International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 62 - 73
https://doi.org/10.5281/zenodo.10202633

Fourth Teoplitz Determinant for Analytic Function Defined by Gegenbauer Polynomial involving the Sine function

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Article Info
Received: 02 August 2023 Revised: 27 October 2023
Accepted: 28 October 2023 Available online: 15 November 2023

Abstract

In this work, a new class of analytic function was defined by the Gegenbauer polynomial involving the sine function. The initial coefficient estimates were obtained and the fourth Toeplitz determinants was presented.

Keywords: Analytic function, Sine function, Gegenbauer polynomial, Toeplitz determinant.
MSC2010: 30C45.

1 Introduction

Orthogonal polynomials were discovered by Legendre in 1784 [1]. Under specific model restrictions, orthogonal polynomials are frequently employed to discover solutions of ordinary differential equations. Moreover, orthogonal polynomials are a critical feature in approximation theory. Two polynomials Pn and Pm, of order n and m, respectively, are orthogonal if

$$\langle P_n,\ P_m \rangle = \int_c^d P_n(x)P_m(x)r(x)dx = 0$$

for $n \neq m$ where $r(x)$ is non-negative function in the interval (c, d); therefore, all finite order polynomials $P_n(x)$ have well-defined integral. An example of an orthogonal polynomial is a Gegenbauer polynomial (GP) [2]. Several authors have carried out research on the Gegenbauer polynomial, see [3], [4], [5], [6], [7], [8] and [9].
Many researchers have studied several Hankel and Toeplitz determinants for various classes of functions. For example, Janteng et al. [10] investigated second Hankel determinant for a function with a positive real part and starlike and convex functions, respectively; Bansal [11], Lee et al. [12] and Shaharuddin et al [13] discussed the second Hankel determinant for certain analytic functions; Zaprawa [14], Zhang et al. [15] and Babalola [16] derived third-order Hankel determinant for certain different univalent functions; Raza and Malik. [17] and Shi et al. [18], [19], and Breaz et al [20], studied upper bounds of the third Hankel determinant for some classes of analytic functions related

to lemniscate of Bernoulli, cardioid domain and exponential function; Mahmood et al. [21] found third Hankel determinant for a subclass of q-starlike functions. Following the above work, Zhang et al. [22] recently considered fourth-order Hankel determinants of starlike functions related to the sine function. On the other hand, Ramachandran and Kavitha [23] and Ali et al. [24] studied Toeplitz matrices whose elements are the coefficients of starlike, close-to-convex, and univalent functions. Besides, Tang et al., [25] studied third-order Hankel and Toeplitz determinant for a subclass of multivalent q-starlike functions of order ; Zhang et al. [26] considered third-order Hankel and Toeplitz determinants of starlike functions, which are defined by using the sine function; Ramachandran et al. [27] derived an estimation for the Hankel and Topelitz determinant with domains bounded by conical sections involving Ruscheweygh derivative; Srivastava et al. [28] found the Hankel determinant and the Toeplitz matrices for a newly defined class of analytic q-starlike functions.
Motivated by the work of Al-Hawary et al [2], Al-Shbeil et al [8], Olatunji et al [29] and Zhang and Tang [30], it is established that Gegenbauer polynomial also promotes the advancement of geometric function theory. In this paper, we aim to investigate the second, third and fourth-order Toeplitz determinant for this function class $K_{\mu,s}$ associated with sine function and obtain the upper bounds for the determinants.

2 Preliminaries

Let $\mathbb{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2.1}$$

in the unit disk $\mathbb{D}$, $\mathbb{D} = (z \in \mathbb{C} : |z| \leq 1)$.
which are analytic in the unit disc $\mathbb{D}$ with conditions $f(0) = f'(0) - 1 = 0$. Recall that, S is representing a univalent function with some of the above conditions. With simple modificaton and differentiation, various subclasses of $\mathbb{A}$ are known such as starlike function, convex function, close-to-convex just to mention but a few with representations below 1001[31].
[32] A function $f(z) \in \mathbb{A}$ is said to be starlike if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Denote this class by $\mathcal{S}^*$.

[32] A function $f(z) \in \mathbb{A}$ is said to be convex if it satisfies the condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Denote this class by $\mathcal{C}$.

[32] A function $f(z) \in \mathbb{A}$ is called close-to-convex, if there exist a convex function ϕ such that

$$\Re\left(\frac{f'(z)}{\phi'(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Kaplan's definition does not require that the function ϕ is normalized, but since the majority of results obtained for close-to-convex functions assume this, we will suppose that ϕ so that $\phi(0) = 0$ and $\phi'(0) = 1$.

1001[32],1001[33],1001[34] and 1001[35]. A function $f(z) \in \mathbb{A}$ is called close-to-convex, if there exist a function $g \in \mathcal{S}^*$ such that

$$\Re\left(\frac{f'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

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Denote this class by $\mathcal{K}_\circ$. Choosing $g(z) = f(z)$, it is clear that $\mathcal{S}^* \subset \mathcal{K}_\circ$ and so $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{K}_\circ$. [32] Let ρ be analytic in $\mathbb{D}$, with $p(0) = 1$. Denote by P the class of functions ρ with Taylor series expansion

$$\rho(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \tag{2.2}$$

satisfying

$$\Re(\rho(z)) > 0 \quad (z \in \mathbb{D}).$$

(Derek *et al.*, 2018). Functions in P are referred to as functions with positive real parts in $\mathbb{D}$ or Caratheodory functions. [32] For two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$ and write

$$f(z) \prec g(z) \tag{2.3}$$

$(z \in \mathbb{U})$ if there exists a Schwartz function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$ such that

$$f(z) = g(w(z))$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For non-zero real constant λ, a generating function of Gegenbauer polynomials.

$$\kappa_\lambda(m, z) = \frac{1}{(1 - 2mz + z^2)^\lambda}, \tag{2.4}$$

where m $\in$ [-1,1] and z $\in U$. For fixed m the function $\kappa_{\lambda,m}$ is analytic in U, so it can be expanded on a Taylor series as

$$\kappa_{\lambda,m} = \sum_{n=0}^{\infty} C_n^\lambda(m) z^n, \tag{2.5}$$

where $C_n^\lambda(m)$ is Gegenbauer polynomial of degree n.

Obviously κ_λ generates nothing when $\lambda = 0$. Therefore, the generating function of the Gegenbauer polynomial is set to be

$$\kappa_0(m, z) = 1 - log(1 - 2mz + z^2) = \sum_{n=0}^{\infty} C_n^0(m) z^n \tag{2.6}$$

for $\lambda = 0$ and Gegenbauer polynomials can also be defined by the following recurrence relations:

$$C_n^\lambda(m) = \frac{1}{n}[2m(n + \lambda - 1)C_{n-1}^\lambda(m) - (n + 2\lambda - 2)C_{n-1}^\lambda(m)] \tag{2.7}$$

with initial values

$$C_0^\lambda(m) = 1, C_1^\lambda(m) = 2\lambda m, \quad C_2^\lambda(m) = 2\lambda(1 + \lambda)m^2 - \lambda$$
$$\text{and} \quad C_3^\lambda(m) = \frac{4\lambda(\lambda + 1)(\lambda + 2)}{3} m^2 - 2\lambda(\lambda + 1)m \tag{2.8}$$

see details in [36]

Szynal [37] introduced the class $T(\lambda)$ as a subclass consisting of functions of the form

$$c(z) = \int_{-1}^{1} k(z, m) d\sigma(m) \tag{2.9}$$

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where

$$k(z,m) = z + \sum_{n=2}^{\infty} c_{n-1}^{\lambda}(m) z^n \tag{2.10}$$

Let

$$\zeta_{\lambda,m} f(z) = k(z,m) * f = z + \sum_{n=2}^{\infty} c_{n-1}^{\lambda}(m) a_n z^n \tag{2.11}$$

(2.11) denotes the Hadarmard product of (2.1) and (2.10).

To prove our desired results, we require the following lemmas and definitions.

Lemma 2.1. *[30]. If $p(z) \in P$ from 2.2, then $|d_n| \leq 2, n = 1, 2, \ldots.$*

Lemma 2.2. *[30] Let $p(z) \in P$, then $2d_2 = d_1^2 + (4 - d_1^2)\xi$. and $4d_3 = d_1^3 + 2d_1(4 - d_1^2)\xi - d_1(4 - d_1^2)\xi^2 + 2(4 - d_1^2)(1 - |\xi|^2)\eta$. for some ξ, η satisfying $|\xi| \leq 1, |\xi| \leq 1$ and $d_1 \in [0, 2]$.*

Lemma 2.3. *[30] If $p(z) \in P$, then*

$$|d_2 - \frac{d_1^2}{2}| \leq 2 - \frac{|d_1^2|}{2}, \tag{2.12}$$

$$|d_{n+k} - \mu d_n d_k| < 2, 0 \leq \mu \leq 1 \tag{2.13}$$

$$|d_{n+2k} - \mu d_n d_k^2| \leq 2(1 + 2\mu). \tag{2.14}$$

Lemma 2.4. *[36]. If $g(z) \in S^*$, then $|b_n| \leq n, n \geq 2$.*

[32] Let ρ be analytic in $\mathbb{D}$, with $p(0) = 1$. Denote by P the class of functions ρ with Taylor series expansion

$$\rho(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \tag{2.15}$$

satisfying

$$\Re(\rho(z)) > 0 \quad (z \in \mathbb{D}).$$

(Derek *et al.*, 2018).

Now we define the following new subclass of $K_{\mu,s}$ as follows [30]. Let $\zeta_{\lambda,m} \in K_{\mu,s}$, if $\zeta_{\lambda,m} \in A$ and there exists $g(z) \in S^*$ such that

$$\left| \frac{z(\zeta_{\lambda,m} f)'(z)}{g(z)} - 1 \right| \prec \sin z$$

where $\lambda \geq 0, m \in [1, -1]$ and $K_{\mu,s}$ denotes the natural close- to-convex analogue of S_{μ}^*. Note that s denotes the Sine function.

3 Main Results

Theorem 3.1. *If $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1, -1]$, then*

$$|a_2| \leq \frac{3}{2|c_1^{\lambda}(m)|}$$

$$|a_3| \leq \frac{5}{3|c_{2|}^{\lambda}(m)|}$$

$$|a_4| \leq \frac{53}{24|c_3^{\lambda}(m)|}$$

$$|a_5| \leq \frac{92}{15|c_4^{\lambda}|(m)}$$

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Proof: From Definition 2.8, $g(z) \in S^*$ and according to subordination relationship, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$\left|\frac{z(\zeta_{\lambda,m}f)'(z)}{g(z)} - 1\right| \prec \sin z$$

$$z(\zeta_{\lambda,m}f)'(z) = g(z)(1 + \sin\omega(z))$$

$$\zeta_{\lambda,m}f(z) = z + \sum_{n=2}^{\infty} c_{n-1}^{\lambda}(m)a_n z^n$$

$$\zeta_{\lambda,m}f'(z) = 1 + \sum_{n=2}^{\infty} nc_{n-1}^{\lambda}(m)a_n z^{n-1}$$

$$z\zeta_{\lambda,m}f'(z) = z + \sum_{n=2}^{\infty} nc_{n-1}^{\lambda}(m)a_n z^n \tag{3.1}$$

Now, if $g(z) \in S^*$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{3.2}$$

$$\exists \;\; p(z) = \frac{1+\omega}{1-\omega} = 1 + d_1 z + d_2 z^2 + \text{...such that } \; p(z) \in P$$

and

$$\omega(z) = \frac{p(z) - 1}{1 + p(z)} = \frac{d_1 z + d_2 z^2 + ...}{2 + d_1 z + d_2 z^2 + ...}$$

$$\Rightarrow \omega(z) = \frac{d_1}{2}z + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)z^2 + \left(\frac{d_1^3}{8} - \frac{d_1 d_2}{2} + \frac{d_3}{2}\right)z^3 + \left(\frac{3d_1^2 d_2}{8} - \frac{d_1 d_3}{2} - \frac{d_1^4}{16} - \frac{d_2^2}{4} + \frac{d_4}{2}\right)z^4 + ...$$

$$(\omega(z))^3 = \left(\frac{1}{48}d_1^3\right)z^3 + \left(\frac{d_2 d_1^2}{16} - \frac{d_1^4}{32}\right)z^4 + \left(\frac{3d_3 d_1^2}{8} - \frac{3d_2 d_1^3}{4} + \frac{d_1 d_2^2}{4} + \frac{3d_1^5}{16}\right)z^5...$$

$$(\omega(z))^5 = \frac{d_1^5}{32}z^5$$

$$\sin\omega(z) = \omega(z) - \frac{\omega(z)^3}{3!} + \frac{\omega(z)^5}{5!} + ...$$

Substituting for $\omega(z)$ in $\sin\omega(z)$,

$$\sin\omega(z) = \frac{d_1}{2}z + \left(\frac{d_2}{2} - \frac{d_1^2}{4}\right)z^2 + \left(\frac{5d_1^3}{48} - \frac{d_1 d_2}{2} + \frac{d_3}{2}\right)z^3 + \left(\frac{5d_1^2 d_2}{16} - \frac{d_1 d_3}{2} - \frac{3d_1^4}{32} - \frac{d_2^2}{4} + \frac{d_4}{2}\right)z^4 + \left(\frac{5d_1^2 d_3}{16} + \frac{5d_1 d_2^2}{16} - \frac{5d_1^3 d_2}{48} + \frac{61d_1^5}{3840} - \frac{d_2 d_3}{32} + \frac{d_5}{2} + \frac{d_1 d_4}{2}\right)z^5...$$

$$1+\sin\omega(z) = 1+\frac{d_1}{2}z+\left(\frac{d_2}{2}-\frac{d_1^2}{4}\right)z^2+\left(\frac{5d_1^3}{48}-\frac{d_1d_2}{2}+\frac{d_3}{2}\right)z^3+\left(\frac{5d_1^2d_2}{16}-\frac{d_1d_3}{2}-\frac{3d_1^4}{32}-\frac{d_2^2}{4}+\frac{d_4}{2}\right)z^4 + \left(\frac{5d_1^2d_3}{16}+\frac{5d_1d_2^2}{16}-\frac{5d_1^3d_2}{48}+\frac{61d_1^5}{3840}-\frac{d_2d_3}{32}+\frac{d_5}{2}+\frac{d_1d_4}{2}\right)z^5\ldots$$

Multiplied by $(g(z))$, we have

$$g(z)\left(1+\sin\omega(z)\right) = z+\left(\frac{d_1}{2}+b_2\right)z^2+\left(\frac{d_2}{2}-\frac{d_1^2}{4}+\frac{d_1b_2}{2}+b_3\right)z^3 + \left(\frac{5d_1^3}{48}-\frac{d_1d_2}{2}+\frac{d_3}{2}+\frac{d_2b_2}{2}-\frac{d_1^2b_2}{4}+\frac{d_1b_3}{2}+b_4\right)z^4+\ldots \tag{3.3}$$

Equating coefficients (3.1) and (3.3); we have that

$$\left(2c_1^\lambda(m)a_2\right)z^2 = \left(\frac{d_1}{2}+b_2\right)z^2 \tag{3.4}$$

$$\left(3c_2^\lambda(m)a_3\right)z^3 = \left(\frac{d_2}{2}-\frac{d_1^2}{4}+\frac{d_1b_2}{2}+b_3\right)z^3 \tag{3.5}$$

$$\left(4c_3^\lambda(m)a_4\right)z^4 = \left(\frac{5d_1^3}{48}-\frac{d_1d_2}{2}+\frac{d_3}{2}+\frac{d_2b_2}{2}-\frac{d_1^2b_2}{4}+\frac{d_1b_3}{2}+b_4\right)z^4 \tag{3.6}$$

$$\left(5c_4^\lambda(m)a_5\right)z^5 = \left(\frac{5d_1^2d_2}{16}+\frac{d_4}{2}-\frac{d_1d_3}{2}-\frac{3d_1^4}{32}-\frac{d_2^2}{4}+\frac{5d_1^3b_2}{48}-\frac{b_2d_1d_2}{2}+\frac{d_1b_4}{2}+\frac{b_2d_3}{2}+\frac{b_3d_2}{2}-\frac{b_3d_1^2}{4}+b_5\right)z^5 \tag{3.7}$$

From equation 3.4

$$a_2 = \frac{1}{2c_1^\lambda(m)}\left(\frac{d_1}{2}+b_2\right)$$
$$|a_2| \le \frac{1}{2|c_1^\lambda(m)|}\left|\frac{d_1}{2}+b_2\right| \tag{3.8}$$
$$|a_2| \le \left|\frac{d_1}{4|c_1^\lambda(m)|}+\frac{b_2}{2|c_1^\lambda(m)|}\right|$$

From equation 3.5

$$a_3 = \frac{1}{3c_2^\lambda(m)}\left(\frac{d_2}{2}-\frac{d_1^2}{4}+\frac{d_1b_2}{2}+b_3\right)$$
$$a_3 = \frac{1}{3c_2^\lambda(m)}\frac{1}{2}\left(d_2-\frac{d_1^2}{2}+\frac{d_1b_2}{2}+b_3\right)$$
$$|a_3| \le \left|\frac{1}{3c_2^\lambda(m)}\right|\left|\frac{1}{2}(d_2-\frac{d_1^2}{2})+\frac{d_1b_2}{2}+b_3\right|$$
$$|a_3| \le \frac{1}{3|c_2^\lambda(m)|}\frac{1}{2}\left|d_2-\frac{d_1^2}{2}\right|+\frac{|d_1||b_2|}{2}+|b_3| \tag{3.9}$$

From (3.6)

$$a_4 = \frac{1}{4c_3^\lambda(m)}\left(\frac{5d_1^3}{48} - \frac{d_1d_2}{2} + \frac{d_3}{2} + \frac{d_2b_2}{2} - \frac{d_1^2b_2}{4} + \frac{d_1b_3}{2} + b_4\right)$$

$$a_4 = \frac{1}{4c_3^\lambda(m)}\left(\frac{5d_1^3}{48} + \frac{d_1b_3}{2} + \frac{1}{2}(d_3 - d_1d_2) + \frac{b_2}{2}(d_2 - \frac{d_1^2}{2}) + b_4\right)$$

$$|a_4| \leq \left|\frac{1}{4c_3^\lambda(m)}\right|\left|\frac{5d_1^3}{48} + \frac{d_1b_3}{2} + \frac{1}{2}(d_3 - d_1d_2) + \frac{b_2}{2}(d_2 - \frac{d_1^2}{2}) + b_4\right|$$

$$|a_4| \leq \frac{1}{4|c_3^\lambda(m)|}\left(\frac{5|d_1^3|}{48} + \frac{|d_1||b_3|}{2} + \frac{1}{2}|d_3 - d_1d_2| + \frac{|b_2|}{2}|d_2 - \frac{d_1^2}{2}| + |b_4|\right) \tag{3.10}$$

Immediately from (3.7)

$$a_5 = \frac{1}{5c_4^\lambda(m)}\left(\frac{5d_1^2d_2}{16} + \frac{d_4}{2} - \frac{d_1d_3}{2} - \frac{3d_1^4}{32} - \frac{d_2^2}{4} + \frac{5d_1^3b_2}{48} - \frac{b_2d_1d_2}{2} + \frac{d_1b_4}{2} + \frac{b_2d_3}{2} + \frac{b_3d_2}{2} - \frac{b_3d_1^2}{4} + b_5\right)$$

$$|a_5| \leq \left|\frac{1}{5c_4^\lambda(m)}\right|\left|\frac{5d_1^2d_2}{16} + \frac{d_4}{2} + \frac{5d_1^3b_2}{48} + b_5 + \frac{d_1b_4}{2} + \frac{b_2d_3}{2} + \frac{b_3d_2}{2} - \frac{d_1d_3}{2} - \frac{3d_1^4}{32} - \frac{d_2^2}{4} - \frac{b_2d_1d_2}{2} - \frac{b_3d_1^2}{4}\right|$$

$$|a_5| \leq \frac{1}{5|c_4^\lambda|(m)}\left|\frac{5d_1^2d_2}{16} + \frac{d_4}{2} + \frac{5d_1^3b_2}{48} + b_5 + \frac{d_1b_4}{2} + \frac{b_2d_3}{2} + \frac{b_3d_2}{2}\right| + |-d_1|\left|\frac{d_3}{2} + \frac{3d_1^3}{32} + \frac{b_2d_2}{2} + \frac{b_3d_1}{4}\right| + \left|-\frac{d_2^2}{4}\right|$$

$$|a_5| \leq \frac{1}{5|c_4^\lambda|(m)}\left[\frac{5|d_1^2||d_2|}{16} + \frac{|d_4|}{2} + \frac{5|d_1^3||b_2|}{48} + |b_5| + \frac{|d_1||b_4|}{2} + \frac{|b_2||d_3|}{2} + \frac{|b_3||d_2|}{2} + |d_1|\left(\frac{|d_3|}{2} + \frac{3|d_1^3|}{32} + \frac{|b_2||d_2|}{2} + \frac{|b_3||d_1|}{4}\right) + \frac{1}{4}|d_2^2|\right] \tag{3.11}$$

If $\zeta_{\lambda,m} \in C_n(z)$, then $|a_2| \leq \frac{3}{2c_1^\lambda(m)}$

Proof:
setting $n = 2$ in (3.1) yeilds (3.8) and applying lemmas 2.1 and 2.4

$$|a_2| \leq \frac{1}{2|c_1^\lambda(m)|}\left|\frac{d_1}{2} + b_2\right|$$

the proof follows. This is the result obtained by Olatunji [29].

If $\zeta_{\lambda,m} \in C_n(z)$, then

$$|a_3| \leq \frac{5}{3c_2^\lambda(m)}$$

Proof:
setting $n = 3$ in (3.1) yeilds (3.9) and applying lemmas 2.1, 2.3 and 2.4, we have

$$|a_3| \leq \frac{1}{3|c_2^\lambda(m)|}\frac{1}{2}\left|d_2 - \frac{d_1^2}{2}\right| + \frac{|d_1||b_2|}{2} + |b_3|$$

the proof follows. This is the result obtained by Olatunji [29].

If $\zeta_{\lambda,m} \in C_n(z)$, then

$$|a_4| \leq \frac{53}{24|c_3^\lambda(m)|}$$

Proof:
setting $n = 4$ in (3.1) yeilds (3.10) and applying lemmas 2.1, 2.3 and 2.4, we have

$$|a_4| \leq \frac{1}{4|c_3^\lambda(m)|}\left(\frac{5|d_1^3|}{48} + \frac{|d_1||b_3|}{2} + \frac{1}{2}|d_3 - d_1 d_2| + \frac{|b_2|}{2}|d_2 - \frac{d_1^2}{2}| + |b_4|\right)$$

hence the proof.

If $\zeta_{\lambda,m} \in C_n(z)$, then

$$|a_5| \leq \frac{92}{15|c_4^\lambda|(m)}$$

Proof:
setting $n = 5$ in (3.1) yeilds (3.11) and applying lemmas 2.1, 2.3 and 2.4, we have

$$|a_5| \leq \frac{1}{5|c_4^\lambda|(m)}\left[\frac{5|d_1^2||d_2|}{16} + \frac{|d_4|}{2} + \frac{5|d_1^3||b_2|}{48} + |b_5| + \frac{|d_1||b_4|}{2} + \frac{|b_2||d_3|}{2} + \frac{|b_3||d_2|}{2}\right.$$
$$\left. + |d_1|\left(\frac{|d_3|}{2} + \frac{3|d_1^3|}{32} + \frac{|b_2||d_2|}{2} + \frac{|b_3||d_1|}{4}\right) + \frac{1}{4}\left|d_2^2\right|\right]$$

the proof follows.

Toeplitz Determinants of $K_{\mu,s}$
The Toeplitz matrix is a matrix in which each descending diagonal from left to right is constant, this means

$$T_q(n) = \begin{pmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_n & a_{n+1} & \dots \\ . & . & \dots & . \\ . & . & \dots & a_{n+1} \\ a_{n+q-1} & \dots & a_{n+1} & a_n \end{pmatrix}$$

$n \leq 1, q \leq 1$

This matrix has computational properties and appearances in various areas, 1001[23],1001[38]. In this work, assume $a_1 = 1$, then the estimates for the Toeplitz determinant in the cases of $q = 2, n = 2$, $q = 3, n = 1$, $q = 3, n = 2$, $q = 4, n = 1$ and $q = 4, n = 2$ of the analytic function having entries from the class $K_{\mu,s}$ is presented in the following Theorems.

Theorem 3.2. *Let $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1, -1]$, and*

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}$$

then

$$\left|a_3^2 - a_2{}^2\right| \leq \frac{9}{4|c_1^{2\lambda}(m)|} + \frac{71}{18|c_2^{2\lambda}(m)|}$$

Proof: The proof follows from equations 3.8 and 3.9

Theorem 3.3. *Let $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1, -1]$, and*

$$T_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_2 \\ a_3 & a_2 & a_1 \end{vmatrix}$$

then,

$$\left|1 + 2a_2^2(a_3 - 1) - a_3{}^2\right| \leq 1 + \frac{1}{36|c_1^{2\lambda}c_2^\lambda(m)|} - \frac{5}{2|c_1^{2\lambda}(m)} - \frac{97}{18|c_2^{2\lambda}(m)|}$$

Proof: The proof follows from (3.8),(3.9),(3.10) and (3.11)

Theorem 3.4. *Let $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1,-1]$, and*

$$T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}$$

then,

$$\left|2a_3^2(a_4 - a_2) - a_2{a_4}^2 + a_2^3\right| \leq \frac{|A|^3}{8|c_1^3(m)|} + \frac{|B|^2|C|}{18|c_2^{2\lambda}|c_3^{\lambda}|(m)} + \frac{|A||B|^2}{9|c_1^{\lambda}||c_2^{2\lambda}(m)|} + \frac{|A||C|^2}{32|c_1^{\lambda}||c_3^{2\lambda}(m)|}$$

Proof: The proof follows.

Theorem 3.5. *Let $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1,-1]$, and*

$$T_4(1) = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_1 & a_2 & a_3 \\ a_3 & a_2 & a_1 & a_2 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}$$

then,

$$\begin{aligned} &\left|1 - 3a_2^2 + 2a_2^2a_3 - 2a_3^2 - 2a_2^2a_3^2 + 4a_2a_3a_4 - 2a_2a_3^2a_4 - 2a_2^3a_4 + a_2^4 + a_3^4 + a_4^2 + a_2^2a_4^2\right| \\ &\quad \leq 1 - \frac{3|A|^2}{4|c_1^2\lambda(m)|} + \frac{|A|^2|B|}{6|c_1^{2\lambda}|c_2^{\lambda}|(m)} + \frac{2|B|^2}{9|c_2^{2\lambda}|(m)} + \frac{|A|^2|B|^2}{36|c_1^{2\lambda}|c_2^2\lambda|(m)} + \frac{|A||B|^2|C|}{36|c_1^{\lambda}||c_2^2\lambda||c_3^{\lambda}|(m)} \\ &\qquad + \frac{|A|^3|C|}{16|c_1^3\lambda||c_3^{\lambda}|(m)} + \frac{|C|^2}{16|c_3^{2\lambda}|(m)} + \frac{|A|^2|C|^2}{64|c_1^{2\lambda}||c_3^2\lambda|(m)} \end{aligned}$$

Proof: The proof follows from (3.8),(3.9),(3.10) and (3.11).

Theorem 3.6. *Let $\zeta_{\lambda,m} \in K_{\mu,s}$ where $m \in [1,-1]$, and*

$$T_4(2) = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix}$$

then,

$$\begin{aligned} &\left|(a_2^2 - a_3^2)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5) - (a_2a_3 - a_3a_4)^2 + (a_4^2 - a_3a_5)^2 - (a_3a_4 - a_2a_5)^2\right| \\ &\leq \frac{|A|^4}{16|c_1^4\lambda(m)|} + \frac{|B|^4}{81|c_2^{4\lambda}|(m)} + \frac{|A||B||C||D|}{30|c_1^{\lambda}||c_2^{\lambda}||c_3^{\lambda}||c_4^{\lambda}|(m)} + \frac{|A||B|^2|C|}{18|c_1^{\lambda}||c_2^2\lambda||c_3^{\lambda}|(m)} + \frac{|C|^4}{256|c_3^{4\lambda}|(m)} + \\ &\frac{|B|^2|D|^2}{225|c_2^{2\lambda}|c_4^2\lambda|(m)} + \frac{|A|^2|B|^2}{12|c_1^2\lambda||c_2^2\lambda|(m)} + + \frac{|A|^2|C|^2}{32|c_1^2\lambda||c_3^2\lambda|(m)} + \frac{|A|^2|D|^2}{100|c_1^2\lambda||c_3^{2\lambda}|(m)} + \frac{|B|^2|C|^2}{72|c_2^{2\lambda}||c_3^2\lambda|(m)} + \\ &\frac{2|B|^3|D|}{135|c_2^{2\lambda}||c_4^{\lambda}|(m)} + \frac{|B||C|^2|D|}{120|c_2^{\lambda}||c_3^2\lambda||c_4^{\lambda}|(m)} \end{aligned}$$

Proof: The proof follows from Theorem 3.1

4 Acknowledgement

The authors sincerely thank the the editor and the anonymous reveiwers for their valuable comments and suggestions. Their insightful feedback significantly improved the clarity and quality of this manuscript.

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 62 - 73
https://doi.org/10.5281/zenodo.10202633

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