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FINANCIAL VALUATION OF EARLY RETIREMENT BENEFIT IN A DEFINED BENEFIT PENSION PLAN

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Abstract

American options can be exercised at any time the holder wishes to do so before its expiry date. It therefore, becomes important to know when this option should be exercised to maximize gain. The study considers one pricing formulation of American options, namely, the optimal stopping formulation as an equivalence of a free-boundary problem. An optimal stopping problem on perpetual American put option was formulated and its solutions found. The solutions were analysed systematically by applying matching value condition, smooth pasting condition, asset equilibrium condition and the boundary condition. We used the free-boundary approach to derive the solution. The result compares favourably with that of [2]). We recommend application of the model by individuals wanting to exercise early retirement option.

Keywords: Matching Value Condition, Smooth Pasting Condition, Continuation Region, Stopping Region

1 Introduction

There comes a time when employees consider starting a pension plan - either on the advice of a friend, a relative, own volition or by law (employer). The plan of choice may depend on various factors, such as the age and salary of the individual, number of years of expected employment, as well as options to retire early or late. Some people would want to continue to work as long as possible, some even alter their dates of birth so as to remain in employment longer than the stipulated period, known as maturity time, T. On the other hand, some people wish to retire at any earlier time $t,\ t < T$ before the maturity time, T. An individual wishes to consider early retirement alternative, but wants to know when it is optimal to retire so as to maximize the expected benefit of the pension plan. We shall formulate a mathematical model equation for the value, $V(S,t)$ at time, t and salary, S of the benefit (benefit of the pension plan) with early retirement provision as an optimal stopping problem.

The theory of optimal stopping is concerned with the problem of choosing a time to take a given action based on sequentially observed random variables in order to maximize an expected payoff (profits, gains, rewards, etc) or to minimize an expected cost (loss, waste, pain, punishment). Problems of this type are found in the area of statistics, where the action taken may be to test an hypothesis or to estimate a parameter; in financial mathematics where early retirement decisions in defined benefit pension plan are taken; in the area of operations research where the action may be to replace a machine, hire a secretary, or reorder stock, etc. In optimal stopping problems we are interested in finding the solution to the given problem such that the solution is the possible optimal payoff.

Pension plans generally fall into two categories: Defined Benefit (DB) plans and Defined Contribution (DC) plans. Both of these are common in countries such as Nigeria, Chile, Canada, the United State of America, the United Kingdom and Australia. A DB pension plan is a type of pension plan in which the benefits on retirement are predetermined by a formula based on the employee's salary history, age, duration of employment and some other personalized factors, rather than investment returns. This (DB) pension benefit is the amount that is typically based on the employee's number of years of service (at the time of retirement) and the salary (final or average salary) over an employment period. For instance, the employee may receive a fraction of the average salary for a certain number of years. In a defined benefit plan the company sponsoring the plan is in charge of the portfolio management and bears the investment risk. The benefits are usually distributed through life annuities. In a life annuity, employees receive equal periodic benefit payments (monthly, quarterly, etc.) for the rest of their lives. It is different from the DC plan where benefits are dependent on the performance of the portfolio in which the contributions are invested.

Early retirement means different things to different folks. Some people retire and then, to while away the time, take up a hobby that evolves into a business. Still, others spend time in activities that they simply love to do for which they may or may not get paid ([1]). Pension planning can be hard to understand at the best of times. If one is in any doubt about what to do, it's important to talk to an independent financial adviser before making any major decisions.

Options are financial tools that give the holder the right, but not the obligation, for a transaction of a certain asset at a given time for a given price. There are two different types of options namely, puts and calls options. American option and European option are two main styles of options. In particular, a *call option* allows its owner to buy and a *put option* to sell its *underlying asset* at a certain time (t) for a fixed *strike price (K)*. *European* options can only be exercised at one given *expiry,* or *maturity, date* (t = T), whereas *American* options can additionally be executed at any time prior to their maturity date ($t < T$). The strike salary is the fixed set of salary, which the employee makes references to when he considers early retirement.

Defiened Benefit (DB) pension plans provide benefits to members which are defined in terms of a member's final salary and the length of service. At age T the member retires and receives a pension equal to 1⁄60 of final salary for each year as a member of the scheme, payable annually in advance for life. Thus,

Annual pension, $P(t) = \frac{N}{60} \times \varpi$

where N =number of years of plan membership

ϖ =final pensionable salary

There are two major ways of formulating the American option pricing problem. The earliest, the free boundary formulation, is due to [7]. The other is the variational inequality formulation, or linear complementarity problem, due to [5]). [7] formulated the American call option as a free boundary problem (see also [10]). The free boundary formulation and the optimal stopping formulation are equivalent ([4], [9], [14).

[6] was among the first to apply the method of no arbitrage to show that the price of an American put option could be treated as optimal stopping problems. [7] then derived a free boundary problem for the discounted American call option with gain function $\Phi(S) = e^{-\tau r(S-K)^+}$.

[12] studied pricing of a perpetual American put option when a stock price is in a geometric Brownian motion. He used the theory of Laplace transform for first passage time of drifted Brownian motion to find a discounting factor, $e^{-r\tau}$. After getting the discounting factor, he came up with the price S_f which is lower than K (strike price) where the owner of the put will take action at that price S_f and get an optimal value, $V(S_f)$

[13] discovered that American put options can be treated as optimal stopping problems but boundary conditions to find the optimal solutions is required. He also said that perpetual American put is always an optimal stopping problem. This is not the case with the American call options which can be considered as optimal stopping problems only when the stock pays dividend. So according to his findings American put option can be of infinite or finite horizon.

[3] presented a partial differential equation (PDE) model governing the value of a defined benefit pension plan without the option for early retirement. They said that it is important to develop mathematical models to compute the value of this liability in order to estimate the financial situation of the institution or company that has the obligation with the member of pension plan.

A mathematical model to determine the value of the pension benefits without early retirement was developed by Calvo-Garrido and Vazquez (see, [3]). In their work, a partial differential equation (PDE) was used to model the Pricing Pension Plan without early retirement.

[2] considered a particular case in which the benefit depends on the average salary using linear complementarity formulation. More precisely, they propose a PDE model to price the employer liabilities with a member of the pension plan and obtain the financial value for a defined benefit pension plan including that of early retirement, thus computing the financial value of the pension plan as well as the region of early retirement.

This paper draws inspiration from the work of [3] to allow for the possible early retirement. The modified equation will enable us determine both early retirement region and optimal time for early retirement. Here instead of average salary used in the existing model equation, we shall use what is called final salary. The main difference in this model is the consideration that the salary is stochastic, so that the pension plan can be handled as an option on the final salary, $S(t)$.

2.0 Assumptions of the Model

a) Retirement payments are based on final salary
b) A member of the plan would retire on maximizing the benefits of retirement among all possible dates (stopping times) to retire.
c) Smooth pasting condition holds
d) Annual dividend yield of the asset (salary), $\delta = 0$ such that $\mu = r$.
e) The arbitrage free price of the perpetual American put option is given by $V(S) = \min_{\tau} E_S\,[e^{-\tau r}(K - S_\tau)^+]$
f) Optimal stopping problem with a value function $V(S_t) = \sup_{\tau \le T} E_S e^{-\tau r} \mathrm{V}_\tau (K - S_t)$ satisfies geometric Brownian motion, $dS_t = \mu S_t dt + \sigma S_t dW_t$
g) The infinitesimal generator of the (strong) Markov process S is given by

$$\mathbb{L}_S V = rS\frac{\partial}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2}.$$

h) Standard Markovian arguments suggest that V from assumption (g) solves the following free boundary problem of parabolic type

$$\mathbb{L}_S V = rV$$

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2.1 Parameters and Variables of the Model

The following parameters (functions) and variables are used in this research work:

$V = V(S, t) = V(S_t)$ is the financial value of retirement benefit in the time interval $0 < t \leq T$;

$S = S_t =$ salary at time, t;

S_f = Optimal salary;

K= strike salary;

$S - K$ = Payoff for the call option (American call option for a fixed K and any given salary, S) representing an employer's option

$K - S =$ Payoff for the put option (American put option for a fixed K and any given salary, S) representing an employee's option

$(S - K)^+ = \max_S(S - K, 0)$ assumed to occur at the optimal boundary (call option)

$(K - S)^+ = \min_S(K - S, 0)$ assumed to occur at the optimal boundary (put option)

$V(S_f)$= Optimal financial value of retirement benefit (is also the same as the optimal retirement benefits) with respect to salary

$t =$ Time (in year) spent with the pension plan

r =the salary growth rate or Accrual rate

μ =$(r - \delta)$ is the expected return of the salary (asset)

δ =annual dividend yield $\delta \geq 0$ of the asset (salary) (when $\delta = 0$, then $\mu = r$)

$\sigma =$ The volatility of the salary (also the standard deviation)

T =Worker's expected retirement time (maturity or expiry time) in years

τ= stopping time

τ_f= optimal stopping time

$\mathcal{C}, \mathcal{D}$ = continuation set and stopping set respectively

W_t =geometric Brownian process, (Disturbance factor)

k_1 , k_2= arbitrary constants,

w_+ ,w_-= respective positive and negative roots of an auxiliary equation

$\mathbb{R}^d = d -$ dimensionalEuclidean space

$\mathbb{L}_S =$ infinitesimal operator of S

V_τ = value function at stopping time, τ,

Φ_τ = gain function at stopping time, τ,

E_S =expectation with respect to S

$(\Omega, \mathcal{F}, \mathcal{P})$ =probability space

$\mathcal{F}_t$ =filtration

2.2 Proposition: Optimum time for Exercising

There exists S_f such that early exercise is worthwhile for $S \leq S_f$, but not for $S > S_f$.

Proof

Let $\pi = V + S$ be a portfolio. As soon as

$V = (K - S)^+$ the option can be exercised since the amount

$\pi = (K - S)^+ + S = K$ at interest rate, r.

For $V > K - S$ it is not worthwhile since the value of portfolio is

$\pi = V + S > (K - S) + S \geq K$ but after exercising, it is equal to K

The value S_f depends on time, and it is termed the free boundary value. We have

$$V(S,t) = (K - S)^+, \quad S \leq S_f(t)$$

$$V(S,t) > (K - S) > S_f(t)$$

Since the free-boundary value is not known, it must be determined with the option price.

For large S, the put option satisfies the Black – Scholes equation ([8]).

That is, $S >> K$,

$V(S,t) \underset{S\to\infty}{\longrightarrow} 0$ and

$$V(S_f(t), t) = K - S_f(t)$$

Additional conditions are required as these are not sufficient.

These are

$S \to \frac{\partial V}{\partial S}(S,t)$ is continuous at $S = S_f(t)$

Since for $S < S_f(t)$

$\frac{\partial V}{\partial S}(S,t) = \frac{\partial}{\partial S}(K - S) = -1$

also $\frac{\partial V}{\partial S}(S_f(t), t) = \frac{\partial}{\partial S_f}(K - S_f) = -1$

This is the smooth pasting condition. According to [8], American Put option can be determined by solving

$S \leq S_f(t)$: $\quad V(S,t) = (K - S)^+$

$S > S_f(t)$: $\quad \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial V^2}{\partial S^2} + (r - \delta)\frac{\partial V}{\partial S} - rV = 0$

With the endpoint condition

$\mathrm{V(S,T) = (K - S)}$

And the boundary conditions

$\lim_{S\to\infty} \mathrm{V(S,T)} = 0$

$V(S_f(t), t) = K - S_f(t),$

$\frac{\partial V}{\partial S}(S_f(t), t) = -1$

Our formulation follows perpetual American put option. Perpetual American put option has value function that is a function of the salary (stock price), $V = V(S)$ only and its optimal stopping boundary is a constant function.

So, the boundary conditions become

$$\lim_{S\to\infty} V(S) = 0$$
$$V(S_f) = K - S_f,$$
$$\frac{\partial V}{\partial S}(S_f) = -1 \quad .$$

2.3 Formulation of the Problem

For a mathematical formulation of the problem, let us consider a probability space, $(\Omega, \mathcal{F}, P)$ carrying a standard one-dimensional Brownian motion, $W = (W_t)_{t\geq 0}$. Let $S = (S_t)_{t\geq 0}$ be the geometric Brownian motion representing the salary of the employee in the pension plan. Also let $\sigma > 0$ and $r > 0$ be the volatility coefficient and accrual rate respectively.

Let S_t be the salary of an employee at time, t (using the American Put option). Suppose that one takes decision to retire at a stopping time, τ then the discounted value of the payoff is:

$e^{-\tau r}(K - S_t)$

This describes the difference between the amount one receives, and the actual value of the projected salary, where

$e^{-\tau r}$is the discount factor

$(K - S_t)$ = payoff

$r$is the accrual rate.

S_t is the salary at time, t

K is the fixed salary for reference purposes

τ is a stopping time

If the employee chooses the stopping time, τ then fundamental theorem of asset pricing says that the payoff has initial value

$$E_S e^{-\tau r}(K - S_t) \tag{1}$$

where $E_S$is the expectation with respect to . Next, we take the supremum over all stopping times $\leq$. That is to say that pricing the American put option is equivalent to solving an optimal stopping problem with a value function ([4]; [9], [14]).

$$V(S_t) = \sup E_S e^{-\tau r} V(K - S_t)$$

where = is the value function at stopping time, , where we consider an optimal stopping problem as:

Find

$$V_\tau(S_t) = \sup_{\tau \le T} E_S e^{-\tau r} V_\tau(K - S_t) \tag{2}$$

We assume that the function S_t satisfies the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3}$$

subject to

$$S(0) = S_0 \tag{4}$$

Further, standard Markovian argument suggests the associated free-boundary problem is,

to solve

$$(\mathbb{L}V - rV)S = 0 \, , \mu = r - \delta, \ \delta \ge 0 \tag{5}$$

Subject to the conditions

$$For\ S > S_f: \quad V(S) > K - S \tag{6a}$$

$$\text{AND} \ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S\frac{\partial V}{\partial S} - rV < 0 \tag{6b}$$

$$\text{For S} < S_f: \quad \text{V(S)} = \text{K} - \text{S} \tag{7a}$$

$$\text{AND} \ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S\frac{\partial V}{\partial S} - rV = 0 \tag{7b}$$

$$\text{For S} = S_f: \quad \text{V(S)} = (K - S)^+ \tag{8}$$

$$\text{AND finally} \ \frac{\partial \text{V}}{\partial \text{S}}(\text{S}_\text{f}) = -1 \tag{9}$$

where $(K - S)$ *is* payoff for American put option,

Equation (5) subject to equations (6a) - (9) is referred as the free boundary problem to be solved.

Here, equation (5) satisfies salary equilibrium condition, equations (6a) and (8) represent value matching conditions in continuation region, while (7a) represent value matching condition in a stopping region. Equation (6b) is the region where early exercise is not worthwhile while (7b) is the region where early exercise is worthwhile. Equation (9) represents smooth pasting condition.

3.0 Methods of Solution

There are generally two approaches of solving optimal stopping problems ([9]) namely, the martingale and Markovian. When the gain process is described by its unconditional finite-dimensional distributions, the appropriate solution technique is the martingale approach; the most important concept being the Snell envelope. Dynamic programming is used when the planning horizon, T is finite and time is discrete.

On the other hand, when the process is determined by a family of (conditional) transition functions leading to a Markovian family of transition probabilities, we use the Markovian method. The solution due to the Markovian method is usually obtained by solving the problem subject to associated free-boundary conditions. In this paper, we shall consider only the Markovian method of solving equation (2) subject to the five free-boundary conditions, (5) - (9).

The results about the relationship between optimal stopping problems for Markov processes and free-boundary problems for partial differential equations give an opportunity to obtain explicit solutions in some particular cases (see e.g. [11])

The basic idea with Markovian optimal stopping problems with the infinite time horizon is that $V(S)$ and $\boldsymbol{\mathcal{C}}$ or $\mathcal{D}$ (continuation set and stopping set respectively) should solve the free- boundary problem

$$\begin{cases} \mathbb{L}_S V \leq 0 \\ V \geq \Phi\,, (V > \Phi \text{ in } \boldsymbol{\mathcal{C}} \text{ and } \; V = \Phi \text{ in } \mathcal{D}), \end{cases}$$

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where $\mathbb{L}_S$ is the infinitesimal operator of S.

When S is a diffusion process,Φ is sufficiently smooth in a neighbourhood of the boundary $\partial\mathcal{C}$, and $\partial\boldsymbol{\mathcal{C}}$ is well defined, then the above system of inequalities splits into two conditions

$$\begin{cases} \mathbb{L}_S V \leq 0 \;\text{ in } \boldsymbol{\mathcal{C}} \\ V = \Phi \quad \text{in } \mathcal{D} \\ \left.\frac{\partial V}{\partial s}\right|_{\partial\mathcal{C}} = \left.\frac{\partial \Phi}{\partial s}\right|_{\partial\mathcal{C}} \;\; \textbf{(smooth fit)} \end{cases} \tag{11}$$

By the smooth pasting condition (fit), S being a homogeneous 1-dimensional process, it is possible to find the solution (V) of this system explicitly, and then verify that it is also a solution of the optimal stopping problem, heuristically.

3.1 Derivation of the Solution by the Markovian Method

We note that, under the value matching condition in the stopping region (8) where

$$V(S) = (K - S_t)^+ \Longrightarrow V(S_t) = \Phi(S_t)$$

therefore, equation $V(S_t) = \sup_{\tau \leq T} E_S e^{-\tau r} V_\tau (K - S_t)^+$ from (2) becomes

$$V_\tau(S_t) = \sup_{\tau \leq T} E_S e^{-\tau r} \Phi_\tau (K - S_t)^+ \tag{12}$$

where K is the strike salary, τ is the stopping time and S_t is the current salary. Henceforth, we will simply write

$$\left.\begin{aligned} V_\tau(K-S_t)^+ &= V(K-S_\tau)^+ \\ \Phi_\tau(K-S_t)^+ &= \Phi(K-S_\tau)^+, \\ \Phi_\tau(S_t) &= \Phi(S_\tau) \\ \text{and } V_\tau(S_t) &= V(S_\tau) \end{aligned}\right\} \tag{13}$$

We need to find the point, S_f which will give the optimal salary in order to find optimal value of $V(S)$ and optimal time τ_f. That means we now seek the optimal stopping time τ_f and the optimal salary, $S_f \in (0, \mathrm{K})$ such that

$$\tau_f = min\{t \geq 0, S_t \leq S_f\} \tag{14}$$

The infinitesimal operator, $\mathbb{L}$ acts on $\Phi(S)$ according to the rule

$$(\mathbb{L}\Phi)(S) = \frac{\sigma^2(S)}{2}\Phi''(S) + \mu S\Phi'(S). \tag{15}$$

for all $S > 0$.

The smooth pasting property states that the value function is differentiable at any threshold at which stopping becomes optimal,

$$V(S) = \Phi(S) \tag{16}$$

so

$$V'(S) = \Phi'(S) \tag{17}$$

where

$$V'(S) = \frac{\partial V}{\partial S} \text{and} \Phi'(S) = \frac{\partial \Phi}{\partial S} \tag{18}$$

We will now look for functions which solve the stated free-boundary problem (5) - (9). We say that a geometric Brownian motion which satisfies the stochastic differential equation

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t,$$

has generator $\mathbb{L}$ such that

$$\mathbb{L}V = rV$$

or

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} = rV \tag{19}$$

We use Euler substitution to *transform (26) to differential equation with constant coefficient before we solve. This is as shown below:*

$$\text{Let } S = e^t \tag{20}$$

$$\frac{dS}{dt} = e^t, then\ \frac{dt}{dS} = e^{-t}$$

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t}\cdot\frac{dt}{dS} = e^{-t}\frac{\partial V}{\partial t} \tag{21}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial t}\left(\frac{\partial V}{\partial t}\right)\frac{dt}{dS}$$

$$= \left(-e^{-t}\frac{\partial V}{\partial t} + e^{-t}.\frac{\partial^2 V}{\partial t^2}\right)\frac{dt}{dS}$$

$= e^{-2t}\left(-\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial t^2}\right)$

$$= e^{-2t}\left(\frac{\partial^2 V}{\partial t^2} - \frac{\partial V}{\partial t}\right) \quad (22)$$

We substitute the values in (20), (21) and (22) in to equation (19), we have

$\frac{\sigma^2}{2}\left(\frac{\partial^2 V}{\partial t^2} - \frac{\partial V}{\partial t}\right) + \mu\frac{\partial V}{\partial t} - rV = 0$

$$\frac{\sigma^2}{2}\frac{\partial^2 V}{\partial t^2} + \left(\mu - \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial S} - rV = 0 \quad (23)$$

The characteristic equation of equation (30) becomes

$$\frac{\sigma^2}{2}\lambda^2 + \left(\mu - \frac{\sigma^2}{2}\right)\lambda - r = 0 \quad (24)$$

We multiply (31) by 2 to eliminate the fraction,

$$\sigma^2\lambda^2 + (2\mu - \sigma^2)\lambda - 2r = 0 \quad (25)$$

The quadratic formula (equation) is

$$\lambda = \frac{-(2\mu-\sigma^2) \pm \sqrt{(2\mu-\sigma^2)^2 + 4\times 2\sigma^2 r}}{2\sigma^2} \quad (26)$$

$\lambda_1 = \frac{-(2\mu-\sigma^2) + \sqrt{(2\mu-\sigma^2)^2 + 8\sigma^2 r}}{2\sigma^2} \equiv w_+$

$\lambda_2 = \frac{-(2\mu-\sigma^2) - \sqrt{(2\mu-\sigma^2)^2 + 8\sigma^2 r}}{2\sigma^2} \equiv w_-$

$\therefore V(t) = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} = k_1(e^t)^{\lambda_1} + k_2(e^t)^{\lambda_2}$

$$\Longrightarrow V(S) = k_1 S^{\lambda_1} + k_2 S^{\lambda_2} \equiv k_1 S^{w_+} + k_2 S^{w_-} \quad (27)$$

We assume that at the optimal boundary, $V(S)$ becomes $V(S_f) = \Phi(S_f)$, that is where equations (16) holds, (that is $V(S) = \Phi(S)$).

Equation (27) becomes

$$V(S_f) = k_1 S^{w_+} + k_2 S^{w_-} \quad (28)$$

If we substitute the solution given in equation (27) in equation (28), we will have three unknowns, namely k_1, k_2 and S_f in three equations.

We obtain the three equations in (16), (16) and (18) by considering the value matching condition, smooth pasting condition and boundary condition respectively.

From the properties of geometric Brownian motion, we let

$$V(0) = 0. \quad (29)$$

Applying equation *(16)* to equation (27) we see that as

$k_2 S^{w_-} \to +\infty, S \to 0$ for $k_2 > 0$.

Likewise we see that $k_2 S^{w_-} \to +\infty$ as $S \to 0$ for $k_2 < 0$. This implies that condition (27) can only hold when $k_2 = 0$.

So equation (27) reduces to

$$V(S) = k_1 S^{w_+} \text{since } k_2 S^{w_-} = 0 \tag{30}$$

Applying *(30)* to (28) and (27) we have

$$k_1 S^{w_+} = \Phi(S_f) \tag{31}$$

and differentiating (37), we have

$$S^{(w_+)-1} = \Phi'(S_f) \tag{32}$$

Substituting the salary of $\Phi(S_f)$ (where S_f is a fixed value) from *(35)* into *(36)* we have

$$\frac{(w_+)\Phi(S_f)}{S_f} = \Phi'(S_f) \tag{33}$$

Hence the optimal salary to this particular problem is

$$S_f = \frac{(w_+)\Phi(S_f)}{\Phi'(S_f)} \tag{34}$$

From this optimal value, we can deduce the optimal stopping time τ_f as

$$\tau_f = min\{t \geq 0: S_t = S_f\} \tag{35}$$

$$= min\left\{t \geq 0: S_t = \frac{(w_+)\Phi(S_f)}{\Phi'(S_f)}\right\} \tag{36}$$

The value function at these optimal stopping values becomes

$$V'(S) = E_S\left(e^{-\mu t}\Phi(S_f)\right) \tag{37}$$

The gain function $\Phi(S_f)$ is usually given or known and it depends on the nature of the problem one is solving.

We need to know the values of w_+ and w_- as far as this problem is concerned. To do this we consider equation (25).

$$w_\pm = \frac{(\sigma^2-2\mu)\pm\sqrt{(2\mu-\sigma^2)^2+8\sigma^2 r}}{2\sigma^2} \tag{38}$$

If we let $\mu = r$ we have

$$w_\pm = \frac{(\sigma^2-2r)\pm\sqrt{(2r-\sigma^2)^2+8\sigma^2 r}}{2\sigma^2}$$

$$= \frac{(\sigma^2-2r)\pm\sqrt{4r^2-2(2r)\sigma^2+\sigma^4+8\sigma^2 r}}{2\sigma^2}$$

$$= \frac{(\sigma^2-2r)\pm\sqrt{4r^2-4r\sigma^2+\sigma^4+8\sigma^2 r}}{2\sigma^2}$$

$$= \frac{(\sigma^2-2r)\pm\sqrt{4r^2+4\sigma^2 r+\sigma^4}}{2\sigma^2}$$

$$= \frac{(\sigma^2-2r)\pm\sqrt{(2r+\sigma^2)^2}}{2\sigma^2}$$

$$= \frac{(\sigma^2-2r)\pm(2r+\sigma^2)}{2\sigma^2}$$

First, we consider

$$\frac{(\sigma^2-2r)+(2r+\sigma^2)}{2\sigma^2} = \frac{\sigma^2+\sigma^2}{2\sigma^2} = \frac{2\sigma^2}{2\sigma^2} = 1$$

Next, we consider

$$\frac{(\sigma^2-2r)-(2r+\sigma^2)}{2\sigma^2} = \frac{\sigma^2-2r-\sigma^2-2r}{2\sigma^2} = \frac{-4r}{2\sigma^2} = \frac{-2r}{\sigma^2}$$

$$w_+ = 1 \text{ and } w_- = \frac{-2r}{\sigma^2}$$

So the general solution of (26) becomes

$$V(S) = k_1 S + k_2 S^{\frac{-2r}{\sigma^2}} \quad (39)$$

In an arbitrage free market the option price of American options is $V(S) \leq K$ for $S > 0$. This implies that the solution in equation (27) is bounded. This means $k_1 S + k_2 S^{\frac{-2r}{\sigma^2}} \leq K$. On the other hand, when S gets very large the first term $\rightarrow \infty$ for $k_1 > 0$ and $-\infty$ for $k_1 < 0$. This implies that the function in equation (27) is no longer bounded. For it to be bounded k_1 must then be zero.

From the above fact, the solution in equation (39) becomes

$$V(S) = k_2 S^{\frac{-2r}{\sigma^2}} \quad (40)$$

where k_2 is a constant to be determined.

The optimal value S_f is obtained by replacing w_+ with w_- from equation (39). Hence

$$S_f = \frac{(K-S)^+(w_-)}{\frac{d}{dS}(K-S)^+} = -(w_-)(K-S)^+ \quad (41)$$

Applying condition in (8) to (41) we have

$$S_f = -(w_-)(K-S)^+$$

$$= \frac{-Kw_-}{1-(w_-)} = \frac{K\frac{2r}{\sigma^2}}{1+\frac{2r}{\sigma^2}} = \frac{K}{1+\frac{2r}{\sigma^2}} \quad (42)$$

Remark1

It was assumed that $S_f \in (0, K)$. This is the optimal salary to exercise the option. This shows that the optimal salary depends on the strike salary, K and the volatility, σ. If the volatility, σ and the strike salary, K in the perpetual American put option are known in advance we would be able to find the specific value of S_f

From equation (35) and (42), the optimal stopping time becomes

$$\tau_f = min\left\{t \geq 0: S_t \leq \frac{K}{1+\frac{2r}{\sigma^2}}\right\} \quad (43)$$

We need to find the value of $V(S)$ at this point. To do this, we need to find the value of k_2 at this particular value of S_f.

From equation (34) we have

$$V(S) = k_2 S^{\frac{-2r}{\sigma^2}} \text{when } k_1 = 0 \quad (44)$$

$$V'(S) = \left(-\frac{2r}{\sigma^2}\right) k_2 \, S_f^{\left(\frac{-2r}{\sigma^2}-1\right)} \tag{45}$$

If we assume smooth pasting condition (17), where

$$V'(S) = \frac{\partial V}{\partial S} = -1$$

We have

$$V'(S) = \left(-\frac{2r}{\sigma^2}\right) k_2 \, S_f^{\left(\frac{-2r}{\sigma^2}-1\right)} = -1 \tag{46}$$

Multiplying both sides by -1 we have

$$\left(\frac{2r}{\sigma^2}\right) k_2 \, S_f^{\left(\frac{-2r}{\sigma^2}-1\right)} = 1$$

We want to make k_2 the subject so that

$$\left(\frac{2r}{\sigma^2}\right) S_f^{\left(\frac{-2r}{\sigma^2}-1\right)} = \frac{1}{k_2}$$

$$k_2 = \left(\frac{\sigma^2}{2r}\right) S_f^{\left(\frac{2r}{\sigma^2}+1\right)}, \text{ but } S_f = \frac{K}{1+\frac{2r}{\sigma^2}}$$

So that

$$k_2 = \left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} \tag{47}$$

Substituting the value of $k_2$into (40) we have

$$V(S) = \left(\left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)}\right) S^{\frac{-2r}{\sigma^2}}$$

or

$$S^{\frac{-2r}{\sigma^2}}\left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} \tag{48}$$

We have applied the three boundary conditions, namely (5) salary equilibrium condition, (8) value matching condition and (9) smooth pasting condition so far to arrive at the solution in (54). From the other two boundary conditions represented by (6) continuation region and (7a) value matching condition, we see that $V(S) \geq K - S$ for $S < S_f$.

Considering all the five boundary conditions (5) to (9) we have

$$V(S) = \begin{cases} S^{\frac{-2r}{\sigma^2}}\left(\frac{\sigma^2}{2r}\right)\left(\frac{K}{1+\frac{2r}{\sigma^2}}\right)^{\left(\frac{2r}{\sigma^2}+1\right)} & \text{for } S \geq S_f \\ K - S & \text{for } S < S_f \end{cases} \tag{49}$$

where $K - S$ is the payoff in the case of American put option at any given value of salary, S.

The equation (55) is used to determine the financial value of retirement benefit, $V(S)$ (free arbitrage price or the option value) for perpetual American put option. It can only be determined if its optimal value, S_f is known. $V(S_f)$ is the value of the pension benefit that is optimal to retire.

4.0 Conclusion

In this paper, we formulated a simple optimal stopping problem. The solution to this problem was found and analysed systematically by applying matching value condition, smooth pasting condition, asset equilibrium condition and the boundary condition. We used the free-boundary approach to derive the solution. Under this arrangement, we formulated an equivalent ordinary differential free-boundary problem for the unknown value function and stopping boundaries. Since such a free-boundary problem usually has a non-unique solution, we assumed that the appropriate solutions must satisfy certain additional conditions (e.g. smooth fit or smooth pasting condition, normal boundary, normal reflection, etc.). Our formulation follows perpetual American put option. The reason we formulate the optimal stopping problem as perpetual American put option is that its value function is a function of the salary, $V = V(S)$ only and its optimal stopping boundary is a constant function. This makes the problem less tedious to solve.

The result compares favourably with that of [2]). In the existing model, the interval in which it is optimal to retire before the retirement date, T were proposed. This model goes beyond interval to find the optimal point for early retirement.

We recommend the appliction of this model to individuals as well as cooperate organisations so as maximise the expected retirement benefit. This can only to achieved if we know the optmal point to retire.

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