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Common Coupled Fixed Point Theorems without Compatibility in Partially Ordered Metric Spaces

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Abstract

A perfect blend of requirements for the proof of common coupled fixed point theorems in partially ordered metric space without the assumptions of (weak) compatibility is accomplished. Previous attempts in this direction involving these assumptions mostly ensure existence of coupled coincidence points. In many existing works in this area, attempt have been made to prove the existence of common coupled fixed points. However, only identity mappings can satisfy the conditions of the theorems. The method of proof presented in this present work is powerful in view of the fact that it guarantees the existence of common coupled fixed points without the imposition of (weak) compatibility conditions and identity mappings. To illustrate the results, an example is provided.

Keywords: Coupled fixed point of mappings, Partially ordered metric spaces, Common coupled fixed points, Identity mappings, Compatibility, Mixed monotone properties.
MSC2010: 37C25

1 Introduction

The following definitions are to be employed in the sequel.
Definition 1.1 [1–3] Let (X, d) be a metric space and let $T : X \times X \to X$ be a mapping. An element $(a, b) \in X \times X$ is said to be a *coupled fixed point* of T if $a = T(a, b)$ and $b = T(b, a)$.
Definition 1.2. [4, 5] Let (X, d) be a metric space and let $T : X \times X \to X$, $S : X \times X \to X$ be mappings. An element $(a, b) \in X \times X$ is said to be a *common coupled fixed point* of T and S if $a = S(a, b) = T(a, b)$ and $b = S(b, a) = T(b, a)$.
Definition 1.3 [9–11]: Consider a function $\psi :\ ^{+} \to ^{+}$ satisfying
(i) ψ is monotone increasing;
(ii) $\psi^n(t) \to 0$, as n $\to \infty$;

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(iii) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$.

(1) A function ψ satisfying (i) and (ii) above is called a *comparison function.*
(2) A function ψ satisfying (i) and (iii) above is called a *(c)-comparison function.*

Remark 1.4 [10, 11] :
(i) Any (c)-comparison function is a comparison function.
(ii) Every comparison function satisfies $\psi(0) = 0$.

The notion of coupled fixed points was introduced by Chang and Ma [13]. Since then, the concept has been of great interest to many researchers in the metrical fixed point theory.
Bhaskar and Lakshmikantham [1] proved a coupled fixed point theorem in a metric space endowed with partial order.
They introduced the following notions of a mapping satisfying the mixed monotone property:
Definition 1.5 [1]: Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$, a mapping. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any a, $b \in X$,

$$a_1, a_2 \in X, \quad a_1 \leq a_2 \Rightarrow F(a_1, b) \leq F(a_2, b)$$

and

$$b_1, b_2 \in X, \quad b_1 \leq b_2 \Rightarrow F(a, b_1) \geq F(a, b_2).$$

They proved the following theorem, considering a function satisfying the mixed monotone property:
Theorem 1.6: [1] Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with

$$d(F(a, b), F(u, v)) \leq \frac{k}{2}[d(a, u) + d(b, v)], \forall\ a \geq u, b \leq v.$$

If there exists $a_0, b_0 \in X$ such that

$$a_0 \leq F(a_0, b_0) \quad and \quad b_0 \geq F(b_0, a_0),$$

then, there exist a, $b \in X$ such that $a = F(a, b) \quad and \quad b = F(b, a)$.
The result has been generalised and extended by Lakshmikantam and Ciric [18], by introducing mixed g-monotone mapping and proving coupled coincidence and coupled common fixed point theorems for such contractive mappings in partially ordered metric spaces.

Several other authors have generalized the results of Bhaskar and Lakshmikantham [1]. Also, for more results on coupled fixed point theorems, interested readers can check [2,3,8,12,14,20–24,26–28] and others in the literature.

The problem of proving common coupled fixed point theorems in the setting of partially ordered metric spaces using the mixed monotone property has been of immense research interest. Lakshmikantam and Ciric [18] established common coupled fixed point theorems by using g-mixed monotone property while one of the mappings is made a proper subset of the other. They proved the existence of coupled coincidence points while coupled fixed point can be established only by assumption of identity on one of the mappings.
In the same line of argument, Abbas *et al.* [4] and Kim and Chandok [17] proved coupled coincidence points of mappings using the mixed monotone property and making one of the mappings a proper subset of the other.
Moerover, Abbas *et al.* [5] proved coupled coincidence point and common coupled fixed points for w-compatible mappings in partially ordered metric spaces. Also Kadelburg *et al.* [16] proved common coupled fixed points for compatible mappings in partially ordered metric spaces. Several researchers have proved common coupled fixed point theorems employing the notions mentioned above. For details on the common coupled fixed point theorems in partially ordered metric spaces, see [6, 7, 15, 17, 19, 25] in the reference section of this paper and others in the literature.

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In this paper, we establish that it is possible to prove common coupled fixed point of mappings in the setting of partially ordered metric space without imposing the above outlined conditions on the mappings but by enhancing the initial assumptions of the theorem of Bhaskar and Lakshmikantam [1] together with the mixed monotone property. Our method is distinct from the existing ones in that we have enlarged the class of mappings that can be investigated due to the following reasons:
(i) We do not assume compatibility and weak compatibility of mappings involved;
(ii) We do not make one of the mappings a proper subset of the other;
(iii) We used our method to prove common coupled fixed point theorem for family of mappings; and
(iv) We do not impose comparison criterion a priori on the sequences involved in the computation of the common coupled fixed points.

2 Main Results

A. Results Involving Rational-type Contractive Conditions.
Theorem 2.1. Let $(X, \leq)$ be a partially ordered set and d a metric on X such that (X, d) is a complete metric space. Let $F, G : X \times X \to X$ be mappings having the mixed monotone property such that for some $\lambda \geq 0$, $\forall a, b, u, v \in X$, $d(u, F(u, v)) + d(a, u) > 0$ and ψ, a (c)-comparison function , we have

$$d(F(a,b),G(u,v)) \leq \frac{\lambda d(a,F(a,b)).d(a,G(u,v)).d(u,F(a,b))}{1+d(u,F(u,v))+d(a,u)} + \psi(d(a,u)). \tag{2.1}$$

Suppose that we endow the product space $X \times X$ with the following partial order:
For $(a, b), (u, v) \in X \times X$, $(u, v) \leq (a, b) \iff a \geq u, b \leq v$.
If there exist $a_0, b_0 \in X$, such that $a_0 \leq F(a_0, b_0) \leq G(a_0, b_0)$ and $b_0 \geq F(b_0, a_0) \geq G(b_0, a_0)$. Then all F and G have a common coupled fixed point.

Proof. Choose $a_0, b_0 \in X$ such that $a_0 \leq F(a_0, b_0) \leq G(a_0, b_0)$ and $b_0 \geq F(b_0, a_0) \geq G(b_0, a_0)$.
Define $a_{2k+1} = F(a_{2k}, b_{2k})$, $b_{2k+1} = F(b_{2k}, a_{2k})$ and $a_{2k+2} = G(a_{2k+1}, b_{2k+1})$, $b_{2k+2} = G(b_{2k+1}, a_{2k+1})$ for $k \geq 0$.
We are to prove that a_k is non-decreasing and b_k is non-increasing. That is, for all $k \geq 0$,

$$a_{2k} \leq a_{2k+1} \leq a_{2k+2}$$

and

$$b_{2k} \geq b_{2k+1} \geq b_{2k+2}.$$

Firstly, $a_0 \leq F(a_0, b_0) = a_1$, and $b_0 \geq F(b_0, a_0) = b_1$.
By the iterative process above,

$$a_2 = G(a_1, b_1),\ b_2 = G(b_1, a_1).$$

Owing to the mixed monotone property of F and G, we have

$$a_1 = F(a_0, b_0) \leq G(a_0, b_0) \leq G(a_1, b_1) = a_2,$$

$$b_1 = F(b_0, a_0) \geq G(b_0, a_0) \geq G(b_1, a_1) = b_2,$$

Moreover,

$$a_2 = G(a_1, b_1) \leq G(a_2, b_2) = a_3,$$
$$b_2 = G(b_1, a_1) \geq G(b_2, a_2) = b_3$$

and

$$a_3 = F(a_2, b_2) \leq F(a_3, b_3) = a_4,$$

$$b_3 = F(b_2, a_2) \geq F(b_3, a_3) = b_4.$$

Therefore, for $n \geq 1$,

$$\begin{aligned} a_{2k+1} &= F(a_{2k}, b_{2k}) \leq F(a_{2k+1}, b_{2k+1}) = a_{2k+2}, \\ b_{2k+1} &= F(b_{2k}, a_{2k}) \geq F(b_{2k+1}, a_{2k+1}) = b_{2k+2}, \end{aligned}$$

and

$$\begin{aligned} a_{2k+2} &= G(a_{2k+1}, b_{2k+1}) \leq G(a_{2k+2}, b_{2k+2}) = a_{2k+3}, \\ b_{2k+2} &= G(b_{2k+1}, a_{2k+1}) \geq G(b_{2k+2}, a_{2k+2}) = b_{2k+3}. \end{aligned}$$

Hence,

$$a_0 \leq a_1 \leq a_2 \leq ... \leq a_{2k} \leq a_{2k+1} \leq ...$$

and

$$b_0 \geq b_1 \geq b_2 \geq ... \geq b_{2k} \geq b_{2k+1} \geq ...$$

Therefore, we deduce by (2.1) that

$$\begin{aligned} d(a_{2k+1}, a_{2k+2}) &= d(F(a_{2k}, b_{2k}), G(a_{2k+1}, b_{2k+1})) \\ &\leq \frac{\lambda d(a_{2k}, F(a_{2k}, b_{2k})).d(a_{2k}, G(a_{2k+1}, b_{2k+1})).d(a_{2k+1}, F(a_{2k}, b_{2k}))}{1 + d(a_{2k+1}, F(a_{2k+1}, b_{2k+1})) + d(a_{2k}, a_{2k+1})} \\ &+ \psi(d(a_{2k}, a_{2k+1})) \\ &= \frac{\lambda d(a_{2k}, a_{2k+1}) d(a_{2k}, a_{2k+2}).d(a_{2k+1}, a_{2k+1})}{1 + d(a_{2k+1}, a_{2k+2}) + d(a_{2k}, a_{2k+1})} + \psi(d(a_{2k}, a_{2k+1})) \\ &= \psi(d(a_{2k}, a_{2k+1})). \end{aligned} \tag{2.2}$$

Similarly,

$$\begin{aligned} d(b_{2k+1}, b_{2k+2}) &= d(F(b_{2k}, a_{2k}), G(b_{2k+1}, a_{2k+1})) \\ &\leq \frac{\lambda d(b_{2k}, F(b_{2k}, a_{2k})).d(b_{2k}, G(b_{2k+1}, a_{2k+1})).d(b_{2k+1}, F(b_{2k}, a_{2k}))}{1 + d(b_{2k+1}, F(b_{2k+1}, a_{2k+1})) + d(b_{2k}, b_{2k+1})} \\ &+ \psi d(b_{2k}, b_{2k+1}) \\ &= \frac{\lambda d(b_{2k}, b_{2k+1}) d(b_{2k}, b_{2k+2}).d(b_{2k+1}, b_{2k+1})}{1 + d(b_{2k+1}, b_{2k+2}) + d(b_{2k}, b_{2k+1})} + \psi d(b_{2k}, b_{2k+1}) \\ &= \psi(d(b_{2k}, b_{2k+1})). \end{aligned} \tag{2.3}$$

Therefore,

$$d(a_{2k+1}, a_{2k+2}) \leq \psi(d(a_{2k}, a_{2k+1})) \tag{2.4}$$

and

$$d(b_{2k+1}, b_{2k+2}) \leq \psi(d(b_{2k}, b_{2k+1})) \tag{2.5}$$

Similarly, proceeding as above, we have

$$d(a_{2k+2}, a_{2k+3}) \leq \psi(d(a_{2k+1}, a_{2k+2})) \tag{2.6}$$

and

$$d(b_{2k+2}, b_{2k+3}) \leq \psi(d(b_{2k+1}, b_{2k+2})). \tag{2.7}$$

Hence, it can be deduced from (2.4),(2.5), (2.6) and (2.7) that

$$\begin{aligned}
d(a_{2k+2},a_{2k+3})+d(b_{2k+2},b_{2k+3}) &\leq \psi(d(a_{2k+1},a_{2k+2}))+\psi(d(b_{2k+1},b_{2k+2}))\\
&\leq \psi^2(d(a_{2k},a_{2k+1}))+\psi^2(d(b_{2k},b_{2k+1}))\\
&\leq ... \leq \psi^n(d(a_0,a_1))+\psi^n(d(b_0,b_1)).
\end{aligned}$$

Thus, it follows that

$$d(a_{n+1},a_n)+d(b_{n+1},b_n) \leq \psi^n(d(a_0,a_1))+\psi^n(d(b_0,b_1)). \tag{2.8}$$

Furthermore, for $n, r \in \mathbb{N}$, using (2.8) inductively and repeated application of triangle inequality, we have

$$\begin{aligned}
d(a_n,a_{n+r})+d(b_n,b_{n+r}) &\leq [d(a_n,a_{n+1})+d(b_n,b_{n+1})]\\
&+[d(a_{n+1},a_{n+2})+d(b_{n+1},b_{n+2})]+...\\
&+[d(a_{n+r-1},a_{n+r})+d(b_{n+r-1},b_{n+r})]\\
&\leq \psi^n(d(a_0,a_1))+\psi^n(d(b_0,b_1))+\psi^{n+1}(d(a_0,a_1))\\
&+\psi^{n+1}(d(b_0,b_1))+...+\psi^{n+r-1}(d(a_0,a_1))+\psi^{n+r-1}(d(b_0,b_1))\\
&=\psi^n(d(a_0,a_1))+\psi^{n+1}(d(a_0,a_1))+...+\psi^{n+r-1}(d(a_0,a_1))\\
&+\psi^n(d(b_0,b_1))+\psi^{n+1}(d(b_0,b_1))+...+\psi^{n+r-1}(d(b_0,b_1))
\end{aligned}$$

$$\begin{aligned}
&=\sum_{k=n}^{n+r-1}\psi^k(d(a_0,a_1))+\sum_{k=n}^{n+r-1}\psi^k(d(b_0,b_1))\\
&=\sum_{k=0}^{n+r-1}\psi^k(d(a_0,a_1))-\sum_{k=0}^{n-1}\psi^k(d(a_0,a_1))+\sum_{k=0}^{n+r-1}\psi^k(d(b_0,b_1))\\
&-\sum_{k=0}^{n-1}\psi^k(d(b_0,b_1)) \quad \to 0 \ as \ n\to\infty
\end{aligned}$$

since ψ is a (c)-comparison function. Therefore, $\{a_n\}$,$\{a_n\}$ are Cauchy sequences in (X, d). Furthermore, since (X, d) is a complete metric space, there exist $a^*, b^* \in X$ such that

$$\lim_{n\to\infty} a_n = a^*$$

and

$$\lim_{n\to\infty} b_n = b^*$$

.

We now show that (a^*, b^*) is a coupled fixed point of F and G.
By condition (2.1) again, and noting that $\psi(0) = 0$, we have

$$\begin{aligned}
d(a^*,G(a^*,b^*)) &\leq d(a^*,a_{2n+1})+d(a_{2n+1},G(a^*,b^*))\\
&=d(a^*,a_{2n+1})+d(F(a_{2n},b_{2n}),G(a^*,b^*))\\
&\leq d(a^*,a_{2n+1})+\frac{\lambda d(a_{2n},F(a_{2n},b_{2n})).d(a_{2n},G(a^*,b^*)).d(a^*,F(a_{2n},b_{2n}))}{1+d(a^*,F(a^*,b^*))+d(a^*,a_{2n})}\\
&+\psi(d(a_{2n},a^*))\\
&=d(a^*,a_{2n+1})+\frac{\lambda d(a_{2n},a_{2n+1})d(a_{2n},G(a^*,b^*)).d(a^*,a_{2n+1}))}{1+d(a^*,F(a^*,b^*))+d(a_{2n},a^*)}+\psi(d(a_{2n},a^*))\\
&\to 0 \ as \ n\to\infty.
\end{aligned}$$

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Therefore $d(a^*, G(a^*, b^*)) = 0$, with the consequence that $a^* = G(a^*, b^*)$.
Again by (2.1), we have

$$\begin{aligned} d(b^*, G(b^*, a^*)) &\leq d(b^*, b_{2n+1}) + d(b_{2n+1}, G(b^*, a^*)) \\ &= d(b^*, b_{2n+1}) + d(F(b_{2n}, a_{2n}), G(a^*, b^*)) \\ &\leq d(b^*, b_{2n+1}) + \frac{\lambda d(b_{2n}, F(b_{2n}, a_{2n})).d(b_{2n}, G(b^*, a^*)).d(b^*, F(b_{2n}, a_{2n}))}{1 + d(b^*, F(b^*, a^*)) + d(b^*, b_{2n})} \\ &+ \psi(d(b_{2n}, b^*)) \\ &= d(b^*, b_{2n+1}) + \frac{\lambda d(b_{2n}, b_{2n+1}) d(b_{2n}, G(b^*, a^*)).d(b^*, b_{2n+1}))}{1 + d(b^*, F(b^*, a^*)) + d(b_{2n}, b^*)} + \psi(d(b_{2n}, b^*)) \\ &\to 0 \ as \ n \to \infty. \end{aligned}$$

Therefore $d(b^*, G(b^*, a^*)) = 0$, implying that $b^* = G(b^*, a^*)$.
Hence, (a^*, b^*) is a coupled fixed point of G.
Next is to show that (a^*, b^*) is a coupled fixed point of F too. By (2.1), we have

$$\begin{aligned} d(F(a^*, b^*), a^*) &= d(F(a^*, b^*), G(a^*, b^*)) \\ &\leq \frac{\lambda d(a^*, F(a^*, b^*)).d(a^*, G(a^*, b^*)).d(a^*, F(a^*, b^*))}{1 + d(a^*, F(a^*, b^*)) + d(a^*, a^*)} + \psi(d(a^*, a^*)) \\ &= \frac{\lambda d(a^*, F(a^*, b^*)).d(a^*, a^*).d(a^*, F(a^*, b^*))}{1 + d(a^*, F(a^*, b^*)) + d(a^*, a^*)} + \psi(0) = 0, \end{aligned}$$

implying that $d(F(a^*, b^*), a^*) = 0$, and by extension, $a^* = F(a^*, b^*)$.

Similarly by (2.1), we have

$$\begin{aligned} d(F(b^*, a^*), b^*) &= d(F(b^*, a^*), G(b^*, a^*)) \\ &\leq \frac{\lambda d(b^*, F(b^*, a^*)).d(b^*, G(b^*, a^*)).d(b^*, F(b^*, a^*))}{1 + d(b*, F(b^*, a^*)) + d(b^*, b^*)} + \psi(d(b^*, b^*)) \\ &= \frac{\lambda d(b^*, F(b^*, a^*))d(b^*, b^*).d(b^*, F(b^*, a^*))}{1 + d(b^*, F(b^*, a^*)) + d(b^*, b^*)} + \psi(0) = 0, \end{aligned}$$

implying that $d(F(b^*, a^*), b^*) = 0$, therefore, $b^* = F(b^*, a^*)$.
Thus, (a^*, b^*) is a common coupled fixed point of F and G.
Example 2.2. Let $X = [0, 4]$ be endowed with the usual metric. Let $\psi(t) = \frac{1}{2}t$, for all $t \in X$. Clearly, $\psi(t)$ is a (c)-comparison function. Define $F, G : [0, 4] \times [0, 4] \to [0, 4]$ by:

$$F(a, b) = 3a - 2b,$$

$$G(a, b) = \begin{cases} 2a - b + 1, & \text{if } a \geq b; \\ \frac{1}{7}, & \text{if } a < b. \end{cases}$$

Clearly, F and G have the mixed monotone property.
Let $a_0 = \frac{2}{3}$, $b_0 = \frac{1}{2} \in X$.
$F(a_0, b_0) = F(\frac{2}{3}, \frac{1}{2}) = 3(\frac{2}{3}) - 2(\frac{1}{2}) = 1$
$F(b_0, a_0) = F(\frac{1}{2}, \frac{2}{3}) = 3(\frac{1}{2}) - 2(\frac{2}{3}) = \frac{1}{6}$.
$G(a_0, b_0) = G(\frac{2}{3}, \frac{1}{2}) = 2(\frac{2}{3}) - (\frac{1}{2}) + 1 = \frac{11}{6}$
$F(b_0, a_0) = G(\frac{1}{2}, \frac{2}{3}) = \frac{1}{7}$.
Thus,
$\frac{2}{3} < 1 < \frac{11}{6}$ and $\frac{1}{2} > \frac{1}{6} > \frac{1}{7}$.
Hence,
$a_0 \leq F(a_0, b_0) \leq G(a_0, b_0)$ and $b_0 \geq F(b_0, a_0) \geq G(b_0, a_0)$.
Consider the contractive condition (2.1) in Theorem 2.1. Let $a = \frac{4}{3}$, $b = \frac{1}{3}$, $\lambda = 2$, $u = \frac{4}{5}$ and $v = \frac{1}{5}$.

Then, $F(a,b) = F(\frac{4}{3}, \frac{1}{3}) = \frac{10}{3}$, $G(u,v) = G(\frac{4}{5}, \frac{1}{5}) = \frac{12}{5}$, $d(F(a,b), G(u,v)) = d(F(\frac{4}{3}, \frac{1}{3}), G(\frac{4}{5}, \frac{1}{5})) = \left|\frac{10}{3} - \frac{12}{5}\right| = \frac{14}{15}$, $F(u,v) = 3(\frac{4}{5}) - 2(\frac{1}{5}) = 2$, $d(a, F(a,b)) = \left|\frac{4}{3} - \frac{10}{3}\right| = 2$, $d(a, G(u,v)) = \left|\frac{4}{3} - \frac{12}{5}\right| = \frac{16}{15}$, $d(u, F(a,b)) = \left|\frac{4}{5} - \frac{10}{3}\right| = \frac{38}{15}$, $d(a,u) = \left|\frac{4}{3} - \frac{4}{5}\right| = \frac{8}{15}$ and $d(u, F(u,v)) = \left|\frac{4}{5} - 2\right| = \frac{6}{5}$.
Hence, contractive condition (2.1), i.e

$$d(F(a,b), G(u,v)) \leq \frac{\lambda d(a, F(a,b)).d(a, G(u,v)).d(u, F(a,b))}{1 + d(u, F(u,v)) + d(a,u)} + \psi(d(a,u)), \tag{2.9}$$

implies that $\frac{14}{15} = 0.9333 \leq \frac{2 \times 2 \times \frac{16}{15} \times \frac{38}{15}}{1 + \frac{6}{5} + \frac{8}{15}} + \frac{1}{2}(\frac{8}{15}) = \frac{2432}{615} + \frac{8}{30} = 4.2211$.
Since all the assumptions of Theorem 2.1 are satisfied, F and G have a common coupled fixed point in $X = [0,4]$.

Corollary 2.3. Let $(X, \leq)$ be a partially ordered set and d a metric on X such that (X,d) is a complete metric space. Let $F, G : X \times X \to X$ be mappings having the mixed monotone property such that for some $\lambda \geq 0$, $\mu \in [0,1)$ and $\forall a,b,u,v \in X$, where $d(u, F(u,v)) + d(a,u) > 0$, we have

$$d(F(a,b), G(u,v)) \leq \frac{\lambda d(a, F(a,b)).d(a, G(u,v)).d(u, F(a,b))}{1 + d(u, F(u,v)) + d(a,u)} + \mu d(a,u). \tag{2.10}$$

Suppose that we endow the product space $X \times X$ with the following partial order:
For $(a,b), (u,v) \in X \times X, (u,v) \leq (a,b) \iff a \geq u, b \leq v$.
If there exist $a_0, b_0 \in X$, such that $a_0 \leq F(a_0, b_0) \leq G(a_0, b_0)$ and $b_0 \geq F(b_0, a_0) \geq G(b_0, a_0)$.
Then all F and G have a common coupled fixed point.

B. Results Involving Contractive Conditions of Non-Rational-Type.

Theorem 2.4. Let (X,d) be a complete metric space and $(X, \leq)$ a partially ordered set. Suppose $\{T_\alpha\}_{\alpha \in J}$ with $T_\alpha : X \times X \to X$ is a family of mappings and J is an index set. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$,

$$\begin{aligned} d(T_\alpha(s,r), T_\beta(u,v)) \leq\ & ad(s,u) + b[d(s, T_\alpha(s,r)) + d(u, T_\beta(u,v))] \\ & + c[d(s, T_\beta(u,v)) + d(u, T_\alpha(s,r))] \end{aligned} \tag{2.11}$$

$\forall s,r,u,v \in X$, for some $a,b,c \geq 0$ and $a + 2b + 2c < 1$.
Suppose that we endow the product space $X \times X$ with the following partial order:
For $(s,r), (u,v) \in X \times X, (u,v) \leq (s,r) \iff s \geq u, r \leq v$.
If $\forall \alpha \in J$, there exist $s_0, r_0 \in X$, such that $s_0 \leq T_\alpha(s_0, r_0) \leq T_\beta(s_0, r_0)$ and $r_0 \geq T_\alpha(r_0, s_0) \geq T_\beta(r_0, s_0)$. Then all T_α have a common coupled fixed point.

Proof. Choose $s_0, r_0 \in X$ such that $s_0 \leq T_\alpha(s_0, r_0) \leq T_\beta(s_0, r_0)$ and $r_0 \geq T_\alpha(r_0, s_0) \geq T_\beta(r_0, s_0)$. and define sequences $\{s_n\}$, $\{r_n\}$ in X in the following way:

$$s_{2n+1} = T_\alpha(s_{2n}, r_{2n}), \ r_{2n+1} = T_\alpha(r_{2n}, s_{2n})$$

$$s_{2n+2} = T_\beta(s_{2n+1}, r_{2n+1}), \ r_{2n+2} = T_\beta(r_{2n+1}, s_{2n+1})$$

We are to prove that $\{s_n\}$ is non-decreasing and $\{r_n\}$ is non-increasing. That is, for all $n \geq 0$, $s_{2n} \leq s_{2n+1}$ and $r_{2n} \geq r_{2n+1}$
Firstly, we set

$$s_0 \leq T_\alpha(s_0, r_0) = s_1,$$

(say) and

$$r_0 \geq T_\alpha(r_0, s_0) = r_1$$

(say).
By the iterative process above,

$$s_2 = T_\beta(s_1, r_1),\ r_2 = T_\beta(r_1, s_1).$$

Owing to the mixed monotone property of T_α and T_β, we have

$$s_1 = T_\alpha(s_0, r_0) \leq T_\beta(s_0, r_0) \leq T_\beta(s_1, r_1) = s_2,$$

$$r_1 = T_\alpha(r_0, s_0) \geq T_\beta(r_0, s_0) \geq T_\beta(r_1, s_1) = r_2.$$

Moreover,

$$\begin{aligned} s_2 &= T_\beta(s_1, r_1) \leq T_\beta(s_2, r_2) = s_3, \\ r_2 &= T_\beta(r_1, s_1) \geq T_\beta(r_2, s_2) = r_3, \end{aligned}$$

and

$$\begin{aligned} s_3 &= T_\alpha(s_2, r_2) \leq T_\alpha(s_3, r_3) = s_4, \\ r_3 &= T_\alpha(r_2, s_2) \geq T_\alpha(r_3, s_3) = r_4. \end{aligned}$$

Therefore, for $n \geq 1$,

$$\begin{aligned} s_{2n+1} &= T_\alpha(s_{2n}, r_{2n}) \leq T_\alpha(s_{2n+1}, r_{2n+1}) = s_{2n+2}, \\ r_{2n+1} &= T_\alpha(r_{2n}, s_{2n}) \geq T_\alpha(r_{2n+1}, s_{2n+1}) = r_{2n+2}, \end{aligned}$$

and

$$\begin{aligned} s_{2n+2} &= T_\beta(s_{2n+1}, r_{2n+1}) \leq T_\beta(s_{2n+2}, r_{2n+2}) = s_{2n+3}, \\ r_{2n+2} &= T_\beta(r_{2n+1}, s_{2n+1}) \geq T_\beta(r_{2n+2}, s_{2n+2}) = r_{2n+3}. \end{aligned}$$

Hence,

$$\begin{aligned} &s_0 \leq s_1 \leq s_2 \leq ... \leq s_{2n} \leq s_{2n+1} \leq ..., \\ &r_0 \geq r_1 \geq r_2 \geq ... \geq r_{2n} \geq r_{2n+1} \geq .... \end{aligned}$$

Therefore, we deduce by (2.11) that

$$\begin{aligned} d(s_{2n+1}, s_{2n+2}) &= d(T_\alpha(s_{2n}, r_{2n}), T_\beta(s_{2n+1}, r_{2n+1})) \\ &\leq ad(s_{2n}, s_{2n+1}) + b[d(s_{2n}, T_\alpha(s_{2n}, r_{2n})) + d(s_{2n+1}, T_\beta(s_{2n+1}, r_{2n+1}))] \\ &+ c[d(s_{2n}, T_\beta(s_{2n+1}, r_{2n+1})) + d(s_{2n+1}, T_\alpha(s_{2n}, r_{2n}))] \\ &\leq ad(s_{2n}, s_{2n+1}) + b[d(s_{2n}, s_{2n+1}) + d(s_{2n+1}, s_{2n+2})] \\ &+ c[d(s_{2n}, s_{2n+2}) + d(s_{2n+1}, s_{2n+1})] \\ &= ad(s_{2n}, s_{2n+1}) + bd(s_{2n}, s_{2n+1}) + bd(s_{2n+1}, s_{2n+2}) + cd(s_{2n}, s_{2n+2}) \\ &\leq ad(s_{2n}, s_{2n+1}) + bd(s_{2n}, s_{2n+1}) + bd(s_{2n+1}, s_{2n+2}) + cd(s_{2n}, s_{2n+1}) \\ &+ cd(s_{2n+1}, s_{2n+2}) \\ &\leq \frac{a+b+c}{1-b-c} d(s_{2n}, s_{2n+1}) \\ &= qd(s_{2n}, s_{2n+1}), \end{aligned} \tag{2.12}$$

where

$$q = \frac{a+b+c}{1-b-c} < 1.$$

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Similarly by (2.11),

$$\begin{aligned}
d(r_{2n+1},r_{2n+2}) &= d(T_\alpha(r_{2n},s_{2n}),T_\beta(r_{2n+1},s_{2n+1}))\\
&\leq ad(r_{2n},r_{2n+1})+b[d(r_{2n},T_\alpha(r_{2n},s_{2n}))+d(r_{2n+1},T_\beta(r_{2n+1},s_{2n+1}))]\\
&+c[d(r_{2n},T_\beta(r_{2n+1},s_{2n+1}))+d(r_{2n+1},T_\alpha(r_{2n},s_{2n}))]\\
&\leq ad(r_{2n},r_{2n+1})+b[d(r_{2n},r_{2n+1})+d(r_{2n+1},r_{2n+2})]\\
&+c[d(r_{2n},r_{2n+2})+d(r_{2n+1},r_{2n+1})]\\
&= ad(r_{2n},r_{2n+1})+bd(r_{2n},r_{2n+1})+bd(r_{2n+1},r_{2n+2})+cd(r_{2n},r_{2n+2})\\
&\leq ad(r_{2n},r_{2n+1})+bd(r_{2n},r_{2n+1})+bd(r_{2n+1},r_{2n+2})+cd(r_{2n},r_{2n+1})\\
&+cd(r_{2n+1},r_{2n+2})\\
&\leq ad(r_{2n},r_{2n+1})+bd(r_{2n},r_{2n+1})+bd(r_{2n+1},r_{2n+2})+c[d(r_{2n},r_{2n+1})\\
&+d(r_{2n+1},r_{2n+2})]\\
&\leq \frac{a+b+c}{1-b-c}d(r_{2n},r_{2n+1}) = qd(r_{2n},r_{2n+1}).
\end{aligned} \tag{2.13}$$

Now, adding (2.12) and (2.13), we have

$$d(s_{2n+1},s_{2n+2})+d(r_{2n+1},r_{2n+2}) \leq q[d(s_{2n},s_{2n+1})+d(r_{2n},r_{2n+1})]$$

Similarly by (2.11),

$$\begin{aligned}
d(s_{2n},s_{2n+1}) &= d(T_\alpha(s_{2n-1},r_{2n-1}),T_\beta(s_{2n},r_{2n}))\\
&\leq ad(s_{2n-1},s_{2n})+b[d(s_{2n-1},T_\alpha(s_{2n-1},r_{2n-1}))+d(s_{2n},T_\beta(s_{2n},r_{2n}))]\\
&+c[d(s_{2n-1},T_\beta(s_{2n},r_{2n}))+d(s_{2n},T_\alpha(s_{2n-1},r_{2n-1}))]\\
&\leq ad(s_{2n-1},s_{2n})+b[d(s_{2n-1},s_{2n})+d(s_{2n},s_{2n+1})]\\
&+c[d(s_{2n-1},s_{2n+1})+d(s_{2n},s_{2n})]\\
&\leq ad(s_{2n-1},s_{2n})+bd(s_{2n-1},s_{2n})+bd(s_{2n},s_{2n+1})+cd(s_{2n-1},s_{2n})\\
&+cd(s_{2n},s_{2n+1})\\
&\leq \frac{a+b+c}{1-b-c}d(s_{2n-1},s_{2n}) = qd(s_{2n-1},s_{2n}).
\end{aligned} \tag{2.14}$$

Furthermore, by (2.11), it follows that

$$\begin{aligned}
d(r_{2n},r_{2n+1}) &= d(T_\alpha(r_{2n-1},s_{2n-1}),T_\beta(r_{2n},s_{2n}))\\
&\leq ad(r_{2n},r_{2n-1})+b[d(r_{2n-1},T_\alpha(r_{2n-1},s_{2n-1}))+d(r_{2n},T_\beta(r_{2n},s_{2n}))]\\
&+c[d(r_{2n-1},T_\beta(r_{2n},s_{2n}))+d(r_{2n},T_\alpha(r_{2n-1},s_{2n-1}))]\\
&\leq ad(r_{2n-1},r_{2n})+b[d(r_{2n-1},r_{2n})+d(r_{2n},r_{2n+1})]\\
&+c[d(r_{2n-1},r_{2n+1})+d(r_{2n},r_{2n})]\\
&\leq ad(r_{2n-1},r_{2n})+bd(r_{2n-1},r_{2n})+bd(r_{2n},r_{2n+1})+c[d(r_{2n-1},r_{2n+1})\\
&\leq \frac{a+b+c}{1-b-c}d(r_{2n},r_{2n+1}) = qd(r_{2n-1},r_{2n}).
\end{aligned} \tag{2.15}$$

Also, from (2.14) and (2.15), we have

$$d(s_{2n},s_{2n+1})+d(r_{2n},r_{2n+1}) \leq q[d(s_{2n-1},s_{2n})+d(r_{2n-1},r_{2n})]. \tag{2.16}$$

If we let

$$\varepsilon_n = d(s_{2n},s_{2n+1})+d(r_{2n},r_{2n+1}),$$

then by (2.16), we have

$$\varepsilon_n \leq q\varepsilon_{n-1}. \tag{2.17}$$

Thus,

$$0 \leq \varepsilon_n \leq q\varepsilon_{n-1} \leq q^2\varepsilon_{n-2} \leq ... \leq q^n\varepsilon_0. \tag{2.18}$$

If $\varepsilon_0 = 0$, then (s_0, r_0) is a coupled fixed point of T. Suppose that $\varepsilon_0 > 0$. Then for each $m \in \mathbb{N}$, we obtain by repeated application of triangle inequality and (2.18) that

$$\begin{aligned} d(s_{2n}, s_{2n+m}) + d(r_{2n}, r_{2n+m}) &\leq [d(s_{2n}, s_{2n+1}) + d(r_{2n}, r_{2n+1})] + \\ &[d(s_{2n+1}, s_{2n+2}) + d(r_{2n+1}, r_{2n+2})] + ... \\ &+ [d(s_{2n+m-1}, s_{2n+m}) + d(r_{2n+m-1}, r_{2n+m})] \\ &= \varepsilon_n + \varepsilon_{n+1} + ... + \varepsilon_{n+m-1} \\ &\leq q^n\varepsilon_0 + q^{n+1}\varepsilon_0 + ... + q^{n+m-1}\varepsilon_0 \\ &\leq q^n\varepsilon_0 + q^{n+1}\varepsilon_0 + ... + q^{n+m-1}\varepsilon_0 \\ &= \frac{q^n(1 - q^{m-1})\varepsilon_0}{1 - q} \to 0 \ as \ n \to \infty. \end{aligned} \tag{2.19}$$

Therefore, $\{s_n\}$,$\{r_n\}$ are Cauchy sequences in (X, d). Since (X, d) is a complete metric space, there exist $s^*, r^* \in X$ such that $\lim_{n\to\infty} s_n = s^*$ and $\lim_{n\to\infty} r_n = r^*$.
We first show that (s^*, r^*) is a coupled fixed point of T_β. By condition (2.11) again, we have

$$\begin{aligned} d(s^*, T_\beta(s^*, r^*)) &\leq d(s^*, s_{2n+1}) + d(s_{2n+1}, T_\beta(s^*, r^*)) \\ &= d(s^*, s_{2n+1}) + d(T_\alpha(s_{2n}, r_{2n}), T_\beta(s^*, r^*)) \\ &\leq d(s^*, s_{2n+1}) + ad(s_{2n}, s^*) + b[d(s_{2n}, T_\alpha(s_{2n}, r_{2n})) + d(s^*, T_\beta(s^*, r^*))] \\ &+ c[d(s_{2n}, T_\beta(s^*, r^*)) + d(s^*, T_\alpha(s_{2n}, r_{2n}))] \\ &= d(s^*, s_{2n+1}) + ad(s_{2n}, s^*) + b[d(s_{2n}, s_{2n+1}) + bd(s^*, T_\beta(s^*, r^*))] \\ &+ c[d(s_{2n}, T_\beta(s^*, r^*)) + d(s^*, s_{2n+1})] \\ &\leq d(s^*, s_{2n+1}) + ad(s_{2n}, s^*) + b[d(s_{2n}, s^*) + d(s^*, s_{2n+1}) + bd(s^*, T_\beta(s^*, r^*))] \\ &+ c[d(s_{2n}, s^*) + d(s^*, T_\beta(s^*, r^*)) + d(s^*, s_{2n+1})] \\ &\leq \frac{d(s^*, s_{2n+1}) + ad(s_{2n}, s^*) + b[d(s_{2n}, s^*) + d(s^*, s_{2n+1})]}{1 - b - c} \\ &+ \frac{c[d(s_{2n}, s^*) + d(s^*, s_{2n+1})]}{1 - b - c} \\ &\to 0 \quad as \ n \to \infty. \end{aligned} \tag{2.20}$$

Therefore $d(s^*, T_\beta(s^*, r^*)) = 0$, implying that $s^* = T_\beta(s^*, r^*)$.
Similarly,

$$\begin{aligned} d(r^*, T_\beta(r^*, s^*)) &\leq d(r^*, r_{2n+1}) + d(r_{2n+1}, T_\beta(r^*, s^*)) \\ &= d(r^*, r_{2n+1}) + d(T_\alpha(r_{2n}, s_{2n}), T_\beta(r^*, s^*)) \\ &\leq d(r^*, r_{2n+1}) + ad(r_{2n}, r^*) + b[d(r_{2n}, T_\alpha(r_{2n}, s_{2n})) + d(r^*, T_\beta(r^*, s^*))] \\ &+ c[d(r_{2n}, T_\beta(r^*, s^*)) + d(r^*, T_\alpha(r_{2n}, s_{2n}))] \\ &= d(r^*, r_{2n+1}) + ad(r_{2n}, r^*) + b[d(r_{2n}, r_{2n+1}) + bd(r^*, T_\beta(r^*, s^*))] \\ &+ c[d(r_{2n}, T_\beta(r^*, s^*)) + d(r^*, r_{2n+1})] \\ &\leq d(r^*, r_{2n+1}) + ad(r_{2n}, r^*) + b[d(r_{2n}, r^*) + d(r^*, r_{2n+1}) + d(r^*, T_\beta(r^*, s^*))] \\ &+ c[d(r_{2n}, r^*) + d(r^*, T_\beta(r^*, s^*)) + d(r^*, r_{2n+1})] \\ &\leq \frac{d(r^*, r_{2n+1}) + ad(r_{2n}, r^*) + b[d(r_{2n}, r^*) + d(r^*, r_{2n+1})]}{1 - b - c} \\ &+ \frac{c[d(r_{2n}, r^*) + d(r^*, r_{2n+1})]}{1 - b - c} \\ &\to 0 \quad as \ n \to \infty, \end{aligned}$$

implying that

$$d(r^*, T_\beta(r^*, s^*)) = 0$$

and thus

$$r^* = T_\beta(r^*, s^*).$$

Therefore, (s^*, r^*) is a coupled fixed point of T_β.
We now show that (s^*, r^*) is a coupled fixed point of $\{T_\alpha\}$ too.
Let $\alpha \in J$ be arbitrary, then by (2.11), we have

$$\begin{aligned} d(T_\alpha(s^*, r^*), s^*) &= d(T_\alpha(s^*, r^*), T_\beta(s^*, r^*)) \\ &\leq ad(s^*, s^*) + bd(s^*, T_\alpha(s^*, r^*)) + b[d(s^*, T_\beta(s^*, r^*)) \\ &+ cd(s^*, T_\beta(s^*, r^*))] + cd(s^*, T_\alpha(s^*, r^*)) \\ &= bd(s^*, T_\alpha(s^*, r^*)) + cd(s^*, T_\alpha(s^*, r^*)) \\ &= (b + c)d(s^*, T_\alpha(s^*, r^*)). \end{aligned}$$

Since $b + c < 1$, we get $d(s^*, T_\alpha(s^*, r^*)) = 0$, that is, $s^* = T_\alpha(s^*, r^*)$.
On further employing (2.11), it follows that

$$\begin{aligned} d(T_\alpha(r^*, s^*), r^*) &= d(T_\alpha(r^*, s^*), T_\beta(r^*, s^*)) \\ &\leq ad(r^*, r^*) + bd(r^*, T_\alpha(r^*, s^*)) + bd(r^*, T_\beta(r^*, s^*)) \\ &+ cd(r^*, T_\beta(r^*, s^*)) + cd(r^*, T_\alpha(r^*, s^*)) \\ &= bd(r^*, T_\alpha(r^*, s^*)) + cd(r^*, T_\alpha(r^*, s^*)) \\ &= (b + c)d(r^*, T_\alpha(r^*, s^*)). \end{aligned}$$

Since $(b + c) < 1$, we get $d(r^*, T_\alpha(r^*, s^*)) = 0$, that is $r^* = T_\alpha(r^*, s^*)$.
Thus, (s^*, r^*) is a coupled fixed point of T_α too.
Hence, (s^*, r^*) is a common coupled fixed point of T_α and T_β.

Remark 2.5. Theorem 2.5 is a variant of Theorem 2.1 of Abbas $et-al$ [5] to the common coupled fixed point setting, involving family of mappings.
Remark 2.6. Furthermore, Theorem 2.5 generalizes and extends Theorems 2.5 and 2.6 of Sabetghadem *et al*. [23], to the common coupled fixed point setting in partially ordered space, the latter consisting of coupled fixed point in cone metric space setting.

3 Competing Interests

The authors of this paper declared that there are no competing interests as regards this article.

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