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Approximate Fixed Point Results for Rational-type Contraction Mappings

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Abstract

In the paper [Tijs, S., Torre, A. and Branzei, P.: Approximate fixed point theorems, Libertas Mathematica, $23(2003), 35-39(2003)$], the authors studied some fixed point theorems by considering weakening of the conditions in the fixed point theorems of Brouwer, Kakutani and Banach which still guarantee the existence of approximate fixed points. Also, in the paper [Berinde, M.: Approximate fixed point theorems, Studia Univ. BABES-BOLYAL, MATHEMATICA, Volume L1, Number $1, pp.\ 11-23(2006)$.], the author gave some qualitative and quantitative approximate fixed point results on metric spaces by introducing two Lemmas, and using some contactive-type operators used by Tijs *etal.*.
The aim of this paper is to establish qualitative and quantitative approximate fixed point results involving rational-type contraction mappings in metric spaces (not necessarily complete). Our results are extensions of several others in the literature. Some examples are provided to illustrate our results.

Keywords: Fixed point, Approximate fixed point, Rational-type contraction, qualitative results, quantitative results.
MSC2010: 47H06, 54H25

1 INTRODUCTION

The famous Banach's contraction mapping principle ensures, under certain conditions, the existence and uniqueness of a fixed point. This theorem provides a technique for solving a variety of applied problems in Mathematical Sciences and Engineering. Several authors have extended and generalized Banach fixed point theorem in many ways:
Dass and Gupta [1] were the first to consider a generalization of the Banach fixed point theorem using a contractive condition of rational-type as follows:
Let $(\mathcal{X}, d)$ be a complete metric space. There exists a fixed point for a mapping $T : \mathcal{X} \to \mathcal{X}$, for which there exists some $\alpha \geq 0, \beta \geq 0$ with $\alpha + \beta < 1$, such that

$$d(Tx, Ty) \leq \frac{\alpha d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y),\ \forall x, y \in \mathcal{X}. \quad (1.1)$$

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Furthermore, Jaggi [2] established a fixed point theorem in complete metric space using a contractive condition of rational-type; namely: There exists a fixed point for a mapping $T : \mathcal{X} \to \mathcal{X}$, for which there exist some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$, such that

$$d(Tx, Ty) \leq \frac{\alpha d(x, Tx).d(y, Ty)}{d(x, y)} + \beta d(x, y), \quad \forall x, y \in \mathcal{X}, \quad x \neq y. \tag{1.2}$$

Using the contractive condition (2), Harjani et al. [3] extended the result of Jaggi [2] to the partially ordered metric spaces. Contractive conditions of rational-type have been greatly employed in the fixed point as well as the coupled fixed point settings. For more details on results pertaining to rational-type contractive conditions, we refer the reader to [4], [5] in the references section and others in the literature.
The theory of fixed point has a lot of applications in many applied areas such as Mathematical economics, dynamic optimization and stochastic games, functional analysis, variational calculus, theory of integro-differential equations, etc [6]. However, in practice, there are many real situations where an approximate solution proved sufficient and hence, existence of fixed points is not strictly necessary. This naturally led to the study of approximate fixed point theory.
Here, an approximate fixed point x of a function f has the property that $f(x)$ is 'near' to x in a sense to be specified [6]. In such situations, weakening the condition of the fixed point setting, by giving up the completeness of the space, the approximate fixed point ($\epsilon-$fixed point) can still be guaranteed for several operators. In 2003, Tijs et al. [6] studied some fixed point theorems by considering weakening of the conditions in the fixed point theorems of Brouwer [7], Kakutani [8] and Banach [9] which still guarantee the existence of approximate fixed points. In 2006, Berinde [10] proved approximate fixed point results by introducing two Lemmas, using some of the operators used by Tijs et al. [6] and gave some qualitative and quantitative results on metric spaces. Furthermore, in 2017, Mohseni Alhosseini [11] proved some approximate fixed point theorems for cyclical contraction mappings. For further results on approximate fixed point, we refer the reader to [12], [13], [14], [15] and others in the literature.
In this paper, we study some qualitative and quantitative results for some mappings satisfying contractive conditions of rational-type in metric spaces. We present the following definitions and Lemma which we are to use in the sequel.

Definition 1.01 [10] Let $(\mathcal{X}, d)$ be a metric space. Let $T : \mathcal{X} \to \mathcal{X}$, $\epsilon > 0$, $x \in \mathcal{X}$. Then x is an $\epsilon-$fixed point (approximate fixed point) of T if $d(Tx, x) < \epsilon$.

Remark 1.02 [10] We denote the set of all $\epsilon-$fixed points of T, for a given ϵ, by $F_\epsilon(T) = \{x \in \mathcal{X} | x \text{ is an } \epsilon-\text{fixed point of } T\}$.

Lemma 1.03 [10] Let $(\mathcal{X}, d)$ be a metric space, $T : \mathcal{X} \to \mathcal{X}$ such that T is asymptotic regular i.e., $d(T^n(x), T^{n+1}(x)) \to 0$ as $n \to \infty$, $\forall x \in \mathcal{X}$. Then, for $\epsilon > 0$, $F_\epsilon(T) \neq \emptyset$.

The following example shows that the set of $\epsilon-$fixed points is indeed larger than the set of fixed points.
Example 1:04. Let $\mathcal{X} \subseteq \mathbb{R}$ be endowed with the usual metric. Suppose $\mathcal{X} = (0, \frac{1}{2}]$. Let T be defined by $T : \mathcal{X} \to \mathcal{X}$ such that $Tx = \frac{x}{4}, \forall x \in \mathcal{X}$.
The fixed point of T is $0 \notin \mathcal{X}$. On the other hand, take $0 < \epsilon < \frac{1}{2}$ and select $y \in \mathcal{X}$, such that

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$y < \frac{1}{4}\epsilon$.

$$\begin{aligned} d(Ty,y) &= |\frac{y}{4} - y| \\ &= |\frac{-3}{4}y| \\ &\leq \frac{3}{4}|y| \\ &< \frac{3}{4}|\frac{1}{4}\epsilon| \\ &= \frac{3\epsilon}{16} < \epsilon. \end{aligned}$$

Hence, T has an approximate fixed point in $\mathcal{X}$, implying that $F_\epsilon(T) \neq \emptyset$ in $\mathcal{X}$ whereas T does not have a fixed point in $\mathcal{X}$.

2 MAIN RESULTS

This section consists of two subsections, namely: The qualitative and the quantitative results. We first deal with the qualitative aspect as follow.

A. QUALITATIVE RESULTS:

Theorem 2.01 : Let $(\mathcal{X}, d)$ be a metric space, $T : \mathcal{X} \to \mathcal{X}$ a self-map on $\mathcal{X}$ such that for some $\alpha, \beta \geq 0,\ \alpha + \beta < 1,\ \forall x, y \in \mathcal{X}$,

$$d(Tx,Ty) \leq \frac{\alpha d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)} + \beta d(x,y). \tag{2.1}$$

Then, T has an ϵ−fixed point.

Proof : Let $\epsilon > 0$ and $x \in \mathcal{X}$.

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)) \\ &\leq \frac{\alpha d(T^n x, T(T^n x))[1 + d(T^{n-1}x, T(T^{n-1}x))]}{[1 + d(T^{n-1}x, T^n x)]} + \beta d(T^{n-1}x, T^n x) \\ &\leq \frac{\alpha d(T^n x, T^{n+1}x))[1 + d(T^{n-1}x, T^n x))]}{[1 + d(T^{n-1}x, T^n x)]} + \beta d(T^{n-1}x, T^n x) \\ &= \alpha d(T^n x, T^{n+1}x) + \beta d(T^{n-1}x, T^n x) \\ &\leq \frac{\beta}{1-\alpha} d(T^{n-1}x, T^n x) \\ &\vdots \\ &\leq (\frac{\beta}{1-\alpha})^n d(x, Tx). \end{aligned}$$

But $\left(\frac{\beta}{1-\alpha}\right) < 1$, therefore

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,\ \forall x \in \mathcal{X}.$$

By Lemma 1.03, it follows that $F_\epsilon(T) \neq \emptyset,\ \forall \epsilon > 0$.

Example 2:02 Let $\mathcal{X} \subseteq \mathbb{R}$ be endowed with the usual metric. Suppose $\mathcal{X} = (0, \infty]$. Let T be defined as $Tx = \frac{x}{2}, \forall x \in (0, \infty]$.

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Certainly, T does not have a fixed point in $(0,\infty]$.
Let $\alpha=\frac{1}{2},\beta=\frac{1}{5}$. Choose $x=\frac{1}{2},y=\frac{1}{4}\in\mathcal{X}$. Let T satisfy the conditions of Theorem 2.01. Then,

$$\begin{aligned} d\Big(T(\frac{1}{2}),T(\frac{1}{4})\Big) &= d\Big(\frac{1}{4},\frac{1}{8}\Big)\\ &= \Big|\frac{1}{4}-\frac{1}{8}\Big| = \frac{1}{8}, \end{aligned}$$

$$d(y,Ty)=d\Big(\frac{1}{4},\frac{1}{8}\Big)=\frac{1}{8},$$

$$d(x,Tx)=d\Big(\frac{1}{2},\frac{1}{4}\Big)=\frac{1}{4},$$

$$d(x,y)=d\Big(\frac{1}{2},\frac{1}{4}\Big)=\frac{1}{4}.$$

Therefore,

$$\begin{aligned} \frac{1}{8} &\leq \frac{\frac{1}{2}(\frac{1}{8})[1+\frac{1}{4}]}{1+\frac{1}{4}}+\frac{1}{5}(1+\frac{1}{4})\\ &= \frac{\frac{1}{16}(\frac{5}{4})}{\frac{5}{4}}+\frac{1}{5}(\frac{5}{4})\\ &= \frac{1}{16}+\frac{1}{4}=\frac{5}{16}. \end{aligned}$$

Hence, T has an approximate fixed point in $\mathcal{X}$.

Theorem 2.03 : Let $(\mathcal{X},d)$ be a metric space, $T:\mathcal{X}\to\mathcal{X}$ a self-map on $\mathcal{X}$ such that for some $\alpha,\beta\in[0,1)$, $\alpha+\beta<1$, we have

$$d(Tx,Ty)\leq\frac{\alpha d(x,Tx).d(y,Ty)}{d(x,y)}+\beta d(x,y), \tag{2.2}$$

$\forall x,y\in\mathcal{X}$, where $d(x,y)>0$. Then T has an $\epsilon-$fixed point.
Proof : Let $\epsilon>0$ and $x\in\mathcal{X}$. If there is $n\in\mathbb{N}$ such that $d(T^{n-1}x,T^nx)=0$, then $T^{n-1}x$ is a fixed point of T and thus an $\epsilon-$fixed point. Suppose now that for all $n\in\mathbb{N}$, $d(T^{n-1}x,T^nx)\neq0$, then,

$$\begin{aligned} d(T^nx,T^{n+1}x) &= d(T(T^{n-1}x),T(T^nx))\\ &\leq \frac{\alpha d(T^{n-1}x,T(T^{n-1}x)).d(T^nx,T(T^nx))}{d(T^{n-1}x,T^nx)}+\beta d(T^{n-1}x,T^nx)\\ &= \frac{\alpha d(T^{n-1}x,T^nx).d(T^nx,T^{n+1}x)}{d(T^{n-1}x,T^nx)}+\beta d(T^{n-1}x,T^nx)\\ &= \alpha d(T^nx,T^{n+1}x)+\beta d(T^{n-1}x,T^nx)\\ &\leq \frac{\beta}{1-\alpha}d(T^{n-1}x,T^nx)\\ &\vdots\\ &\leq \big(\frac{\beta}{1-\alpha}\big)^n d(x,Tx). \end{aligned}$$

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But $\alpha, \beta \in [0,1)$ with $\alpha+\beta<1$, then $(\frac{\beta}{1-\alpha}) \in [0,1)$. Thus, $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. By Lemma 1.03, it follows that $F_\epsilon(T) \neq \emptyset,\ \forall \epsilon > 0$.

Theorem 2.04 : Let $(\mathcal{X}, d)$ be a metric space, $T : \mathcal{X} \to \mathcal{X}$ a self-map on $\mathcal{X}$ such that for some $\alpha \geq 0$, $\beta \in [0,1)$, and $\forall x, y \in \mathcal{X}$, such that $d(y,Ty)+d(x,y) > 0$, we have

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(x,Ty)d(y,Ty)}{d(y,Ty)+d(x,y)} + \beta d(x,y). \tag{2.3}$$

Then, T has an $\epsilon-$fixed point $\forall \epsilon > 0$.
Proof : Let $\epsilon > 0$ and $x \in \mathcal{X}$. If there is $n \in \mathbb{N}$ such that $d(T^{n-1}x, T^n x) = 0$, then $T^{n-1}x$ is a fixed point of T and thus an $\epsilon-$fixed point. Suppose now that for all $n \in \mathbb{N}$, $d(T^{n-1}x, T^n x) \neq 0$, then,

$$\begin{aligned}
d(T^{n+1}x, T^n x) &= d(T(T^n)x, T(T^{n-1}x)) \\
&\leq \frac{\alpha d(T^n x, T(T^n x)) d(T^n x, T(T^{n-1}x)) d(T^{n-1}x, T(T^{n-1}x))}{d(T^{n-1}x, T(T^{n-1}x)) + d(T^n x, T^{n-1}x)} + \beta d(T^n x, T^{n-1}x) \\
&= \frac{\alpha d(T^n x, T^{n+1}x) d(T^n x, T^n x) d(T^{n-1}x, T^n x)}{d(T^{n-1}x, T^n x) + d(T^n x, T^{n-1}x)} + \beta d(T^n x, T^{n-1}x) \\
&\leq \beta d(T^n x, T^{n-1}x) \\
&\leq \\
&\vdots \\
&\leq \beta^n d(Tx, x).
\end{aligned}$$

But $\beta \in [0,1)$, then, $\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0$. By Lemma 1.03, it follows that $F_\epsilon(T) \neq \emptyset,\ \forall \epsilon > 0$.

In what follows, we have a theorem which is a generalization of Theorem 2.04. Meanwhile, the following definition is to be used in the sequel.

Definition 2.05: Consider a function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$. ψ is called a comparison function if it satisfies the conditions:
$(i)\psi$ is monotone increasing, and
$(ii)\psi^n(t)$ converges to 0 as $n \to \infty, \forall t \in \mathbb{R}_+$.

Theorem 2.06: Let $(\mathcal{X}, d)$ be a metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a self-map on $\mathcal{X}$ such that for some $\alpha \geq 0$ and ψ, a comparison function satisfying $\psi(t) < t,\ \forall t > 0$, we have

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(x,Ty)d(y,Ty)}{d(y,Ty)+d(x,y)} + \psi(d(x,y)). \tag{2.4}$$

$\forall x, y \in \mathcal{X}$ such that $d(y,Ty)+d(x,y) \neq 0$. Then, $F_\epsilon(T) \neq \emptyset,\ \forall \epsilon > 0$.

Proof : Let $\epsilon > 0$ and $x \in \mathcal{X}$. If there is $n \in \mathbb{N}$ such that $d(T^{n-1}x, T^n x) = 0$, then $T^{n-1}x$ is a fixed point of T and thus an $\epsilon-$fixed point. Suppose now that for all $n \in \mathbb{N}$, $d(T^{n-1}x, T^n x) \neq 0$, then,

$$\begin{aligned}
d(T^{n+1}x, T^n x) &= d(T(T^n x), T(T^{n-1}x)) \\
&\leq \frac{\alpha d(T^n x, T(T^n x)) d(T^n x, T(T^{n-1}x)) d(T^{n-1}x, T(T^{n-1}x))}{d(T^{n-1}x, T(T^{n-1}x)) + d(T^n x, T^{n-1}x)} + \psi(d(T^n x, T^{n-1}x)) \\
&= \frac{\alpha d(T^n x, T^{n+1}x) d(T^n x, T^n x) d(T^{n-1}x, T^n x)}{d(T^{n-1}x, T^n x) + d(T^n x, T^{n-1}x)} + \psi(d(T^n x, T^{n-1}x)) \\
&= \psi(d(T^n x, T^{n-1}x)) \\
&\leq \\
&\vdots \\
&\leq \psi^n(d(Tx, x)).
\end{aligned}$$

But ψ is a comparison function, $\psi^n(d(Tx,x)) \to 0$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = 0, \ \forall x \in \mathcal{X}.$$

Then, $F_\epsilon(T) \neq \emptyset, \ \forall \epsilon > 0$.

B. QUANTITATIVE RESULTS FOR MAPPINGS IN METRIC SPACES

In this subsection, we will obtain quantitative results for some of the operators we have studied in the previous subsection.

Theorem 2.07 **:** Let $(\mathcal{X}, d)$ be a metric space and $T : \mathcal{X} \to \mathcal{X}$ a mapping satisfying all the conditions of Theorem 2.01. Then, for each $\epsilon > 0$, the diameter of $F_\epsilon(T)$ is not larger than

$$\frac{6\epsilon + 4(\alpha - \beta - \alpha\beta)\epsilon + 4(1 + \alpha - \alpha\beta)\epsilon^2}{2(1-\beta)}.$$

Proof : Let $x, y \in F_\epsilon(T)$. By triangle inequality and condition (3),

$$\begin{aligned}
d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\
&\leq 2\epsilon + d(Tx,Ty) \\
&\leq 2\epsilon + \frac{\alpha d(y,Ty)[1 + d(x,Tx)]}{1 + d(x,y)} + \beta d(x,y) \\
&\leq 2\epsilon + \frac{\alpha\epsilon(1+\epsilon)}{1 + d(x,y)} + \beta d(x,y)
\end{aligned}$$

$$(1-\beta)[d(x,y)]^2 + (1 - \beta - 2\epsilon) d(x,y) \leq 2\epsilon + \alpha\epsilon + \alpha\epsilon^2$$

$$\big[d(x,y)\big]^2 + \left(\frac{1-\beta-2\epsilon}{1-\beta}\right) d(x,y) \leq \frac{2\epsilon + \alpha\epsilon + \alpha\epsilon^2}{1-\beta}.$$

Completing the square, we have

$$\left[d(x,y) + \frac{1-\beta-2\epsilon}{2(1-\beta)}\right]^2 \leq \frac{2\epsilon + \alpha\epsilon + \alpha\epsilon^2}{1-\beta} + \left(\frac{1-\beta-2\epsilon}{2(1-\beta)}\right)^2.$$

Then, it follows that

$$\begin{aligned}
d(x,y) &\leq \frac{\beta + 2\epsilon - 1}{2(1-\beta)} + \frac{1}{2(1-\beta)}\sqrt{1 + 4(1 + \alpha - \beta - \alpha\beta)\epsilon + 4(1 + \alpha - \alpha\beta)\epsilon^2 + \beta^2 - 2\beta} \\
&< \frac{1}{2(1-\beta)}\Big(\beta + 2\epsilon - 1 + 1 + 4(1 + \alpha - \beta - \alpha\beta)\epsilon + 4(1 + \alpha - \alpha\beta)\epsilon^2 + \beta^2 - 2\beta\Big).
\end{aligned}$$

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Hence,

$$\begin{aligned} d(x,y) &\leq \frac{\beta+2\epsilon+4(1+\alpha-\beta-\alpha\beta)\epsilon+4(1+\alpha-\alpha\beta)\epsilon^2+\beta^2-2\beta}{2(1-\beta)} \\ &= \frac{6\epsilon+4(\alpha-\beta-\alpha\beta)\epsilon+4(1+\alpha-\alpha\beta)\epsilon^2+\beta^2-\beta}{2(1-\beta)} \\ &< \frac{6\epsilon+4(\alpha-\beta-\alpha\beta)\epsilon+4(1+\alpha-\alpha\beta)\epsilon^2}{2(1-\beta)}. \end{aligned}$$

Example 2.08. Let $\mathcal{X}=(0,\frac{1}{2}]$ be endowed with the usual metric. Let $T:\mathcal{X}\to\mathcal{X}$ be defined by $Tx=\frac{x}{3},\forall x\in\mathcal{X}$. We want to prove that the conditions of Theorem 2.07 are satisfied. Consider the contractive condition (3) such that $\alpha=\frac{1}{4},\beta=\frac{1}{5}$ and $\epsilon=\frac{1}{2}$. Choose $x=\frac{1}{2},y=\frac{1}{4}\in\mathcal{X}$.

$$d(x,Tx)=|\frac{1}{2}-\frac{1}{6}|=\frac{1}{3}<\frac{1}{2}.$$

Also,

$$d(y,Ty)=|\frac{1}{4}-\frac{1}{8}|=\frac{1}{8}<\frac{1}{2}$$

Thus, $x,y\in F_\epsilon(T)$.

$$\begin{aligned} d(x,y) &= d(\frac{1}{2},\frac{1}{4}) \\ &= \frac{1}{4} \\ &< \frac{6\epsilon+4(\alpha-\beta)\epsilon+4(1+\alpha-\alpha\beta)\epsilon^2+\beta^2-\beta}{2(1-\beta)} \\ &= \frac{6(\frac{1}{2})+4(\frac{1}{4}-\frac{1}{5})\frac{1}{2}+4(1+\frac{1}{4}-\frac{1}{20})\frac{1}{4}+\frac{1}{25}-\frac{1}{5}}{2(1-\frac{1}{5})} \\ &= \frac{3+\frac{26}{20}-\frac{4}{25}}{2(\frac{4}{5})} \\ &= \frac{207}{80}, \end{aligned}$$

hence, the result.

Theorem 2.09 **:** Let the conditions of Theorem 2.03 be satisfied. Then, for each $\epsilon>0$, the diameter of $F_\epsilon(T)$ is not larger than $\left(\alpha+\frac{2}{1-\beta}\right)\epsilon$.

Proof : Let $x,y\in F_\epsilon(T)$. By triangle inequality and condition (4),

$$\begin{aligned} d(x,y) &\leq d(x,Tx)+d(Ty,y)+d(Tx,Ty) \\ &\leq 2\epsilon+d(Tx,Ty) \\ &\leq 2\epsilon+\frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)}+\beta d(x,y) \end{aligned}$$

$$[d(x,y)]^2\leq 2\epsilon d(x,y)+\alpha d(x,Tx)d(y,Ty)+\beta[d(x,y)]^2$$

$$(1-\beta)[d(x,y)]^2\leq 2\epsilon d(x,y)+\alpha\epsilon^2$$

$$[d(x,y)]^2-\frac{2\epsilon d(x,y)}{1-\beta}\leq\frac{\alpha\epsilon^2}{1-\beta}.$$

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Completing the square, we have

$$\left[d(x,y) - \frac{\epsilon}{1-\beta}\right]^2 \leq \frac{\alpha\epsilon^2}{1-\beta} + \left(\frac{\epsilon}{1-\beta}\right)^2.$$

Hence,

$$\begin{aligned} d(x,y) &\leq \frac{\epsilon}{1-\beta} + \sqrt{\frac{\alpha\epsilon^2 + \epsilon^2(1-\alpha\beta)}{(1-\beta)^2}} \\ &= \frac{\epsilon}{1-\beta} + \frac{\epsilon}{1-\beta}\sqrt{1+\alpha-\alpha\beta} \\ &= \frac{\epsilon}{1-\beta}\left(1+\sqrt{1+\alpha-\alpha\beta}\right) \\ &< \frac{\epsilon}{1-\beta}\left(1+1+\alpha-\alpha\beta\right) \\ &= \epsilon\left(\frac{2+\alpha(1-\beta)}{1-\beta}\right) \\ &= \left(\alpha + \frac{2}{1-\beta}\right)\epsilon \end{aligned}$$

which completes the proof.

Theorem 2.10 : Let $(\mathcal{X}, d)$ be a metric space and $T : \mathcal{X} \to \mathcal{X}$ satisfy the conditions of Theorem 2.04. Then, for each $\epsilon > 0$, the diameter of $F_\epsilon(T)$ is not larger than

$$\frac{1}{2(1-\beta)}[\epsilon(\beta+1) + \epsilon^2(\alpha+10) + \epsilon^3(6\alpha - 2\alpha\beta) + \epsilon^4\alpha^2].$$

Proof : Let $x, y \in F_\epsilon(T)$. By triangle inequality and condition (5),

$$\begin{aligned} d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\ &\leq 2\epsilon + d(Tx,Ty) \\ &\leq 2\epsilon + \frac{\alpha d(x,Tx)d(x,Ty)d(y,Ty)}{d(y,Ty)+d(x,y)} + \beta d(x,y). \end{aligned}$$

Also, by triangle inequality,

$$\begin{aligned} d(x,y) &\leq 2\epsilon + \frac{\alpha d(x,Tx)[d(x,y)+d(y,Ty)]d(y,Ty)}{d(y,Ty)+d(x,y)} + \beta d(x,y) \\ &\leq 2\epsilon + \frac{[\alpha d(x,Tx)d(x,y) + \alpha d(x,Tx)d(y,Ty)]d(y,Ty)}{d(y,Ty)+d(x,y)} + \beta d(x,y) \\ &\leq 2\epsilon + \frac{[\epsilon\alpha d(x,y) + \alpha\epsilon^2]\epsilon}{\epsilon + d(x,y)} + \beta d(x,y) \end{aligned}$$

$$(1-\beta)d(x,y) \leq 2\epsilon + \frac{\epsilon^2\alpha d(x,y) + \alpha\epsilon^3}{\epsilon + d(x,y)}$$

$$(1-\beta)d(x,y) \leq \frac{2\epsilon(\epsilon + d(x,y)) + \epsilon^2\alpha d(x,y) + \alpha\epsilon^3}{\epsilon + d(x,y)}$$

$$\begin{aligned} (1-\beta)[d(x,y)]^2 + \epsilon(1-\beta)d(x,y) &\leq 2\epsilon(\epsilon + d(x,y)) + \epsilon^2\alpha d(x,y) + \alpha\epsilon^3 \\ &= 2\epsilon^2 + 2\epsilon d(x,y) + \epsilon^2\alpha d(x,y) + \alpha\epsilon^3 \end{aligned}$$

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$$(1-\beta)[d(x,y)]^2+\epsilon(1-\beta)d(x,y)-2\epsilon d(x,y)-\epsilon^2\alpha d(x,y)\leq 2\epsilon^2+\alpha\epsilon^3$$

$$(1-\beta)[d(x,y)]^2+[\epsilon(1-\beta)-2\epsilon-\alpha\epsilon^2]d(x,y)\leq 2\epsilon^2+\alpha\epsilon^3$$

$$[d(x,y)]^2+\frac{[-\epsilon\beta-\epsilon-\alpha\epsilon^2]d(x,y)}{1-\beta}\leq\frac{2\epsilon^2+\alpha\epsilon^3}{1-\beta}.$$

Completing the square, we have

$$\left[d(x,y)+\frac{-\epsilon\beta-\epsilon-\alpha\epsilon^2}{2(1-\beta)}\right]^2\leq\frac{2\epsilon^2+\alpha\epsilon^3}{1-\beta}+\left[\frac{-\epsilon\beta-\epsilon-\alpha\epsilon^2}{2(1-\beta)}\right]^2$$

Implying that,

$$\begin{aligned}
d(x,y) &\leq \frac{\epsilon\beta+\epsilon+\alpha\epsilon^2}{2(1-\beta)}+\sqrt{\frac{2\epsilon^2+\alpha\epsilon^3}{1-\beta}+\left[\frac{-\epsilon\beta-\epsilon-\alpha\epsilon^2}{2(1-\beta)}\right]^2}\\
&= \frac{\epsilon\beta+\epsilon+\alpha\epsilon^2}{2(1-\beta)}+\sqrt{\frac{2\epsilon^2+\alpha\epsilon^3}{1-\beta}+\frac{(-\epsilon\beta-\epsilon-\alpha\epsilon^2)^2}{4(1-\beta)^2}}\\
&= \frac{\epsilon\beta+\epsilon+\alpha\epsilon^2}{2(1-\beta)}+\sqrt{\frac{4(1-\beta)(2\epsilon^2+\alpha\epsilon^3)+(-\epsilon\beta-\epsilon-\alpha\epsilon^2)^2}{4(1-\beta)^2}}\\
&= \frac{1}{2(1-\beta)}\left[\epsilon\beta+\epsilon+\alpha\epsilon^2+\sqrt{4(1-\beta)(2\epsilon^2+\alpha\epsilon^3)+(-\epsilon\beta-\epsilon-\alpha\epsilon^2)^2}\right]\\
&< \frac{1}{2(1-\beta)}\left[\epsilon\beta+\epsilon+\alpha\epsilon^2+4(1-\beta)(2\epsilon^2+\alpha\epsilon^3)+(-\epsilon\beta-\epsilon-\alpha\epsilon^2)^2\right]\\
&= \frac{1}{2(1-\beta)}[\epsilon\beta+\epsilon+\alpha\epsilon^2+\epsilon^2\beta^2+9\epsilon^2+6\alpha\epsilon^3+\alpha^2\epsilon^4-6\epsilon^2\beta-2\alpha\beta\epsilon^3]\\
&\leq \frac{1}{2(1-\beta)}[\epsilon(\beta+1)+\epsilon^2(\alpha+10)+\epsilon^3(6\alpha-2\alpha\beta)+\epsilon^4\alpha^2].
\end{aligned}$$

Theorem 2.11 : Let $(\mathcal{X},d)$ be a metric space and $T:\mathcal{X}\to\mathcal{X}$ satisfy the conditions of Theorem 2.06. Then, for each $\epsilon>0$, the diameter of $F_\epsilon(T)$ is indeterminate.
Proof : $x,y\in F_\epsilon(T)$. By triangle inequality and condition (6),

$$\begin{aligned}
d(x,y) &\leq d(x,Tx)+d(Tx,Ty)+d(Ty,y)\\
&\leq \epsilon+d(Tx,Ty)+\epsilon\\
&= 2\epsilon+d(Tx,Ty)\\
&\leq 2\epsilon+\frac{\alpha d(x,Tx)d(x,Ty)d(y,Ty)}{d(y,Ty)+d(x,y)}+\psi(d(x,y)).
\end{aligned}$$

Also, by triangle inequality,

$$d(x,y)\leq 2\epsilon+\frac{\alpha d(x,Tx)[d(x,y)+d(y,Ty)]d(y,Ty)}{d(y,Ty)+d(x,y)}+\psi(d(x,y)).$$

Since $\psi(t)<t,\forall t>0$, then,

$$\begin{aligned}
d(x,y) &< 2\epsilon+\frac{\alpha d(x,Tx)[d(x,y)+d(y,Ty)]d(y,Ty)}{d(y,Ty)+d(x,y)}+d(x,y)\\
&\leq 2\epsilon+\frac{\alpha\epsilon[d(x,y)+\epsilon]\epsilon}{\epsilon+d(x,y)}+d(x,y).
\end{aligned}$$

It implies that

$$-\frac{\alpha\epsilon[d(x,y)+\epsilon]\epsilon}{\epsilon+d(x,y)} \leq 2\epsilon$$

$$-\alpha\epsilon[d(x,y)+\epsilon]\epsilon \leq 2\epsilon^2 + 2\epsilon d(x,y)$$

$$-\alpha\epsilon^2 d(x,y) - \alpha\epsilon^3 \leq 2\epsilon^2 + 2\epsilon d(x,y)$$

$$(-\alpha\epsilon^2 - 2\epsilon)d(x,y) \leq 2\epsilon^2 + \alpha\epsilon^3$$

$$(-\alpha\epsilon - 2)d(x,y) \leq 2\epsilon + \alpha\epsilon^2$$

$$(\alpha\epsilon + 2)d(x,y) \geq -2\epsilon - \alpha\epsilon^2$$

$$d(x,y) \geq \frac{-2\epsilon - \alpha\epsilon^2}{\alpha\epsilon + 2}.$$

3 CONCLUSION

We have proved qualitative and quantitative results involving contractive conditions of rational type. It is interesting to note that apart from a quantitative result involving a comparison function, the value of ϵ is directly proportional to the diameter estimate of the set containing the $\epsilon-$fixed points in all the cases considered in this paper. When ϵ approaches zero, we approach the more restricted fixed point setting.
The theory of $\epsilon-$fixed points is therefore not less important than that of fixed points as several results given in the latter can be formulated in a weaker setting to guarantee existence of the $\epsilon-$fixed points.

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The authors of this paper state that there are no competing interests.

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