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Stability of Nonlocal Stochastic Volterra Equations

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Abstract

In this paper, we study Ulam-Hyers-Rassias stability of solutions for nonlocal stochastic Volterra equations. Sufficient conditions for the existence and stability of solutions are derived using the Gronwall lemma. The advantage of our model equation is that it allows for additional measurements leading to better results compared to models with local initial conditions. Examples are solved to illustrate the applications of the results.

Keywords: Ulam-Hyers-Rassias stability, Stochastic Volterra equation, Gronwall lemma, Nonlocal conditions.
MSC2010: 37L05, 47J35, 47H10, 93D20

1 Introduction

The study of Volterra differential and integral equations with classical initial conditions have been of interest due to their several applications in various fields of science such as semiconductors, population dynamics, heat conduction, fluid flow, etc., see [1, 2, 7, 9–20, 23]. Differential equations with nonlocal and functional conditions have become an active area of research. The study of nonlocal problems is driven not only by theoretical interest but also because they occur naturally when modelling real-world applications. For example several phenomena in engineering, physics and life sciences can be described by employing differential equations subject to nonlocal initial conditions. See [3–6, 9, 13, 15, 17, 20–23]. However, there are not much literature on the theory of stochastic Volterra equations with nonlocal conditions (see [17, 19, 23] and the references therein). Some properties of the solutions to differential equations of Volterra type were studied by [2, 8, 9, 11, 12, 14, 16, 17, 23]. [7, 8, 10–12] analised some types of stability (Ulam, Hyers-Ulam, and Hyers-Ulam-Rassias) for nonlinear integro-differential Volterra equations and Volterra integral equations by using fixed-point arguments and the Bielecki metric techniques. [14] extended the work of [8] to a class of nonlinear stochastic integral equation of Volterra type. Of interest here is the work of [14], where existence and stability of the solution of stochastic Volterra integral equation is guaranteed if the following conditions are satisfied: given a group G_1, a metric group (G_2, d), and $\epsilon > 0$, there exists a $\delta > 0$ such that the map $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) > \delta \ \forall \ x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists and is stable provided a solution exists for such a problem. Ulam-Hyers-Rassias (U-H-R) stability of a nonlocal stochastic Volterra integral equation (NSVIE)

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is studied. We use Gronwall's lemma and some other analytical tools to establish the results. Our results extend and improve corresponding theorems in the existing literature concerning stochastic Volterra equations. In the sequel, we give some background definitions and some technical results in section 2, while section 3 is devoted to proving the main theorem and in section 4, we present some examples.

2 Preliminaries

We introduce and study the stability of the following nonlocal stochastic Volterra equation

$$\begin{aligned} dy_t &= F(t,y_t)dt + G(t,y_t)dW_t \\ y(0) &= y_0 + g(y), t \in [0,T]. \end{aligned} \qquad (2.1)$$

$F(t,y)$ and $G(t,y)$ are Σ_t-measurable functions. $g(y(.))$ is the nonlocal condition. W_t is a Brownian motion defined on a given probability space. y_0 is an Σ_0-measurable random variable independent of W_t, and $y_t \in C$.
Let $(\Omega, \Sigma, \mathbf{P})$ be a complete probability space equipped with a filtration $\{\Sigma_t\}_{0\le t \le T}$ and let $\|.\|_p = (E|.|^p)^{\frac{1}{p}}$ be the norm of the space $L^p(\Omega, \mathbf{P})$, $p > 0$.
Let $C := C(I, L^p_{ad}(\Omega, \Sigma, \mathbf{P})) : I \to L^p_{ad}(\Omega, \Sigma, \mathbf{P})$ denote the space of all continuous stochastic processes $x(t,\omega)$ such that each $x(t,\omega)$ is adapted to the filtration $\{\Sigma_t\}$ and $E(\int_0^t |x(s)|^p ds) < \infty$, $I := [0,T]$.
Equation (2.1) is better understood in integral form as

$$y_t = y_0 + g(y(.)) + \int_0^t F(s,y_s)ds + \int_0^t G(s,y_s)dW_s, t \in I \qquad (2.2).$$

Definition 2.1. Let $v > 0$. The problem (2.2) is said to be Ulam-Hyers-Rassias (U-H-R) stable with respect to $\epsilon > 0$ if for every solution $y_t \in C$ of the inequality

$$\left\| y_t - [y_0 + g(y(.))] - \int_0^t F(s,y_s)ds + \int_0^t G(s,y_s)dW_s \right\|_p \le \epsilon, \qquad (2.3)$$

we can find another solution $z_t \in C$ of (2.2) such that

$$\|y_t - z_t\|_p \le v\epsilon, t \in I.$$

Definition 2.2. The problem (2.2) is said to be Ulam-Hyers-Rassias (U-H-R) stable with respect to $\varphi(t)$ if we can find a constant $M_\varphi > 0$ such that for every solution $y_t \in C$ of the inequality

$$\left\| y_t - [y_0 + g(y(.))] - \int_0^t F(s,y_s)ds + \int_0^t G(s,y_s)dW_s \right\|_p \le \varphi(t), \qquad (2.4)$$

there exists another solution $z_t \in C$ of (2.2) such that

$$\|y_t - z_t\|_p \le M_\varphi \varphi(t).$$

Lemma 2.1 (Gronwall Lemma) Let $\varphi(t), \psi(t) \in C([0,T],_+)$, where $\varphi(t)$ is nondecreasing. If $u(t) \in C([0,T],_+)$ is a solution of the following inequality

$$u(t) \le \varphi + \int_0^t \psi(s)u(s)ds, s,t \in [0,T],$$

then

$$u(t) \le \varphi exp\left(\int_0^t \psi(s)ds\right), s,t \in [0,T].$$

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The following result guarantees the existence of a unique continuous solution of problem (2.2).
Theorem 2.2. Assume that $F(t,y)$, $G(t,y)$ and $g(y)$ are real valued-measurable functions on C and $F(t,y)$, $G(t,y)$ satisfy the Lipschitz and linear growth conditions in y. Suppose that also, the nonlocal initial condition $y_0 + g(y)$ is such that $\|y_0 + g(y)\|^p < \infty$. Then the nonlocal problem (2.2) has a unique continuous solution y_t on $[0,T]$.
Proof: The proof is simply an adaptation of the proof of Theorem 1 in [14], to this present setting.
Theorem 2.3. ([14]) Let $p \geq 2$ and let $f \in C$ be such that

$$E\left[\int_a^b |f(t)|^2 dt\right] < \infty,$$

then

$$E\left|\int_a^b f(t)dW_t\right|^p \leq \gamma_1 . E\left[\int_a^b |f(t)|^p dt\right], \tag{2.5}$$

where $\gamma_1 = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} (b-a)^{\frac{p-2}{2}}$.
We state the following assumptions for $y, z \in C$ and $a, b \in I$.
(H_1) $F(t,y)$ and $G(t,y)$ are measurable functions taking values in $I \times C$.
Let $K > 0, L > 0$, and $L_g > 0$ be positive constants such that
(H_2) $|F(t,y) - F(t,z)| \leq K|y-z|$, $|G(t,y) - G(t,z)| \leq K|y-z|$;
(H_3) $F(t,y) \leq L(1+|y|)$, $G(t,y) \leq L(1+|y|)$;
(H_4) $|g(y(.)) - g(z(.))| \leq L_g|y-z|$, $\zeta := \max_{y\in C} ||g(y(.))||$;
(H_5) $(1 - L_g) < 1$.
(H_6) Let $\varphi(t) > 0$ and $\varphi^p(t)$ be a non decreasing function.

3 Major Results

In this section, we discuss the existence of solution of the nonlocal problem (2.2) and the Ulam-Hyers-Rassias (U-H-R) stability.
Theorem 3.1. Assume that the hypotheses (H_1)-(H_6) are satisfied. Then the nonlocal problem (2.2),
(i) has a unique continuous solution $z_t \in C$ and
(ii) has the U-H-R stability with respect to $\varphi \in C$.
Proof.
(i) By the Lyapunov's inequality ([18]), $\|.\|_2 \leq \|.\|_p$. Therefore,
$\|y_0 + \zeta\|_2 \leq \|y_0 + \zeta\|_p$ holds for all $p \geq 2$. Thus, equation (2.2) has a unique continuous solution z_t.
Next we prove that $z_t \in C$.
If z_t is a continuous solution of (2.2), then

$$z_t = z_0 + g(y(.)) + \int_a^b F(s, z_s)ds + \int_a^b G(s, z_s)dW_s.$$

By assumption (H_3) and the inequality $(a+b+c)^p \leq 3^{p-1}(a^p + b^p + c^p)$, we obtain

$$\begin{aligned} |z_t|^p \;\leq\; & 4^{p-1}\{|z_0|^p + |g(z)|^p \\ & + \left|\int_a^b F(s, z_s)ds\right|^p \\ & + \left|\int_a^b G(s, z_s)dW_s\right|^p\}. \end{aligned} \tag{3.1}$$

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(3.2)

Next, we treat the right- hand side of (3.1) term by term as follows.

$$E|z_0|^p + |g(z)|^p \leq E|z_0|^p + E|\xi|^p.$$

$$\begin{aligned} \left|\int_a^b F(s, z_s)ds\right|^p &\leq \left(\int_a^b L(1+|z_s|)ds\right)^p \\ &\leq L^p 2^{p-1}\left(\left(\int_a^b ds\right)^p + \left(\int_a^b |z_s|ds\right)^p\right) \\ &\leq L^p 2^{p-1}\left((b-a)^p + \left(\int_a^b |z_s|ds\right)^p\right). \end{aligned} \quad (3.2)$$

Applying Holder's inequality to the last term on the right-hand side of (3.2) yields

$$\begin{aligned} \int_a^b |z_s|ds &\leq \left(\int_a^b ds\right)^{\frac{p-1}{p}} \left(\int_a^b |z_s|^p ds\right)^{1/p} \\ &\leq (b-a)^{\frac{p-1}{p}} \left(\int_a^b |z_s|^p ds\right)^{1/p}, \end{aligned}$$

and

$$\left|\int_a^b F(s, z_s)ds\right|^p \leq L^p 2^{p-1}(b-a)^{p-1}\left((b-a) + \int_a^b |z_s|^p ds\right).$$

Now applying (2.5) to the last term on the right-hand of (5.1) yields

$$\begin{aligned} E\left|\int_a^b G(s, z_s)dW_s\right|^p &\leq \gamma_1 E\int_a^b |G(s, z_s)|^p ds \\ &\leq \gamma_1 L^p E\int_a^b (1+|z_s|)^p ds \\ &\leq \gamma_1 L^p E\int_a^b 2^{p-1}(1+|z_s|^p)ds \\ &\leq \gamma_1 L^p 2^{p-1}\left((b-a) + \int_a^b E|z_s|^p ds\right). \end{aligned}$$

Hence,

$$E|z_t|^p \leq [\gamma_2 + \gamma_3 \int_a^b E|z_s|^p ds],$$

where

$$\gamma_2 = 4^{p-1}\left(E|z_0|^p + E|\xi|^p + L^p 2^{p-1}[(b-a)^p + \gamma_1(b-a)]\right)$$

and

$$\gamma_3 = 4^{p-1}\left(L^p 2^{p-1}[(b-a)^{p-1} + \gamma_1]\right).$$

Now by Lemma 2.1, we obtain

$$E|z_t|^p \leq \gamma_2 e^{\int_a^b \gamma_3 ds} \leq \gamma_2 e^{(b-a)\gamma_3} < \infty.$$

Therefore, $z_t \in C$.
(ii) Assume y_t is a solution of (2.4) and z_t is a solution of (2.2). Then

$$\begin{aligned} |y_t - z_t|^p &\leq 4^{p-1}\{|y_t - [y_0 + g(y)] \\ &- \int_a^b F(s, y_s)ds - \int_a^b G(s, y_s)dW_s|^p \\ &+ |g(y) - g(z)|^p \\ &+ |\int_a^b (F(s, y_s) - F(s, z_s))ds|^p \\ &+ |\int_a^b G(s, y_s) - G(s, z_s)dW_s|^p\} \, (3.3). \end{aligned}$$

Again, taking the expectation of each term of (3.3) we have

$$E|y_t - [y_0 + g(y)] + \int_a^b F(s, y_s)ds + \int_a^b G(s, y_s)dW_s|^p \leq \varphi^p(t).$$

By (H_5) we obtain

$$E|g(y) - g(z)|^p \leq L_g^p E|y - z|^p,$$

where $y_0 = z_0$. By (H_2), (H_3) and Holder's inequality, we obtain

$$\begin{aligned} \left|\int_a^b (F(s, y_s) - F(s, z_s))ds\right|^p &\leq \left(\int_a^b K|y_s - z_s|ds|\right)^p \\ &\leq K^p \left(\int_a^b ds\right)^{p-1} \int_a^b |y_s - z_s|^p ds \\ &\leq K^p (b-a)^{p-1} \int_a^b |y_s - z_s|^p ds. \end{aligned}$$

By applying (2.5), we obtain

$$\begin{aligned} E\left|\int_a^b (G(s, y_s) - G(s, z_s))dWs\right|^p &\leq \gamma_1 E\int_a^b |G(s, y_s) - G(s, z_s)|^p ds \\ &\leq \gamma_1 K^p \int_a^b E|y_s - z_s|^p ds. \end{aligned}$$

Therefore,

$$E|y_t - z_t|^p \leq [\gamma_4 \varphi^p(t) + \gamma_5 \int_a^b E|y_s - z_s|^p ds],$$

where $\gamma_4 = 4^{p-1}, \gamma_5 = 4^{p-1}K^p((b-a)^{p-1} + \gamma_1)$ and $(1 - L_g^p) < 1$.
By Lemma 2.1, we get

$$E|y_t - z_t|^p \leq \gamma_4 \varphi^p(t) e^{\left(\int_a^b \gamma_5\right)ds} \leq \gamma_4 \varphi^p(t) e^{(\gamma_5(b-a))}.$$

Thus,

$$E|y_t - z_t|^p \leq M_\varphi \varphi(t),$$

where $M_\varphi = \gamma_4{}^{\frac{1}{p}} e^{\left(\frac{\gamma_5(b-a)}{p}\right)}$. Hence, the nonlocal problem (2.1) has U-H-R stability and the proof is complete.

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4 Examples

To illustrate the established results, we consider the following nonlocal problems:
1. Given

$$x_t = \frac{\eta}{12}x(\eta) + \int_0^t x_s^3 ds + \int_0^t x_s^2 dWs. \tag{4.1}$$

$t \in [a, b] \equiv [0, 1]$, $0 < \eta < 1$, and $p = 2$.
Here $F(t, x) = x^3$, $G(t, x) = x^2$, and $g(x) = \frac{\eta}{12}x(\eta)$.
Next, we show that the hypotheses $(H_1) - (H_6)$ are satisfied.
$|F(t, x) - F(t, y)| \leq |x^3 - y^3|$ and $|G(t, x) - G(t, y)| \leq |x^2 - y^2|$ with $K = 1$.
$|F(t, x)| \leq (1 + |x^3|)$ and $|G(t, x)| \leq (1 + |x^2|)$ with $L = 1$.
$|g(x) - g(y)| \leq \frac{1}{12}|x - y|$, $\eta < 1$ with $L_g = \frac{1}{12}$.
Also, the function $\varphi(t) > 0$ satisfies the hypothesis (H_6) and
$1 - L_g = 1 - \frac{1}{12} < 1$.
Therefore, the problem (4.1) is stable in the sense of U-H-R.

2. Given

$$x_t = \frac{1}{4}x + \int_0^t (\frac{3}{2}x_s^5)ds - \int_0^t x_s^3 dWs. \tag{4.2}$$

$F(t, x) = \frac{3}{2}x_t^5$, $G(t, x) = x^3$, and $g(x) = \frac{1}{4}x$. Here, the hypotheses $(H_1) - (H_6)$ will hold if

$$\gamma_5 = 4^{p-1}(K_F^p(b - a)^{p-1} + K_G^p\gamma_1),$$

where $K_F = \frac{3}{2}$, $K_G = 1$, $L = 1$, and $L_g = \frac{3}{2}$. Showing that the problem (4.2) has the U-H-R stability.

Conclusion
We have used the Gronwall lemma approach to study the existence and U-H-R stability of a continuous solution of a nonlocal stochastic Volterra integral equation. The established results are subject to the condition (H_5), that is, $1 - L_g < 1$. The nonlinear examples have further justified the application of our model and have enriched the theory of stochastic Volterra equations.

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Competing financial interests

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