quad x \in [0,2] \\ &y(0) = 0, \quad y'(0) = \frac{1}{2}, \quad y''(0) = -\frac{1}{4} \end{aligned} \tag{4.7}$$

whose exact solution is $y(x) = \sqrt{1+x}$.

Table 7 shows the error for Problem 7 using the S3HBM and compared with hybrid method in [11]. Though results of the error in [11] was given for all values of the grid points $x = 0.1(0.1)2$, but we have given the values of the errors for $x = 0.2(0.2)2$. The errors obtained demonstrates the efficiency in terms of accuracy of the solution obtained when compared to the exact solution

Table 7: Comparison of errors for Problem 7

x	Error in S3HBM	Error in [11]
0.2	8.43637×10^{-15}	2.181300×10^{-11}
0.4	2.81807×10^{-14}	7.069234×10^{-11}
0.6	5.26285×10^{-14}	1.348290×10^{-10}
0.8	8.06737×10^{-14}	2.105969×10^{-10}
1.0	1.11941×10^{-13}	2.964273×10^{-10}
1.2	1.46179×10^{-13}	3.913760×10^{-10}
1.4	1.83172×10^{-13}	4.947245×10^{-10}
1.6	2.22711×10^{-13}	6.058536×10^{-10}
1.8	2.64603×10^{-13}	7.242000×10^{-10}
2.0	3.08660×10^{-13}	8.492393×10^{-10}

5 Conclusion

A block of 3 mid-point hybrid method based on a 3 step continuous linear multistep method is proposed and applied to solve third-order linear and non linear IVPs and BVPs in ordinary differential equations. The method has order 9 of accuracy and have been tested using some problems in literature. The method has been shown to be less ambiguous and easy to derive. It has been tested to solve diverse kinds of third-order ODEs as shown in the numerical examples. The method show a very high accuracy when compared to the exact solution and hence with existing methods in the literature cited.

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Competing financial interests

The authors declare no competing financial interests. 0

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