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A modified iterative Method for Solving Split variational Inclusion Problems and Fixed Point Problems for Nonexpansive Semigroup

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Abstract

In this paper, we introduce and study a modified iterative method for approximating a common solution of split variational inclusion problems and fixed point problems for nonexpansive semigroup in real Hilbert spaces. We prove that the proposed method converges strongly to the solution of the mentioned problem under some mild assuptions. A new inertial extrapolation is introduced which is known to speed up the rate of covergence of iterative algorithms. Our method uses self-adaptive stepsize that is generated at each iteration and does not depend norm of the bounded linear operator which is difficult in practice. We give numerical illustrations of the proposed scheme in comparison with other existing methods in the literature to further justify the applicability and efficiency of our proposed algorithm.

Keywords: Split variational inclusion problem, Inertial method, Fixed point problem, Hilbert spaces.
MSC2010: 47H09, 47J25.

1 Introduction

Let B_1 and B_2 be two maximal monotone mappings, $A : H_1 \to H_2$ is a bounded linear operator while H_1 and H_2 are two real Hilbert spaces. We are interested to study a problem with the following architecture: find a point $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1(x^*) \\ \text{and} \quad y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2 \\ \text{and} \quad x^* \in F(T), \end{cases} \tag{1.1}$$

where $T : H_1 \to H_1$ is a nonexpansive mapping. Problem (1.1) contains essential optimization problems. Before we go in details with (1.1), we will highlight some important earlier results in this

direction.
The problem of image reconstruction and signal processing from projections can be represented in the following system of linear equations

$$Ax = b. \tag{1.2}$$

Equation (1.2) is the typical inverse problem, where A is the measurement operator, b is the given data and x is the unknown quantity, the problem of interest. A common property of a vast majority of inverse problems is their ill-posedness. In the sense of Hadamard, a mathematical problem (we can think of an equation or optimization problem) is well-posed if it satisfies the following properties:
1.**Existence:** for all (suitable) data, there exists a solution of the problem (in an appropriate sense).
2.**Uniqueness:** for all (suitable) data, the solution is unique.
3.**Stability:** the solution depends continuously on the data.

It is worth mentioning that most inverse problems fail to satisfy at least one of these criteria. Therefore, a problem is ill-posed if one of these three conditions is violated. From the above definitions, we see that, the system (1.2) is often inconsistent, and one usually seeks a point which minimizes $x \in \mathbb{R}^N$ by some predetermined optimization criterion. The problem is frequently ill-posed and there may be more than one optimal solution. If the stability condition is violated, the numerical solution of the inverse problem by standard methods is difficult and often yields instability, even if the data are exact (since any numerical method has internal errors acting like noise). Therefore, special techniques, so-called regularization methods have to be used in order to obtain a stable approximation of the solution. Solving an inverse problem is the task of computing an unknown physical quantity that is related to given, indirect measurements via a forward model. Inverse problems appear in a vast majority of applications, including imaging (Computed Tomography (CT), Positron Emission Tomography (PET), Magnetic Resonance Imaging (MRI), Electron Tomography (ET), microscopic imaging, geophysical imaging), signal- and image-processing, computer vision, machine learning and (big) data analysis in general, among others.

The well-known convex feasibility problem (CFP) is an appraoch of formulating an inverse problem. It is of the form:

$$\text{find a point} \quad x^* \in C, \tag{1.3}$$

where $C := \cap_i C_i \neq \emptyset$ and $C_i = C_1, ..., C_{\mathbb{N}}$ are finitely many closed convex subset of a real Hilbert space H. An attempt to solve two inverse problems (IPs), the popular Split Feasibility Problem (SFP) was introduced and has the following problem structure:

$$\text{find } x^* \in C \subseteq \mathbb{R}^N \;\text{ such that } Ax \in Q \subseteq \mathbb{R}^N. \tag{1.4}$$

The SFP (1.4) was a novel article of Censor and Elfving [1] that was published in 1994, for modeling inverse problems which arises from phase retrievals and in medical image reconstruction. Since then, the SFP has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy (IMRT), approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [1–4] and related literatures.

A number of image reconstruction problems can be formulated as the SFP; see, e.g. [4] and the reference therein. Recently, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP; see, e.g., [3–11] and the references therein. Image restoration and image reconstruction are the two main sub-branches of image recovery. The term image restoration usually applies to the problem of estimating the original form h of a degraded image x. Hence, in image restoration the data consist of measurements taken directly on the image to be estimated, x being a blurred and noise-corrupted version of h. The blurring operation can be induced by the image transmission medium, e.g., the atmosphere in astronomy, or by the recording device, e.g., an out-of-focus or moving camera. On the other hand, image reconstruction refers to

Vol. 8, No. 2, pp. 109 - 130
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problems in which the data x are indirectly related to the form the original image h.
The SFP by Censor and Elfving [1] has been the foundation for solving numerous problems medical imaging. A special case of (1.4) is the convex constrained linear inverse problem (CLIP), which was studied by Landweber [12] in a finite dimensional real Hilbert space. It is characterized by:

$$\text{find} \quad x^* \in C \quad \text{such that} \quad Ax^* = b, \tag{1.5}$$

where C is a nonempty closed convex subset of a Hilbert space H_1 while b is an element in another Hilbert space H_2. The Landweber iterative algorithm is given below:

$$x_{n+1} = x_n + \gamma A^T(b - Ax_n), n \geq 1. \tag{1.6}$$

It is easy to see that $x^* \in C$ solves (1.4) if and only if it solves the following fixed point equation:

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*. \tag{1.7}$$

where P_C and P_Q are the orthogonal projections onto nonempty closed and convex subsets C and Q respectively of a Hilbert space, $\gamma > 0$, and A^* is an adjoint of the matrix A.
The popular iterative method for solving the (1.4) is called the CQ algorithm and has the following iterative step:

$$x^{k+1} = P_C(x^k + \gamma A^T(P_Q - I)Ax^k), \tag{1.8}$$

where $\gamma \in (0, 2/L)$ with L the largest eigenvalue of the matrix A^TA and P_C and P_Q denote the orthogonal projeections onto C and Q, respectively; that is, $P_C(x)$ minimizes $\|c - x\|$, over all $c \in C$. The CQ algorithm converges to a solution of the SFP, or , more generally, to a minimizer of $\|P_QAc - Ac\|$ over $c \in C$, whenever such exists.

Another important and more general problem is the Split Variational Inequality Problem (SVIP) studied by Censor et al. [13]. Let H_1 and H_2 be two real Hilbert spaces. Let $f : H_1 \to H_1$, $g : H_2 \to H_2$ be given operators, and a bounded linear operator $A : H_1 \to H_2$. Let C and Q be two nonnempty closed convex subets of H_1 and H_2 respectively. Then, the SVIP is formaluated as follows:

$$\text{find a point} \quad x^* \in C \quad \text{such that} \quad \langle f(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.9}$$

and such that

$$\text{the point} \quad y^* = Ax^* \in Q \quad \text{solves} \quad \langle g(y^*), y - y^* \rangle \geq 0 \quad \forall \quad y \in Q. \tag{1.10}$$

When we consider (1.9) and (1.10) separately, we observe that (1.9) is the well known Variational Inequality Problem (VIP) and the solution set is denoted by $SOL(f, C)$. Thus, the SVIP is made up of a pair of VIP which has to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A, is the solution of another VIP in another space H_2.

Remark 1: See that if the operator f and g are identically zero, that is, $f \equiv 0$ and $g \equiv 0$ in (1.9) and (1.10) respectively, we recover the SFP (1.4).

Yet, another interesting problem of study is that of Split Zero problem (SZP) (see, Censor et al. [13]). To construct this problem, we let H_1 and H_2 to be two real Hilbert spaces. Given operators $f : H_1 \to H_1$, $g : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$, then, the SZP is formulated as follows:

$$\text{find a point} \quad x^* \in H_1 \quad \text{such that} \quad f(x^*) = 0 \quad \text{and} \quad g(Ax^*) = 0. \tag{1.11}$$

We see that SZP is a special case of SVIP in which the bounded linear operator A is a surjective mapping. Consider $C = H_1$ and $Q = H_2$ in (1.9) and (1.10) respectively, choose $x := x^* - f(x^*) \in H_1$ in (1.9) such that $y := Ax^* - g(Ax^*) \in H_2$ in (1.10).

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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The more general class of problem is the Split Monotone Inclusion Problem (SMVIP). Before we introduce this problem, let us explain some concepts. Given a nonempty closed convex subset of a real Hilbert space H, its indicator function is defined as:

$$\begin{cases} \delta_C(x) = 0, if \quad x \in C, \\ +\infty, otherwise. \end{cases} \tag{1.12}$$

The normal cone to C at a point v is given by:

$$N_C(v) = \{d \in H : \langle d, y - v \rangle \leq 0\} \forall \quad y \in C, \tag{1.13}$$

and a set valued mapping B is defined by:

$$B(v) = \begin{cases} f(v) + N_C(v), v \in C \\ \emptyset. \end{cases} \tag{1.14}$$

where f is a given operator with some imposed continuity assumptions. The operator B is proven to be a maximal monotone by Rockafellar ([14], Theorem 3) and $B^{-1} = \{x \in C : 0 \in B(x)\} = SOL(f, C)$. This motivated Moudafi [15] to introduce the popular SMVIP which is formulated below:

$$\text{find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in f(x^*) + B_1(x^*), \tag{1.15}$$

and such that

$$\text{a point} \quad y^* = Ax^* \in H_2 \text{ solves} \quad 0 \in g(y^*) + B_2(y^*), \tag{1.16}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued mappings, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two single valued operators and $A : H_1 \to H_2$ is a bounded linear operator.
We quickly observe that if we set $B_1 = N_C$ and $B_2 = N_Q$, we recover the SVIP (1.9)-(1.10). The SMVIP is a generalization of the SFP, SVIP, SZP. It has widely been used to model intensity modulated radiation therapy treatment, for details kindly see [1,3]. It has also been used to model significally many inverse problems, in particular, the phase retrieval and among others, for instance, in sensor networks, in computerized tomography and data compression; for details see, e.g. [8,14,16,17].

Furthermore, another problem of interest is the Split Variational Inclusion Problem (SVIP). Suppose $f \equiv 0$ and $g \equiv 0$, then the SMVIP ((1.15)- (1.16)) reduces to the following SVIP defined as follows:

$$\text{find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in B_1(x^*) \tag{1.17}$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \tag{1.18}$$

Notice that (1.17) is the classical variational inclusion (VI). We denote the solution set of (1.17) by SOLVIP(B_1). Hence, SVIP (1.17)-(1.18) is made up a pair of variational inclusion problems. Let the solution set of (1.17)-(1.18) be denoted by $\Gamma := \{x^* \in H_1 : x^* \in SOLVIP(B_1)$ and $Ax^* \in SOLVIP(B_2)\}$. Observe also that (1.17)-(1.18) is one part of (1.1). Moreso, problem (1.1) involves the study of fixed point problem (FPP) which have widely study in the literature and finds valuable applications in edifferent fields of study.
The weak convergence of the problems (1.16)-(1.17) was recently studied by Byrne et al [18]. They provided the following iterative algorithm: for an initial guess, choose $x_0 \in H_1$, then the iterative sequence $\{x_n\}$ is defind by:

$$x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \tag{1.19}$$

where A^* is the adjoint of A, L is the spectral radius of the operator A^*A and $\gamma \in (0, 2/L)$ for some $\lambda > 0$.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
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The result of [18] recieved a great improvement in 2014 by the work of Kazmi and Rizvi [19]. This is due to the fact that strong convergence is desirable in applications. They considered the following iterative method:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_1} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (I - \alpha_n D)S_n u_n, \end{cases} \tag{1.20}$$

where $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A while λ is a positive number and α_n is a sequence of positive number. They proved that the sequence $\{x_n\}$ converged strongly to the solution set Γ of (1.17)-(1.18).

In 2015, Sithithakerngkiet et al [20] in the same line of research, proposed the hybrid steepest decent method and constructed the following iterative method:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \beta f(x_n) + (I - \alpha_n T)S_n u_n, \end{cases} \tag{1.21}$$

where S_n is a sequnce of nonexpansive mappings, T is a positive operator, A^* is the adjoint of the bounded linear operator $A, \lambda > 0, \gamma \in (0, 1/L)$, where L is the spectral radius of A^*A. They proved that the iterative sequnce $\{x_n\}$ strongly converges to a point p, where $p = P_\Omega(I - T + \beta f)(p)$ is a unique solution of the variational inequality:

$$\langle (T - \beta f)p, p - x \rangle \le 0, x \in \Omega. \tag{1.22}$$

Remark 1: We notice that the algorithms (1.19), (1.20) and (1.21) have two major drawbacks.
1). To excute the algorithms, one needs to compute the operator norm or at least, estimate it. This is generally difficult to implement and in most cases, it is impossible. In numerical and practical sense, it is time consuming. To circumvent this challenge, some authors [21] and [22] provided a tip on how to choose step sizes that do not depend on the operator norm.

Motivated by this, we construct a new stepsize $\{\gamma_n\}$ which does not depend on the operator as follows:

$$\gamma_n \in \{\epsilon, \frac{\|(J_\lambda^{B_2} - I)Ax_n\|^2}{\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2} - \epsilon\}. \tag{1.23}$$

With this variable stepsize, we do not need the knowledge of operator norm and the sequence converges strongly as expected. Following the work of Che and Li [23] , Moudafi [24], Byre et al. [25] and Kazmi and Rizvi [19], we study an iterative method for approximating a common solution of split variotional inclusion problem and fixed point problem for nonexpansive semigroups in Hilbert spaces. Further, it is desirable in applications to have a speedy convergence of iterations. Based on this, inertial extrapolation [35] has been of interest due to the fact that it improves rate convergence (see e.g, [26–29,36–40]) and references therein. Based on the forgoing, we introduce an inertial term in the propose algorithm to speed up the rate of conversgence.

We organize the remaining part of the paper as follows: In section 2, we present some Lemmas, Theorems and basic definitions relevant to our reseult. In section 3, we present our algorithm and some standard assumptions and the convergence analysis is discuss in section 4. The numerical examples are provide in section 5 while the conclusion is in section 6.

2 Preliminaries

Let C denote a nonempty closed convex subset of a real Hilbert space H. Let "$\to$" and "$\rightharpoonup$" denote a strong and weak convergence of a sequence respectively. Let $T : H \to H$ be a mapping. Then, a point $x \in H$ is called the fixed point of T if $Tx = x$. The set of fixed point of T is denoted by $F(T) := \{x \in H : Tx = x\}$. Let $\omega_n(x_n) = \{x : \exists \ x_{n_j} \rightharpoonup x\}$ be the weak $\omega-$limit set of x_n.

We recall some important properties of a projection operator associated with C. For any $x \in C$, there exists a unique point $P_C(x) \in C$ such that

$$\|x - P_C(x)\| \leq inf\{\|x - y\|, \forall y \in C\}.$$

P_C is called the metric projection of H onto C, that is, $P_C : H \to C$. It is well known that P_C has the following properties

$$\langle x - y, P_C(x) - P_C(y)\rangle \geq \|P_C(x) - P_C(y)\|^2, \forall x, y \in C.$$

More so, for $P_C(x) \in C$,

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \forall y \in C.$$

The following basic definitions are important to our study:
Definition 2.1: A map $T : H \to H$ is called a contraction if there exists $\alpha \in (0, 1)$ such that

$$\|Tx - Ty\| \leq \alpha\|x - y\|, \forall x, y, \in H$$

If $\alpha = 1$, then it is called a nonexpansive mapping.
Definition 2.2: A one-parameter family $\Gamma = \{T(t) : 0 \leq t < \infty\}$ on H is called a nonexpansive semigroup if it satisfies the following conditions:
i) $T(0)x = x, \forall x \in H$,
ii) $T(s + t) = T(s)T(t), \forall s, t \geq 0$,
iii) for each $x \in H$, the mapping $T(t)x$ is continous,
iv) $\|T(t)x - T(t)y\| \leq \|x - y\|$, for all $x, y \in H$ and $t > 0$.
Example 2.3 [23] Let $H = R$ and $\Gamma := \{T(t) : 0 \leq t < \infty\} : H \to H$ defined by $T(t)x = (1/10^t)$. Then, F is nonexpansive semigroup. Thus, we see below that Γ satusfies the four conditions:
i) $T(0)x = (1/10^0)x = x$ for all $x \in H$,
ii) $T(s + t) = (1/10^s)(1/10^t) = T(s)T(t)$ for all $t, s \geq 0$,
iii) for each $x \in H$, the mapping $T(t)x = (1/10^t)x$ is continous,
iv) $\|T(t)x - T(t)y\| = \|(1/10^t)x - (1/10^t)y\| = \|(1/10^t)(x - y)\| = (1/10^t)\|x - y\| \leq \|x - y\|$ for all $x, y \in H$ and $t \geq 0$.
Definition 2.4: An operator T is said to be:
(i) $\alpha-$ inverse strongly monotone, if there exists $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2 \quad \forall x, y \in H,$$

(ii) Monotone, if

$$\langle Tx - Ty, x - y\rangle \geq 0 \;\; \forall x, y \in H,$$

Remark 2.5: It is clear that every $\alpha-$inverse strongly monotone operators are monotone.
Definition 2.6: A multi-valued map $T : H \to 2^H$, is called monotone, if

$$\langle x - y, u - v\rangle \geq 0, \;\; \forall \;\; x, y \in H, u \in T(x), v \in T(y).$$

Definition 2.7: The operator $T : H \to 2^H$, is called maximal monotone, if the graph $G(T)$ of T defined by

$$G(T) := \{(x, y) \in H \times H : y \in T(x)\}$$

is not properly contained in the graph of any other monotone operator. It is well known that T is maximal monotone if and only if for $(x, u) \in H \times H, \langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(T)$ implies $u \in T(x)$.
Definition 2.8: The resolvent operator J^T_λ associated with a multivalued operator T and $\lambda > 0$ is the map $J^T_\lambda : H \to 2^H$ defined by $J^T_\lambda(x) = (I + \lambda T)^{-1}(x), x \in H, \lambda > 0$, where I is the identity operator in H. It is known that if T is monotone, then J^T_λ is a singlevalued, nonexpansive and firmly nonexpansive mapping.
Definition 2.9: A map $T : H \to H$ is said to be demiclosed at the origin if for each sequence $\{x_n\}$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
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converges weakly to x, and the sequnce $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.
Definition 2.10: T is said to be semicompact, if for any bounded sequence $\{x_n\} \subset H, \lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_n\}$ converges strongly to some point $x^* \in H$.

The following Lemmas will be relevant in our work.
Lemma 2.1(see [30]): Let H be a real Hilbert space. Then, for $x, y \in H$ the following results hold:
(a) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2$,
(b) $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x - y\rangle$.
(c) for any $\lambda \in [0, 1]$; $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 2.2 (see for details [31]:) Let C be a nonempty bounded closed and convex subset of a real Hilbert space H. Let $\{T(s) : s \geq 0\}$ defined C to itself be a nonexpansive semigroup. Then for all $h \geq 0$,

$$\limsup_{t\to\infty}, {}_{x\in C}\|\frac{1}{t}\int_0^t T(x)xds - T(h)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.3(see [24]:). The SVIP defined in (1.17) and (1.18) is equivalent to finding $x^* \in H_1$ such that $y^* = Ax^* \in H_2$,

$$x^* = J_\lambda^{B_1}(x^*),$$
$$y^* = J_\lambda^{B_2}(y^*)$$

for some $\lambda > 0$.
Lemma 2.4([31]:) Let C be a nonempty bounded closed and convex subset of a real Hilbert space H, let $\{x_n\}$ be a sequence, and let $\{T(s) : s \geq 0\}$ defined on C into itself be a nonexpansive semigroup. Suppose the following conditions hold:
(i) $\{x_n\} \rightharpoonup z$,
(ii) $\limsup_{s\to\infty} \limsup_{n\to\infty} \|T(s)x_n - x_n\| = 0$,
then, $z \in F(T)$.
Lemma 2.5(see [32]) Let H be a Hilbert space and let $\{u_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying the following conditions:
(i) for every $w \in W, \lim_{n\to\infty} \|u_n - w\|$ exists.
(ii) Each weak cluster point of the sequence $\{w_n\}$ is in W.
Then, there exists $w^* \in W$ such that $\{u_n\}$ weakly converges to w^*.
Lemma 2.6:(see [23]) Let $\Gamma = \{T(s) : s \geq 0\}$ be a one parameter family of nonexpansive semigroup defined on H, then for $p \in F(\Gamma), x \in H$ and $t \geq 0$,

$$\|\frac{1}{t}\int_0^t T(s)xds - x\|^2 \leq 2\langle x - p, x - \frac{1}{t}\int_0^t T(s)xds\rangle.$$

Lemma 2.7:([33]) Let $\{\alpha_n\}$ be a sequence of non-negative real numbers, $\{\beta_n\}$ be a sequence of real numbers in $(0, 1)$ with condition $\sum_{n=1}^\infty \beta_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that

$$\alpha_n \leq (1 - \beta_n)\alpha_n + \beta_n d_n, n \geq 0.$$

If the $\limsup_{n\to\infty} d_n \leq 0$ and $\sum_{n=1}^\infty \beta_n = \infty$, then $\lim_{n\to} \alpha_n = 0$.
Lemma 2.8([34]) Let $\{\alpha_n\}$ be a sequence of non-negative real numbers satisfying the following:

$$\alpha_n \leq (1 - \beta_n)\alpha_n + \sigma_n + \gamma_n, n \geq 1,$$

where $\{\beta_n\}$ is a sequnece in $(0, 1)$ and $\{\sigma_n\}$ is a real sequence. Suppose that$\sum_{n=1}^\infty \gamma_n < \infty$ and $\sigma_n \leq \beta_n M$ for some $M \geq 0$. Then, $\{\alpha_n\}$ is a bounded sequence.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
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3 The Main Result

In this section, we present our algorithm and the conditions for the strong convergence.
Assuptions 3.1: H_1 and H_2 are taken to be real Hilbert spaces. We assume that the following hold:
i) $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_1 \to 2^{H_2}$ are maximal monotone operators.
ii) $A : H_1 \to H_2$ is a bounded linear operator such that $A \neq 0$ and $A^* : H_2 \to H_1$, the adjoint of A such that $\langle Ax, y\rangle \leq \langle x, A^*y\rangle$.
iii) $T : H_1 \to H_1$ is a nonexpansive semigroup.
iv) $\Phi = F(T) \cap \Omega \neq \emptyset$, where $\Omega := \{x^* \in H_1 : x^* \in SOLVIP(B_1)$ and $Ax^* \in SOLVIP(B_2)\}$.

Assumption 3.2: Suppose $\{\beta_n\}, \{\alpha_n\}$ and $\{\epsilon_n\}$ are all positive sequences satisfying conditions:
1) $0 < \liminf_{n\to\infty} \alpha_n < \limsup_{n\to\infty} \alpha_n < 1$.
2) $\{\theta_n\} \subset (a, 1 - \beta_n)$ for some $a > 0$.
3) $\lim_{n\to\infty} \dfrac{\varepsilon_n}{\beta_n} = 0$.
4) $\{\beta_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$.

Algorithm 3.3
1: Step 0 Choose sequences $\{\beta_n\}, \{\theta_n\}$ and $\{\varepsilon_n\}$ such that the conditions from Assumption 3.2 hold and let $\lambda > 0, \alpha \geq 0$ and $x_0, x_1 \in H$ be chossen arbitrarily. Set $n := 1$.
Iteration Steps: Step 0: Given the iterate x_{n-1} and $x_n (n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \overline{\alpha}_n$, where

$$\overline{\alpha}_n := \begin{cases} \min\{\dfrac{n-1}{n+\alpha-1}, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } \; x_n \neq x_{n+1}, \\ \dfrac{n-1}{n+\alpha-1}, & otherwise, \end{cases} \tag{3.1}$$

and compute

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n), \\ x_{n+1} = (1 - \theta_n - \beta_n)w_n + \theta_n \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds, \end{cases} \tag{3.2}$$

where

$$\gamma_n \in (\epsilon, \frac{\|(J_\lambda^{B_2} - I)Aw_n\|^2}{\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2} - \epsilon \;),$$

for some large enough $\epsilon > 0$.
Update: **Set** $n := n + 1$ and go to **Step 1.**

Remark 3.3: (a) Notice that the stepsize $\{\gamma_n\}$ is independent of operator norm of A^*A which is a big improvement over others whose stepsize is dependent of operator norms.
(b) The inertial term $\theta_n(x_n - x_{n-1})$ is introduced to speed up the rate of convergence which is also an improvement over algorithms (1.19), (1.20) and (1.21). Furthermore, the computation of our algorithm is simple. It does not require the computation of $\|x_n - x_{n-1}\|$ or $\sum_{n=1}^{\infty} \|x_n - x_{n-1}\| < \infty$ before choosing the inertial factor θ_n.

4 Convergence Analysis

Theorem 4.1: Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 such that 3 and 3 are satisfied, then the sequence $\{x_n\}$ generated recursively by the Algorithm 3.3 strongly converges to the solu-

tion set.
Proof: We break the proof into several steps.
Step I: We first show that the sequence is bounded. Take $p \in \Phi$, we that $J_\lambda^{B_1} p = p, Ap = J^{B_2} Ap$, and $T(s)p = p$

$$\begin{aligned} \|w_n - p\| &= \|x_n + \alpha_n(x_n - x_{n-1}) - p\| \\ &= \|x_n - p + \alpha_n(x_n - x_{n-1})\| \\ &\leq \|x_n - p\| + \alpha_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \beta_n \frac{\alpha_n}{\beta_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{4.1}$$

But $\alpha_n\|x_n - x_{n-1}\| \leq \varepsilon_n, \forall\ n \in \mathbb{N}$, which implies that,
$\frac{\alpha_n}{\beta_n}\|x_- x_{n-1}\| \leq \frac{\epsilon_n}{\beta_n} \to 0$ as $n \to \infty$.
Thus, there exist $M > 0$ such that
$\frac{\alpha_n}{\beta_n}\|x - x_{n-1}\| \leq M, \forall n \in \mathbb{N}$.
It follows from this fact and (4.1) that

$$\|w_n - p\| \leq \|x_n - p\| + \beta_n M. \tag{4.2}$$

Now, from the Algorithm 3.3, $\forall p \in \Phi$, and nonexpansivess of $J_\lambda^{B_1}$ that

$$\begin{aligned} \|u_n - p\|^2 &= \|J^{B_1}(w_n + \gamma_n A^*(J^{B_2}) - I)Aw_n - p\|^2 \\ &= \|J^{B_1}(w_n + \gamma_n A^*(J^{B_2} - I)Aw_n - J^{B_1}(p)\|^2 \\ &\leq \|w_n + \gamma_n A^*(J^{B_2} - I)Aw_n - p\|^2 \\ &= \|w_n - p + \gamma_n A^*(J^{B_2} - I)Aw_n\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2 + 2\gamma_n\langle w_n - p, A^*(J_\lambda^{B_2} - I)Aw_n\rangle \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2 + 2\gamma_n\langle Aw_n - Ap, (J_\lambda^{B_2} - I)Aw_n\rangle. \end{aligned} \tag{4.3}$$

From (4.3), we obtain,

$$\begin{aligned} 2\gamma_n\langle Aw_n - Ap, (J_\lambda^{B_2} - I)Aw_n\rangle &= 2\gamma_n\langle Aw_n - Ap + (J_\lambda^{B_2} - I)Aw_n - (J_\lambda^{B_2} - I)Aw_n, (J_\lambda^{B_2} - I)Aw_n\rangle \\ &= 2\gamma_n\langle J^{B_2}Aw_n - Ap, (J_\lambda^{B_2} - I)Aw_n\rangle - 2\gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &= \gamma_n\|J^{B_2}Aw_n - Ap\|^2 + \gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &\quad -\gamma_n\|Aw_n - Ap\|^2 - 2\gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &= \gamma_n\|J^{B_2}Aw_n - J^{B_2}(Ap)\|^2 - \gamma_n\|Aw_n - Ap\|^2 - \gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &\leq \gamma_n\|Aw_n - Ap\|^2 - \gamma_n\|Aw_n - Ap\|^2 - \gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &= -\gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2. \end{aligned} \tag{4.4}$$

It follows from (4.2), (4.3) and the condition on $\gamma_n$that

$$\begin{aligned} \|u_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2 - \gamma_n\|(J_\lambda^{B_2} - I)Aw_n\|^2 \\ &= \|w_n - p\|^2 - \gamma_n(\|(J_\lambda^{B_2} - I)Aw_n\|^2 - \gamma_n\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2) \\ &= \|w_n - p\|^2. \end{aligned} \tag{4.5}$$

$\|(1 - \theta_n - \beta_n)(w_n - p) + \theta_n(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p)\|^2$

$$
\begin{aligned}
&= (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)\|^2 \\
&\quad +2\theta_n(1-\theta_n-\beta_n)\langle w_n-p,\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\rangle \\
&= (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(s)(p)\|^2 \\
&\quad +2\theta_n(1-\theta_n-\beta_n)\langle w_n-p,\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(s)(p)\rangle \\
&\leq (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|u_n-p\|^2 \\
&\quad +2\theta_n(1-\theta_n-\beta_n)\|w_n-p\|\|u_n-p\| \\
&\leq (1-\beta_n)^2\|w_n-p\|^2. \qquad (4.6)
\end{aligned}
$$

Next, from the algorithm 3.3, (4.6) and (4.2) we estimate that for all $p\in\Phi$,

$$
\begin{aligned}
\|x_{n+1}-p\| &= \|(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)-\beta_np\| \\
&\leq \|(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)\|+\beta_n\|p\| \\
&\leq (1-\beta_n)\|w_n-p\|+\beta_n\|p\| \\
&\leq (1-\beta_n)[\|x_n-p\|+\beta_nM]+\beta_n\|p\| \\
&= (1-\beta_n)|x_n-p\|+(1-\beta_n)\beta_nM+\beta_n\|p\| \\
&\leq (1-\beta_n)|x_n-p\|+\beta_nM+\beta_n\|p\| \\
&= (1-\beta_n)\|x_n-p\|+\beta_n(M+\|p\|). \qquad (4.7)
\end{aligned}
$$

It follows from Lemma 2.8 and (4.7) that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ and $\{u_n\}$ are bounded sequences.

Step II: We show that the limit exists and that the sequence $\{x_n\}$ is asymptotically regular, that is, $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
To establish this result, we consider the following two cases:
Case 1: Assume $\{\|x_n-p\|\}$ is nonincreasing sequence, then $\{\|x_n-p\|\}$ is convergent. Clearly

$$\lim_{n\to\infty}(\|x_n-p\|^2-\|x_{n+1}-p\|^2)=0.$$

Observe also that from the Algorithm 3.3.

$$
\begin{aligned}
\|w_n-x_n\| &= \theta_n\|x_n-x_{n-1}\| \\
&= \beta_n\frac{\theta_n}{\beta_n}\|x_n-x_{n-1}\|\to 0. \qquad (4.8)
\end{aligned}
$$

It follows from (4.8) that

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \qquad (4.9)$$

From the Algorithm 3.3, we know that

$$\gamma_n\|A^*(J^{B_2}-I)Aw_n\|^2\leq\|(J^{B_2}-I)Aw_n\|^2-\epsilon\|A^*(J^{B_2}-I)Aw_n\|^2. \qquad (4.10)$$

Next, we show that $\lim_{n\to\infty}\|A^*(J^{B_2}-I)Aw_n\|^2=0$.
For all $p\in\Phi$, using Assumption 3.2(2),we estimate that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)-\beta_np\|^2\\
&\le \|(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)\|^2+\beta_n^2\|p\|^2\\
&\quad -2\beta_n\langle(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p),p\rangle\\
&\le \|(1-\theta_n-\beta_n)(w_n-p)+\theta_n(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)\|^2+\beta_n^2\|p\|^2\\
&\le (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2\\
&\quad +2\theta_n(1-\theta_n-\beta_n)\langle w_n-p,\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\rangle+\beta_n^2\|p\|^2\\
&\le (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2\\
&\quad +2\theta_n(1-\theta_n-\beta_n)\|w_n-p\|.\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|+\beta_n^2\|p\|^2\\
&\le (1-\theta_n-\beta_n)^2\|w_n-p\|^2+\theta_n^2\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2\\
&\quad +\theta_n(1-\theta_n-\beta_n)[\|w_n-p\|^2+\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2]+\beta_n^2\|p\|^2\\
&= [(1-\theta_n-\beta_n)^2+\theta_n(1-\theta_n-\beta_n)]\|w_n-p\|^2+[\theta_n^2+(1-\theta_n-\beta_n)]\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2\\
&\quad +\beta_n^2\|p\|^2\\
&= (1-\beta_n)(1-\theta_n-\beta_n)\|w_n-p\|^2+\theta_n(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2+\beta_n^2\|p\|^2\\
&\le (1-\beta_n)\|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2+\beta_n^2\|p\|^2\\
&= (1-\beta_n)\|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(s)p\|^2+\beta_n^2\|p\|^2\\
&\le (1-\beta_n)\|w_n-p\|^2+(1-\beta_n)\|u_n-p\|^2+\beta_n^2\|p\|^2\\
&\le \|w_n-p\|^2+\|u_n-p\|^2+\beta_n^2\|p\|^2. \qquad (4.11)
\end{aligned}$$

From (4.2),(4.5),(4.10) and (4.11), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \|w_n-p\|^2+\|w_n-p\|^2+\gamma_n[\gamma_n\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2-\|(J_\lambda^{B_2}-I)Aw_n\|^2]+\beta_n^2\|p\|^2\\
&\le 2\|w_n-p\|^2-\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2+\beta_n^2\|p\|^2\\
&= 2[\|x_n-p\|^2+2\beta_nM\|x_n-p\|+\beta_n^2M^2]-\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2+\beta_n^2\|p\|^2\\
&= 2\|x_n-p\|^2+4\beta_nM\|x_n-p\|+2\beta_n^2M^2-\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2+\beta_n^2\|p\|^2\\
&= \|x_n-p\|^2+\frac{\beta_n}{\beta_n}\|x_n-p\|^2+4\beta_nM\|x_n-p\|+2\beta_n^2M^2-\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2+\beta_n^2\|p\|^2\\
&= \|x_n-p\|^2+\beta_n[\frac{\|x_n-p\|}{\beta_n}+4M\|x_n-p\|+2\beta_nM^2+\beta_n\|p\|]\\
&\quad -\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2. \qquad (4.12)
\end{aligned}$$

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Now, using the condition of β_n in Assumption 3.2(4) and (4.12), we obtain

$$\gamma_n\epsilon\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n[\frac{\|x_n-p\|}{\beta_n}+4M\|x_n-p\|+2\beta_nM^2+\beta_n\|p\|]. \tag{4.13}$$

Since $\{\|x_n-p\|\}$ is convergent, we get from (4.13) that

$$\lim_{n\to\infty}\|A^*(J_\lambda^{B_2}-I)Aw_n\|^2=0. \tag{4.14}$$

It follows from (4.10) and (4.14) that

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Aw_n\|^2=0. \tag{4.15}$$

Again, since $J_\lambda^{B_1}$ is nonexpansive, then, it is firmly nonexpansive. Using Algorithm 3.3, we get

$$\begin{aligned}
\|u_n-p\|^2 &= \|J_\lambda^{B_1}(w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n-J_\lambda^{B_1}(p)\|^2\\
&\leq \langle J_\lambda^{B_1}(w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n-J_\lambda^{B_1}(p),w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\
&= \langle u_n-p,w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\
&= \frac{1}{2}(\|u_n-p\|^2+\|w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n-p\|^2-\|u_n-(w_n+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n)\|^2)\\
&= \frac{1}{2}(\|u_n-p\|^2+\|w_n-p+\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n\|^2-\|u_n-w_n-\gamma_nA^*(J_\lambda^{B_2}-I)Ax_n)\|^2)\\
&= \frac{1}{2}(\|u_n-p\|^2+\|w_n-p\|^2+\gamma_n^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2+2\gamma_n\langle w_n-p,A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\quad -\|u_n-w_n\|^2-\gamma_n^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2+2\gamma_n\langle u_n-w_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle)\\
&= \frac{1}{2}(\|u_n-p\|^2+\|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\langle w_n-p,A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\quad +2\gamma_n\langle u_n-w_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle.
\end{aligned} \tag{4.16}$$

Hence,
$\|u_n-w_n\|^2$

$$\begin{aligned}
&\leq \|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\langle w_n-p,A^*(J_\lambda^{B_2}-I)Ax_n\rangle+2\gamma_n\langle u_n-w_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\leq \|w_n-p\|^2-\|u_n-p\|^2+2\gamma_n\langle w_n-p,A^*(J_\lambda^{B_2}-I)Ax_n\rangle+2\gamma_n\langle u_n-w_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\leq 2\gamma_n\langle w_n-p,A^*(J_\lambda^{B_2}-I)Ax_n\rangle+2\gamma_n\langle u_n-w_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle\\
&\leq 2\gamma_n(\|w_n-p\|\|A^*(J_\lambda^{B_2}-I)Ax_n\|+\|u_n-w_n\|\|A^*(J_\lambda^{B_2}-I)Ax_n\|).
\end{aligned} \tag{4.17}$$

Therefore, using (4.14) in (4.17) yields that

$$\lim_{n\to\infty}\|u_n-w_n\|=0. \tag{4.18}$$

It follows from (4.9) and (4.18) that

$$\|u_n-x_n\|\leq\|u_n-w_n\|+\|w_n-x_n\|. \tag{4.19}$$

Taking limit in (4.19) gives that

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{4.20}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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We know from (4.11) that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\beta_n)\|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p)\|^2+\beta_n^2\|p\|^2\\
&\leq \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2+\beta_n^2\|p\|^2\\
&= \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n+u_n-p\|^2+\beta_n^2\|p\|^2\\
&= \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+(1-\beta_n)\|u_n-p\|^2\\
&\quad +2(1-\beta_n)\langle u_n-p,\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\rangle+\beta_n^2\|p\|^2\\
&= \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+(1-\beta_n)\|u_n-p\|^2\\
&\quad -2(1-\beta_n)\langle p-u_n,\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\rangle+\beta_n^2\|p\|^2\\
&\leq \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+(1-\beta_n)\|u_n-p\|^2+\beta_n^2\|p\|^2\\
&\leq \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+(1-\beta_n)\|w_n-p\|^2+\beta_n^2\|p\|^2\\
&= \|w_n-p\|^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+\beta_n^2\|p\|^2. \qquad (4.21)
\end{aligned}
$$

Now, using (4.2) and (4.21), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (\|x_n-p\|^2+\beta_nM)^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2+\beta_n^2\|p\|^2\\
&= \|x_n-p\|^2+2\beta_nM\|x_n-p\|^2+\beta_n^2M^2+(1-\beta_n)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2\\
&\quad +\beta_n^2\|p\|^2.
\end{aligned}
\qquad (4.22)
$$

Therefore, (4.22) implies that

$$
\begin{aligned}
(\beta_n-1)\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad +\beta_n[2M\|x_n-p\|^2+\beta_n(M^2+\|p\|^2)]. \qquad (4.23)
\end{aligned}
$$

Invoking the condition on β_n of Assumption 3.2(4), we get

$$
\lim_{n\to\infty}\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-u_n\|=0. \qquad (4.24)
$$

Observe that

$$
\begin{aligned}
\|u_n-T(u)u_n\| &\leq \|u_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|+\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(u)\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|\\
&\quad +\|T(u)\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(u)u_n\|\\
&\leq 2\|u_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|+\|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-T(u)\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|.
\end{aligned}
\qquad (4.25)
$$

Applying Lemma 2.2 in (4.25) above, we obtain

$$\lim_{n\to\infty}\|u_n - T(u)u_n\| = 0. \tag{4.26}$$

We further estimate that

$$\|x_{n+1} - u_n\| \quad \leq \quad (1-\theta_n-\beta_n)\|w_n - u_n\| + \theta_n\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - u_n\| \tag{4.27}$$

It follows from (4.18),(4.24) and (4.27) that

$$\lim_{n\to\infty}\|x_{n+1} - u_n\| = 0. \tag{4.28}$$

Furthermore, using (4.18) and (4.9), we get

$$\|u_n - x_n\| \quad \leq \quad \|u_n - w_n\| + \|w_n - x_n\|. \tag{4.29}$$

Now, taking limit in (4.29), we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{4.30}$$

To show that x_n is asymptotically regular, we use the estimates (4.28), (4.29) and obtain

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \to 0. \tag{4.31}$$

It follows from (4.31) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{4.32}$$

And,

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| \leq \lim_{n\to\infty}(\|u_{n+1} - x_{n+1}\| + \|x_{n+1} - u_n\|) = 0. \tag{4.33}$$

Step III: We prove that $q \in \Phi$.
Since $\{w_n\}$ and $\{u_n\}$ are bounded sequences in H_1, let q be a weak cluster point. Without loss of generality, we may assume that the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ weakly converges to a point q. We obtained from (4.18) that $\{u_{n_j}\}$ of $\{u_n\}$ converges weakly to q. Hence, from the Algorithm 3.3, the sequence $u_{n_j} = J_\lambda^{B_1}(w_{n_j} + \tau_{n_j}A^*(J_\lambda^{B_2} - I)Aw_{n_j})$ can be rewritten as:

$$(\frac{w_{n_j} - u_{n_j} + \tau_{n_j}A^*(J_\lambda^{B_2} - I)Aw_{n_j}}{\lambda} \in B_1 u_{n_j}. \tag{4.34}$$

Taking $\lim_{j\to\infty}$ in (4.34), using the estimates in (4.14) and (4.18), together with the fact that the graph of a maximal monotone operator is weakly strongly closed, we conclude that $0 \in B_1(q)$ which further implies that $q \in SOLVIP(B_1)$. Also, using the asymptotical behaviour of the sequences $\{x_n\}$ and $\{u_n\}$, we get that $\{Ax_{n_j}\}$ weakly converges to Aq. Furthrmore, using Lemma 2.3, the nonexpansiveness of the resolvent operator J^{B_2} and the estimate in (4.14), we deduce that $Aq \in B_2(Aq)$; that is, $Aq \in SOLVIP(B_2)$. Therefore, we conclude that $q \in \Omega$.

Step IV: We prove that $\{x_n\}$ strongly converges to the solution set, Φ.
we first show that $q \in F(\Gamma)$. Suppose for contradiction $T(u)q \neq q$. It follows from Opial condition, Lemma 2.4 and (4.26) that

$$\begin{aligned}
\liminf_{j\to\infty}\|u_{n_j} - q\| &\leq \liminf_{j\to\infty}\|u_{n_j} - T(u)q\| \\
&\leq \liminf_{j\to\infty}\{\|u_{n_j} - T(u)u_{n_j}\| + \|T(u)u_{n_j} - T(u)q\|\} \\
&\leq \liminf_{j\to\infty}\{\|u_{n_j} - T(u)u_{n_j}\| + \|u_{n_j} - q\|\} \\
&\leq \liminf_{j\to\infty}\|u_{n_j} - q\|. \qquad (4.35)
\end{aligned}$$

International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
https://doi.org/10.6084/m9.figshare.22347139

This is a contracdiction. Hence, $q \in F(\Gamma)$ and it follows that $q \in \Phi$. From Lemma 2.5, we get that $x_n \rightharpoonup q$. Also, $u_n \rightharpoonup q$ and $q \in \Phi$.

Next: See that From the estimates (4.15) and (4.24), we know that

$$\lim_{n\to\infty} \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - w_n\| \leq \lim_{n\to\infty} (\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - u_n\| + \|u_n - w_n\|) \to 0. \tag{4.36}$$

Using this fact in (4.36), we obtain that

$$\begin{aligned}
&\|(1-\theta_n)w_n + \theta_n \tfrac{1}{t_n}\textstyle\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&= \|(1-\theta_n)(w_n - p) + \theta_n(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p)\|^2 \\
&= (1-\theta_n)^2\|w_n - p\|^2 + \theta_n^2\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&\quad +2\theta_n(1-\theta_n)\langle w_n - p, \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\rangle \\
&\leq (1-\theta_n)^2\|w_n - p\|^2 + \theta_n^2\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&\quad +2\theta_n(1-\theta_n)\|w_n - p\|.\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\| \\
&\leq (1-\theta_n)^2\|w_n - p\|^2 + \theta_n^2\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&\quad +\theta_n(1-\theta_n)\|w_n - p\|^2 \\
&\quad +\theta_n(1-\theta_n)\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&= [(1-\theta_n)^2 + \theta_n(1-\theta_n)]\|w_n - p\|^2 \\
&\quad +[\theta_n^2 + \theta_n(1-\theta_n)]\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - p\|^2 \\
&= (1-\theta_n)\|w_n - p\|^2 + \theta_n\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(s)p\|^2 \\
&\leq (1-\theta_n)\|w_n - p\|^2 + \theta_n[\|u_n - p\|^2 \\
&\leq (1-\theta_n)\|w_n - p\|^2 + \theta_n\|w_n - p\|^2 \\
&= \|w_n - p\|^2 \\
&\leq \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle x_n - p, x_n - x_{n-1}\rangle \\
&\leq \|x_n - p\|^2 + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\alpha_n\|x_n - p\|.\|x_n - x_{n-1}\| \\
&= \|x_n - p\|^2 + \alpha_n\|x_n - x_{n-1}\|\big[2\|x_n - p\| + \alpha_n\|x_n - x_{n-1}\|\big] \\
&\leq \|x_n - p\|^2 + 3\alpha_n\|x_n - x_{n-1}\|M_3
\end{aligned} \tag{4.37}$$

where $M_3 = sup\{\|x_n - p\|, \|x_n - x_{n-1}\|\}$

Further more, using (4.37), (4.36), Lemma 2.1(b) and the Algorithm, we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&= \|(1-\beta_n)[(1-\theta_n)w_n+\theta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p]-[\beta_n\theta_n(w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds)+\beta_np]\|^2 \\
&\leq (1-\beta_n)^2\|(1-\theta_n)w_n+\theta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2-2\langle\beta_n\theta_n(w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds)+\beta_np,x_{n+1}-p\rangle \\
&\leq (1-\beta_n)\|(1-\theta_n)w_n+\theta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2+2\langle\beta_n\theta_n(w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds),p-x_{n+1}\rangle \\
&\quad +2\beta_n\langle p,p-x_{n+1}\rangle \\
&\leq (1-\beta_n)\|(1-\theta_n)w_n+\theta_n\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-p\|^2+2\beta_n\theta_n\|w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|.\|x_{n+1}-p\| \\
&\quad +2\beta_n\langle p,p-x_{n+1}\rangle \\
&\leq (1-\beta_n)[\|x_n-p\|^2+3\alpha_n\|x_n-x_{n-1}\|M_3]+2\beta_n\theta_n\|w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|.\|x_{n+1}-p\| \\
&+ 2\beta_n\langle p,p-x_{n+1}\rangle \\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n(3\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|M_3+2\theta_n\|w_n-\frac{1}{t_n}\int_0^{t_n}T(s)u_nds\|.\|x_{n+1}-p\| \\
&\quad +2\langle p,p-x_{n+1}\rangle) \\
&= (1-\beta_n)\|x_n-p\|^2+\beta_nd_n
\end{aligned} \tag{4.38}$$

where $d_n=(3\frac{\alpha_n}{\beta_n}\|x_n-x_{n-1}\|M_3+2\theta_n\|y_n-Sy_n\|.\|x_{n+1}-p\|+2\langle p,p-x_{n+1}\rangle)$.
Since the $\{x_n\}$ is bounded, thus, there exists a subsequnce $\{x_{nj}\}$ of $\{x_n\}$ that weakly converges to a point $q\in H_1$ such that

$$\limsup_{j\to\infty}\langle p,p-x_{n_j}\rangle=\lim_{j\to\infty}\langle p,p-x_{n_j}\rangle=\langle p,p-q\rangle\leq 0. \tag{4.39}$$

It follows from (4.39) that

$$\limsup_{j\to\infty}\langle p,p-x_{n_j+1}\rangle=\langle p,p-q\rangle\leq 0. \tag{4.40}$$

The fact that $\limsup_{n\to\infty}d_n\leq 0$ follows from (4.9), (4.36) and (4.40). Therefore, we obtain from the concluding part of Lemma 2.7 that $\lim_{n\to\infty}\|x_n-p\|=0$. Hence, $\{x_n\}$ strongly converges to $p\in P_\Phi 0$.
Case 2: Suppose that $\{\|x_n-p\|\}$ is not monotone decreasing sequence. Denote $\Omega_n=\|x_n-p\|^2$ and let $\tau:N\to N$ be a mapping for all $n\geq n_0$(for sufficiently large n_0) defined by:

$$\tau(n):=\quad\max\{k\in N:k\leq n,\Omega_k\leq\Omega_{k+1}\}$$

. Then, it is easy to see that τ is a non-decreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and

$$\Omega_{\tau(n)}\leq\Omega_{\tau(n)+1},\ \text{for}\ \ n\geq n_o.$$

It follows from (4.22)and (4.23) that

$$\begin{aligned}
0 &\leq \|x_{\tau(n)}-p\|^2-\|x_{\tau(n)}-p\|^2 \\
&\leq \beta_{r_n}[2M\|x_{\tau_n}-p\|^2+\beta_{\tau_n}(M^2+\|p\|)]-(1-\beta_{\tau_n})\|\frac{1}{t_{\tau_n}}\int_0^{t_{\tau_n}}T(s)u_{\tau_n}ds-u_{\tau_n}\|^2.
\end{aligned} \tag{4.41}$$

This implies that

$$(\beta_{\tau(n)}-1)|\frac{1}{t_{\tau_n}}\int_0^{t_{\tau_n}}T(s)u_{\tau_n}ds-u_{\tau_n}\|^2\leq\beta_{r_n}[2M\|x_{\tau_n}-p\|^2+\beta_{\tau_n}(M^2+\|p\|)]\to 0.$$

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International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
https://doi.org/10.6084/m9.figshare.22347139

using the same argument as above (4.8) -(4.38), as in **Case 1** above, we deduce that $\{x_{\tau(n)}\}, \{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ are all weakly convergent to $p \in \Omega \cap F(\Gamma) = \Phi$. Now for all $n \geq n_0$,

$$\begin{aligned} 0 &\leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \\ &\leq \beta_{\tau(n)} M_1 + \beta_{\tau(n)}^2 \|p\|^2 + 2\beta_{\tau(n)} M_2 - \|x_{\tau(n)} - p\|^2 \\ &= \beta_{\tau(n)}[M_1 + \beta_{\tau(n)} + 2\beta_{\tau(n)}] - \|x_{\tau(n)} - p\|^2. \end{aligned} \tag{4.42}$$

Thus, using the condition on β_n, we get

$$\|x_{\tau(n)} - p\|^2 \leq \beta_{\tau(n)}[M_1 + \beta_{\tau(n)} + 2\beta_{\tau(n)}] \to 0. \tag{4.43}$$

Hence,

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\|^2 = 0.$$

It follows that

$$\lim_{n\to\infty} \Omega_{\tau(n)} = \lim_{n\to\infty} \Omega_{\tau(n)+1}$$

Furthermore, for $n \geq n_0$, we see that $\Omega_{\tau(n)} \leq \Omega_{\tau(n)+1}$ if $\tau(n) < n$,
Since, $\Omega_j \geq \Omega_{j+1}$ for $\tau(n+1) \leq j \leq n$. Consequently, $\forall n \geq n_0$,

$$0 \leq \Omega_n \leq \quad \max \quad \{\Omega_{\tau(n)}, \Omega_{\tau(n)+1}\} = \Omega_{\tau(n)+1}$$

Therefore,

$$\lim_{n\to\infty} \Omega_n = 0.$$

We conclude that $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to $p \in \Omega \cap F(T) \;\; \forall n \geq n_0$. $\square$

Corollay 4.1: Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be maximal monotone operators. Let $T : H_1 \to H_1$ be a nonexpansive mapping such that the solution set $\Phi \neq \emptyset$. If the Assumptions 3.1 and 3.2 are satisfied, then $\{x_n\}$ generated by the Algorithm 4.1 below strongly converges to a point $q \in \Phi \neq \emptyset$.

Algorithm 4.1

1: Step 0 Choose sequences $\{\beta_n\}, \{\theta_n\}$ and $\{\varepsilon_n\}$ such that the conditions from Assumption 3.2 hold and let $\lambda > 0, \alpha \geq 0$ and $x_0, x_1 \in H$ be chossen arbitrarily. Set $n := 1$.

Iterative Steps: Step 1: Given the iterate x_{n-1} and $x_n (n \geq 1)$, choose α_n such that $0 \leq \alpha_n \leq \overline{\alpha}_n$, where

$$\overline{\alpha}_n := \begin{cases} min\{\dfrac{n-1}{n+\alpha-1}, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\} & \text{if } \; x_n - x_{n+1} > 0, \\ \dfrac{n-1}{n+\alpha-1}, & otherwise. \end{cases} \tag{4.44}$$

Step 2: and compute

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ u_n = J_\lambda^{B_1}(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n), \\ x_{n+1} = (1 - \theta_n - \beta_n)w_n + \theta_n T u_n, \end{cases} \tag{4.45}$$

where

$$\gamma_n \in (\epsilon, \frac{\|(J_\lambda^{B_2} - I)Aw_n\|^2}{\|A^*(J_\lambda^{B_2} - I)Aw_n\|^2} - \epsilon \;),$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
https://doi.org/10.6084/m9.figshare.22347139

for some large enough $\epsilon > 0$.
Update: **Set** $n := n + 1$ and go to **Step 1.**

Proof. Proof: Clearly, the result follows from Theorem 4.1. □

5 Numerical examples

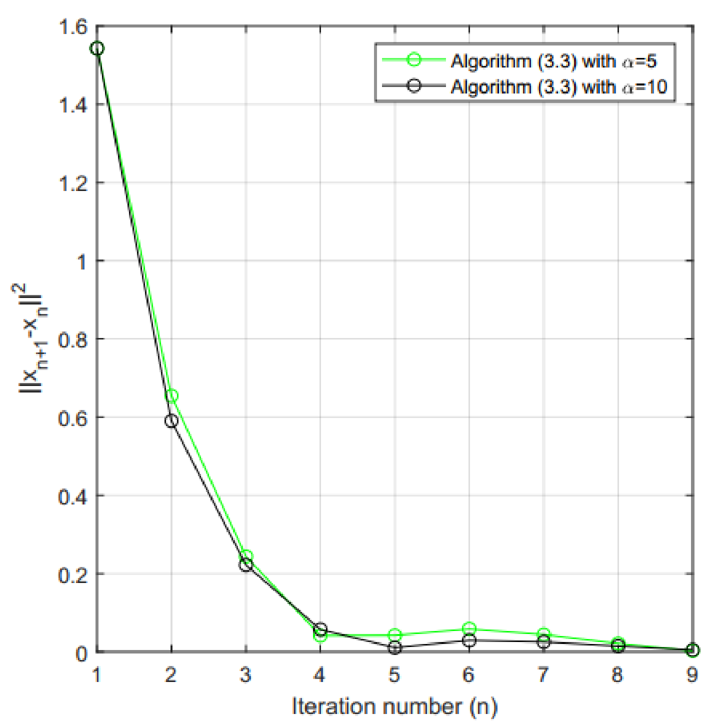
In this section, we provide some numerical examples to demonstrate the efficiency of our algorithm. All the codes were written in MATLAB R2019a. All the computations were performed on personal computer with Intel(R) Core (TM) i5-4300U CPU at 1.90Ghz 2.49GHz with 8.00 Gb-RAM and 64-OS. The performance is tested with the existing results (1.20) and (1.21). The stopping criterion is $\|x_{n+1} - I)x_n\| \leq 10^{-10}$ as in the case of [19, 20]. We shall assume that λ has a fixed value of $0.5, \alpha_n = \overline{\alpha}_n$ with different choices of $\alpha = 5, 10, \beta_n = \frac{1}{n^2}, \theta_n = 1 - \beta_n$ and $\varepsilon_n = \beta_n / n^{1/5n}$. For the sake of algorithms (1.20) and (1.21), let $\gamma = 1/2L$ where is the spectral radius of the operator A^*A, while in our algorithm, it has a variable stepsize γ_n that is generated at each iteration.
Example 5.1 [20]: Let $H_1 = H_2 = \mathbb{R}^2$, and let two operators of matrix multiplication $B_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $B_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $B_1(x) = T_1(x)$ and $B_2(x) = T_2(x,)$ where

$$T_1 = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix},$$

and

$$T_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}.$$

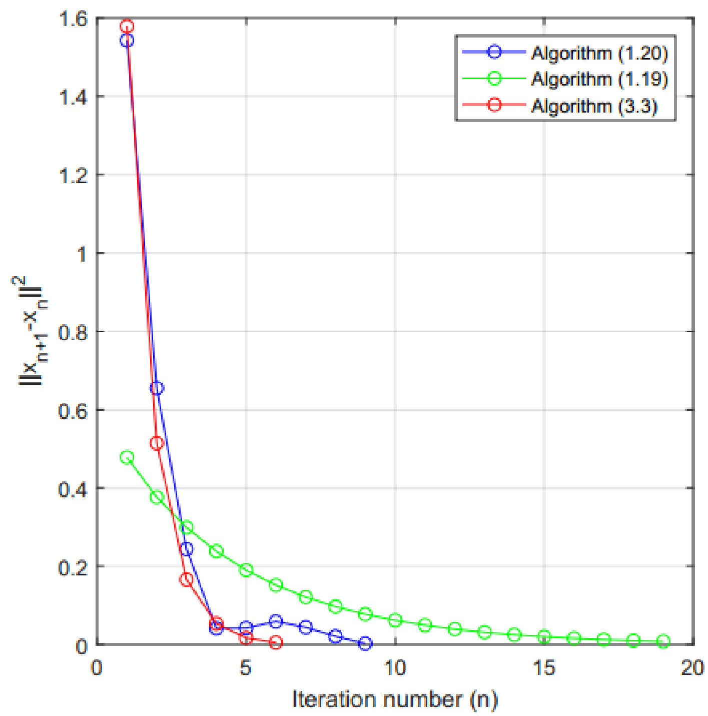
See that T_1 and T_2 are positive linear operators; then they are maximal monotones. Thus, we can define the resolvent mappings associated with maximal monotone as follows: $J_\lambda^{B_1}(I + \lambda B_1)^{-1}$ and $J_\lambda^{B_2}(I + \lambda B_2)^{-1}$, where $\lambda > 0$. Let $A \in \mathbb{R}^{2\times 2}$ be a nonsingular matrix operator in which the elements are randomly selected and let A^* be the adjoint of A.



0.PNG

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
https://doi.org/10.6084/m9.figshare.22347139

Figure 1



1.PNG

Figure 2

Interpretation of the result: In figure 1, we tested different choices of $\alpha = 5, 10$ and the rate of convergence does not show significant different in the choice of α. While in figure 2, we compare the rate of convergence of our algorithm [Algorithm 3.3] and that of Sitthithakerngkiet et al. [20] [Algorithm 1.20] and Kazmi and Rizvi [19] [Algorithm 1.19] and see that our result outperforms the mentioned results.

6 Conclusion

Irrespective of the vast applications of nonexpansive mapping, we have considered in this research, a more valuable mapping, the nonexpansive semigroup in real Hilbert space. A new inertial-based iterative algorithm for solving variational inclusion problem and fixed point problem is constructed. The algorithm improved the work of [18–20, 23, 24] among other results that have already been announced. To achieve this milestone, we carefully constructed an algorithm that the stepsize does not depend on the operator norm, a difficult task in applications; the strong convergence was obtained under some mild conditions. Furthermore, in order to establshed a faster convergence, we incorporated an inertial extrapolation technique in the spirit of Polyak [35] and have deduced from our numerical illustration that our algorithm is efficient and applicable.

7 Competing Interests

The authors declare that they have no competing interests.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 109 - 130
https://doi.org/10.6084/m9.figshare.22347139

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