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Some new results on the stability and boundedness of solutions of certain class of second order vector differential equations

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Abstract

In this paper, we provide certain conditions that guarantee the stability of the zero solution when $P(t, X, Y) = 0$ and boundedness of all solutions when $P(t, X, Y) \neq 0$ of a certain system of second order differential equation using a suitable Lyapunov function. The results in this paper are quite new and complement those in the literature. Examples are given to demonstrate the correctness of the established results.

Keywords: Second order, Stability, Boundedness, Lyapunov function.
MSC2010:34D05, 34C11

1 Introduction

Consider the following second order nonlinear vector differential equation given by:

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}) \tag{1.1}$$

or its equivalent system:

$$\dot{X} = Y, \quad \dot{Y} = -AY - H(X) + P(t, X, Y) \tag{1.2}$$

in which $X, Y : \mathbb{R}^n \to \mathbb{R}^n$, $H(X) : \mathbb{R}^n \to \mathbb{R}^n$, $P(t, X, Y) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, A is an $n \times n$ constant matrix and the dots as usual indicate differentiation with respect to t, $t \in \mathbb{R}^+ = [0, \infty)$, $\mathbb{R}$ denote the real line $(-\infty, \infty)$ and $\mathbb{R}^n$ denote the real $n-$dimensional Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$. It is assumed that the functions $H(X)$ and $P(t, X, Y)$ are continuous for the arguments displayed explicitly and $H(X)$ is not necessarily differentiable. Furthermore, the existence and uniqueness of solutions of Eq. (1.1) will be assumed [see [1]].

The study of qualitative behaviour of solutions of second order differential equation has received notable attention from many researchers. For instance, Tejumola [2] in 1976 used the second method

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of Lyapunov to examine the stability, ultimate boundedness of all solutions and the existence of periodic solution of a certain matrix differential equation of the form

$$\ddot{X} + A\dot{X} + H(X) = P(t, X, \dot{X}), \tag{1.3}$$

where A is an $n \times n$ constant matrix and $H(X)$ is a differentiable vector function. Kroopnick ([3], [4]) used the approach of the integral test to prove the stability and boundedness of solutions of a second order scalar linear differential equation

$$x'' + a(t)x = 0, \tag{1.4}$$

where $a(t) > 0$ and $a'(t) \geq 0$. Tunç and Tunç [5] used integral test to investigate the boundedness and stability properties of a second order vector linear differential equation given by

$$\ddot{X} + a(t)X = P(t). \tag{1.5}$$

And recently, in 2017, Tunç and Tunç [6] used Lyapunov's direct method to study the stability and boundedness of a second order nonlinear vector differential equation of the form

$$\ddot{X} + a(t)X = P(t, \dot{X}), \tag{1.6}$$

where both $a(t)$ and $P(t, \dot{X})$ are continuous in their respective arguments and in addition, $P(t, \dot{X})$ satisfies Lipschitz condition in $\dot{X}$. Many other interesting results have also been obtained concerning solutions of second order and even higher order differential equations. We refer interested readers to the following few papers and the references cited in them: Ademola *et al.* [7], Adeyanju [8], Afuwape [9], Alaba and Ogundare [10], Cartwright and Littlewood [11], Ezeilo [12], Grigoryan [13], Kroopnick [14], Meng [15], Ogundare and Afuwape [16], Ogundare *et al.* [17] , Ogundare *et al.* [18], Omeike [19], Omeike *et al.* [20], Tejumola [21], Tunç [22], Tunç and Tunç ([23], [24], [5], [6]), Wang and Zhu [25].

2 Notation

Henceforth, δ's, Δ's and K's with or without suffixes will denote positive constants whose magnitudes depend on an $n \times n$ constant matrix A and vector functions $H(X)$, $P(t, X, \dot{X})$. The δ's, Δ's and K's with numerical or alphabetical suffixes shall retain fixed magnitudes, while those without suffixes are not necessarily the same at each occurrence.

We also denote the scalar product $\langle X, Y \rangle$ of any vectors X, Y in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X \rangle = \|X\|^2$.

3 Preliminary results

In this section, we state without proofs some standard results needed in the proofs of our main results.

Lemma 3.1. *Let A be a real symmetric positive definite $n \times n$ matrix. Then, for $X \in \mathbb{R}^n$*

$$\delta_a \|X\|^2 \leq \langle AX, X \rangle \leq \Delta_a \|X\|^2, \tag{3.1}$$

where δ_a and Δ_a are respectively the least and greatest eigenvalues of the matrix A.

Proof. See ([26], [27]) □

Remark 3.2. *It should be note that a similar case where $H(X)$ is not necessarily differentiable for a certain third order differential equations was considered by [Afuwape [9], Meng [15]].*

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4 Stability

In this section, we consider the following equation.

$$\ddot{X} + A\dot{X} + H(X) = 0, \tag{4.1}$$

or its equivalent system:

$$\dot{X} = Y, \quad \dot{Y} = -AY - H(X). \tag{4.2}$$

Theorem 4.1. *Suppose that $H(0) = 0$ and that,*

(i) there exists an $n \times n$ real continuous operator $B(X,Y)$ for any vectors $X, Y \in R^n$ such that,

$$H(X) = H(Y) + B(X,Y)(X - Y), \tag{4.3}$$

whose eigenvalues $\lambda_i(B(X,Y))$, $(i = 1, 2, \ldots, n)$ satisfy

$$0 < \delta_b \leq \lambda_i(B(X,Y)) \leq \Delta_b; \tag{4.4}$$

(ii) the constant symmetric matrix A has positive eigenvalues for any vectors $X, Y \in R^n$ and

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a. \tag{4.5}$$

Then, the trivial solution of Eq. (4.1) or system (4.2) is asymptotically stable.

Proof. The main tool in proving our result is the differentiable function $V = V(t, X, Y)$ defined by

$$2V(t,X,Y) = \langle \beta(1-\beta)\delta_a^2 X, X\rangle + \langle \alpha\delta_a Y, Y\rangle + (1-\beta)^2\|Y + \delta_a X\|^2, \tag{4.6}$$

where $\alpha > 0$, $0 < \beta < 1$ are some constants.

Clearly, V(t,0,0) = 0 and

$$2V(t;X,Y) \quad \geq \quad \langle \beta(1-\beta)\delta_a^2 X, X\rangle + \langle \alpha\delta_a Y, Y\rangle.$$

Thus, there exist a positive constant $K = \min\{\beta(1-\beta)\delta_a^2;\ \alpha\delta_a\}$, such that,

$$2V(t,X,Y) \geq K\{\| X \|^2 + \| Y \|^2\} \tag{4.7}$$

for all $t \geq 0$, X, Y. From (4.7), $V(t;X,Y) = 0$ if and only if $\|X\|^2 + \|Y\|^2 = 0$ and $V(t;X,Y) > 0$ if and only if $\|X\|^2 + \|Y\|^2 \neq 0$. It then follows that,

$$V(t;X,Y) \to +\infty \text{ as } \|X\|^2 + \|Y\|^2 \to +\infty.$$

Next, we derive the derivative of V along the trajectories of the system (4.2) and show that it is negative definite for $X \neq 0$ and $Y \neq 0$. By differentiating V, we obtain

$$\begin{aligned}
\dot{V}_{(4.2)} &= \langle \beta(1-\beta)\delta_a^2 X, Y\rangle + \langle \alpha\delta_a Y, -AY - H(X)\rangle + (1-\beta)^2\langle Y + \delta_a X, -AY - H(X) + \delta_a Y\rangle \\
&\leq -\alpha\delta_a^2\langle Y, Y\rangle - (1-\beta)^2\delta_a\langle X, H(X)\rangle + (1-\beta)\delta_a^2\langle X, Y\rangle - (1-\beta)^2\delta_a\langle X, AY\rangle \\
&\quad - \alpha\delta_a\langle Y, H(X)\rangle - (1-\beta)^2\langle Y, H(X)\rangle \\
&\leq -\alpha\delta_a^2\langle Y, Y\rangle - (1-\beta)^2\delta_a\langle X, H(X)\rangle - \{\alpha\delta_a + (1-\beta)^2\}\langle Y, H(X)\rangle \\
&\quad - \langle \delta_a(1-\beta)\{A(1-\beta) - \delta_a I\}X, Y\rangle.
\end{aligned}$$

We can write $\dot{V}$ as

$$\dot{V}_{(4.2)} \leq -U_1 - U_2 - U_3 \tag{4.8}$$

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where

$$U_1 = \frac{1}{2}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{2}(1-\beta)^2\delta_a\langle X, H(X)\rangle,$$

$$U_2 = \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\langle X, H(X)\rangle + \langle\delta_a(1-\beta)\{A(1-\beta)-\delta_a I\}X, Y\rangle$$

and

$$U_3 = \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\langle X, H(X)\rangle + \{\alpha\delta_a + (1-\beta)^2\}\langle Y, H(X)\rangle.$$

In what follows, we will show that $U_2 \geq 0$ and $U_3 \geq 0$. Let K_1 and K_2 be any two strictly positive constants which are carefully chosen, we have

$$\begin{aligned}
&\langle\delta_a(1-\beta)\big(A(1-\beta)-\delta_a I\big)X, Y\rangle \\
&= \|K_1^{-1}\delta_a(1-\beta)(A(1-\beta)-\delta_a I)X + \frac{1}{2}K_1 Y\|^2 - K_1^{-2}\delta_a^2(1-\beta)^2\big(A(1-\beta)-\delta_a I\big)^2\|X\|^2 - \frac{1}{4}K_1^2\|Y\|^2 \\
&\geq -K_1^{-2}\delta_a^2(1-\beta)^2\big(A(1-\beta)-\delta_a I\big)^2\|X\|^2 - \frac{1}{4}K_1^2\|Y\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\langle(\alpha\delta_a + (1-\beta)^2)H(X), Y\rangle \\
&= \|K_2^{-1}(\alpha\delta_a + (1-\beta)^2)H(X) + \frac{1}{2}K_2 Y\|^2 - K_2^{-2}(\alpha\delta_a + (1-\beta)^2)^2\|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2 \\
&\geq -K_2^{-2}(\alpha\delta_a + (1-\beta)^2)^2\|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
U_2 &\geq -\frac{1}{4}K_1^2\|Y\|^2 - K_1^{-2}\delta_a^2(1-\beta)^2\big(A(1-\beta)-\delta_a I\big)^2\|X\|^2 + \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\langle X, H(X)\rangle \\
&\geq \frac{1}{4}\Big(\alpha\delta_a^2 - K_1^2\Big)\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\Big(\delta_b - 4K_1^{-2}\delta_a\Delta_a^2\beta^2\Big)\langle X, X\rangle,
\end{aligned}$$

where we have applied Eq. (4.3) with $Y = 0$, $H(Y) = 0$ and Lemma 3.1. Therefore, for all $X, Y \in \mathbb{R}^n$,

$$U_2 \geq 0,$$

if $K_1^2 \leq \alpha\delta_a^2$ with

$$\Delta_a^2 \leq \frac{K_1^2\delta_b}{4\delta_a\beta^2} \leq \frac{\alpha\delta_a\delta_b}{4\beta^2}.$$

Also,

$$U_3 \geq \frac{1}{4}\alpha\delta_a^2\langle Y, Y\rangle + \frac{1}{4}(1-\beta)^2\delta_a\langle X, H(X)\rangle - K_2^{-2}(\alpha\delta_a + (1-\beta)^2)^2\|H(X)\|^2 - \frac{1}{4}K_2^2\|Y\|^2.$$

On applying Eq. (4.3) with $Y = 0$, $H(0) = 0$ and Lemma 3.1, we have

$$U_3 \geq \frac{1}{4}(\alpha\delta_a^2 - K_2^2)\langle Y, Y\rangle + \frac{1}{4}\Big(\delta_a\delta_b(1-\beta)^2 - \frac{4\big(\alpha\delta_a + (1-\beta)^2\big)^2\Delta_b^2}{K_2^2}\Big)\langle X, X\rangle.$$

Thus, for all $X, Y \in R^n$,

$$U_3 \geq 0,$$

if $K_2^2 \leq \alpha\delta_a^2$ with

$$\Delta_b^2 \leq \frac{K_2^2\delta_a\delta_b(1-\beta)^2}{4\Big(\alpha\delta_a + (1-\beta)^2\Big)^2} \leq \frac{\alpha\delta_a^3\delta_b(1-\beta)^2}{4\Big(\alpha\delta_a + (1-\beta)^2\Big)^2}.$$

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Lastly, we are left with the estimate for U_1. From (4.8) and (4.3), we have

$$\begin{aligned} U_1 &= \frac{1}{2}\alpha\delta_a^2\langle Y,Y\rangle + \frac{1}{2}(1-\beta)^2\delta_a\langle X,H(X)\rangle \\ &= \frac{1}{2}\alpha\delta_a^2\langle Y,Y\rangle + \frac{1}{2}(1-\beta)^2\delta_a\langle X,BX\rangle \\ &\geq \frac{1}{2}\alpha\delta_a^2\langle Y,Y\rangle + \frac{1}{2}(1-\beta)^2\delta_a\delta_b\langle X,X\rangle \\ &\geq \delta_2\{\|X\|^2+\|Y\|^2\}, \end{aligned}$$

where $\delta_2 = \frac{1}{2}\min\{\alpha\delta_a^2; (1-\beta)^2\delta_a\delta_b\}$.

Therefore,

$$\dot{V}_{(4.2)} \leq -\delta_2\{\|X\|^2+\|Y\|^2\} \leq 0. \tag{4.9}$$

So far, we have been able to show that the trivial solution is stable. Now, consider the set defined by

$$W = \{(X,Y) : \dot{V}(X,Y) = 0\}.$$

By applying LaSalle's invariance principle (see [5]), we observe that $(X,Y) \in W$ implies that

$$X = Y = 0. \tag{4.10}$$

This, i.e (4.10), shows that the largest invariant set contained in W is $(0,0) \in W$. Therefore, we conclude that the trivial solution of system (4.2) or Eq. (4.1) is asymptotically stable and this completes the proof of Theorem 4.1. □

Example 4.2. *As a special case of equation (1.1) with $n = 2$, we provide the following example. In (1.1), let*

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \; A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } H(X) = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = BX,$$

where b_1, b_2 are constants satisfying $0 < b_2 < b_1$. Then, by some simple calculations, $0 < \delta_a = 3 \leq \lambda_i(A) \leq \Delta_a = 5$, and $0 < \delta_b = b_1 - b_2 \leq \lambda_i(B) \leq \Delta_b = b_1 + b_2$.
Thus, all the conditions of the theorem are satisfied.

Remark 4.3. *Meng [15], considered the Ultimate boundedness results of a certain third order nonlinear linear differential equation in which the nonlinear term is not necessary differentiable and gave a similar example to our Example 4.2 above.*

5 Boundedness

We now state and prove the result on the boundedness of solutions for the case $P(t,X,Y) \neq 0$.

Theorem 5.1. *Suppose in addition to the conditions (i) and (ii) of Theorem 4.1, the following holds:*

(iii) for all t, X and Y, there exist a finite constant $\delta_1 > 0$ and a continuous function $\theta = \theta(t)$ such that the vector $P(t,X,Y)$ satisfies

$$\|P(t,X,Y)\| \leq \theta(t), \tag{5.1}$$

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where $\int_0^t \theta(s)ds \le \delta_1 < \infty$ *for all* $t \ge 0$. *Then there exists a constant* $D > 0$ *such that every solution* $(X(t), Y(t))$ *of* (1.2) *determined by*

$$X(0) = X_0, \; Y(0) = Y_0$$

satisfies

$$\|X(t)\| \le D, \; \|Y(t)\| \le D \tag{5.2}$$

for all $t \ge 0, \; X, Y \in \mathbb{R}^n$.

Proof. To establish this result, we make use of the Lyapunov function defined in (4.6). We note that under the assumptions of Theorem 5.1, the estimate (4.7) for $V(t, X, Y)$ is still the same. But now that $P(t, X, Y) \neq 0$, we obtain the following as the derivative of V:

$$\begin{aligned}
\dot{V}_{(1.2)} &\le -\delta_2\{\|X\|^2 + \|Y\|^2\} + \langle\{\alpha\delta_a + (1-\beta)^2\}Y + (1-\beta)^2\delta_a X, P\rangle \\
&\le \Big(\{\alpha\delta_a + (1-\beta)^2\}\|Y\| + (1-\beta)^2\delta_a\|X\|\Big)\|P(t, X, Y)\| \\
&\le \delta_3\{\|X\| + \|Y\|\}\|P(t, X, Y)\| \\
&\le \delta_3\theta(t)\{\|X\| + \|Y\|\}.
\end{aligned}$$

where $\delta_3 = \max\{\{\alpha\delta_a + (1-\beta)^2\}; (1-\beta)^2\delta_a\}$. On applying the inequalities $\|X\| \le 1 + \|X\|^2$ and $\|Y\| \le 1 + \|Y\|^2$, we have

$$\begin{aligned}
\dot{V}_{(1.2)} &\le \delta_3\theta(t)\{2 + \|X\|^2 + \|Y\|^2\} \\
&= 2\delta_3\theta(t) + \delta_3\theta(t)\{\|X\|^2 + \|Y\|^2\}.
\end{aligned}$$

By using the inequality (4.7), we have

$$\dot{V}(t)_{(1.2)} \le 2\delta_3\theta(t) + 2\delta_3\theta(t)K^{-1}V(t, X, Y).$$

Integrating the last inequality from 0 to $t, (t > 0)$, we obtain

$$\begin{aligned}
V(t) - V(0) &\le 2\delta_3\int_0^t \theta(s)ds + 2\delta_3 K^{-1}\int_0^t \theta(s)V(s)ds \\
&\le 2\delta_1\delta_3 + \delta_3 K^{-1}\int_0^t \theta(s)V(s)ds.
\end{aligned}$$

If we let $2\delta_1\delta_3 + V(0) = \delta_4$ and $2\delta_3 K^{-1} = \delta_5$, we obtain

$$V(t) \le \delta_4 + \delta_5\int_0^t \theta(s)V(s)ds.$$

Applying Grownwall Bellman's inequality (see Rao [1]) in the above inequality, we get

$$V(t) \le \delta_4 \exp\left(\delta_5\int_0^t \theta(s)ds\right) \le \delta_4\exp(\delta_6) \tag{5.3}$$

where $\delta_6 = \delta_1\delta_5$.
Thus, (5.3) implies that all the solutions of the Eq. (1.1) or it's equivalent system (1.2) are bounded. □

Example 5.2. *Suppose, in addition to Example 1, we have*

$$P(t; X, Y) = \begin{pmatrix} \frac{1}{1+e^t+x_1^2+y_1^2} \\ \frac{1}{1+e^t+x_2^2+y_2^2} \end{pmatrix}.$$

It is very clear that

$$\|P(t; X, Y)\| \leq \frac{2}{e^t} = \theta(t) \text{ and } \max \theta(t) = 2 < \infty.$$

$$\int_0^\infty \theta(s)ds = 2\int_0^\infty \frac{1}{e^s}ds = 2.$$

Also, all the conditions of the theorem are satisfied.

Remark 5.3. *We have thus far, established stability of the trivial solution and boundedness of all solutions of the systems (4.2) and (1.2) respectively without imposing differentiability condition on the vector function $H(X)$ as in papers ([27], [2]).*

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