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Stock Option Price Computation under Economic Recession-Induced Stochastic Volatility Heston Model

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Abstract

We present a model for an option pricing with economic recession-induced stochastic volatility in a univariate Heston setting. The recession-induced volatility concept of Bankole and Ugbebor is extended to the Heston model to account for uncertainty effect of recession on option returns on an underlying stock asset in a recessed economy. The model formulated is subject to two economic states which allows regime switching based on the economic state under consideration. The characteristic function for the model is derived and subjected to fast Fourier transform method of Carr and Madan for option price computation. The numerical integration approximations based on Trapezoidal rule and Simpson's rule is applied and simulation of the model is carried out to obtain European-type call option prices. The option prices obtained following the assumptions and modifications incorporated, shows significant improvement on the existing Heston model especially with the economic recession parameters inclusion.

Keywords: Option returns forecast, Recession induced-volatility, Heston model.
MSC2010: 91G20, 60E05, 91G30.

1 Introduction

The studies on options trading as popular financial derivatives has been taken very serious by various financial players in the financial market including academic researchers in the related fields of studies. Attention of researchers have been devoted to stochastic volatility modeling for option valuation over the deterministic volatility models. One of the deterministic volatility models which has been condemned for the fact that it is not suitable for stock asset volatility smile due to the volatility term structure assumed to be constant is the popular Black-Scholes Model [1]. However, other shortcomings were found in the model of Black and Scholes. This has led to stochastic volatility modelling in option pricing.

There are several stochastic volatility models reported in literature (see [2], [3]). Heston [3] gave a single factor stochastic volatility model for options pricing with application to bond and

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currency options. His model has been extended by other researchers in number of ways. Charlotte, Mung'atu, Abiodun, and Adjei [4] modified the Heston model to forecast Stock prices. Among other stochastic volatility models and stochastic interest rate models in existence are the studies of: Grzelak and Oosterlee [5], Djeutcha & Fono [6], Guohe [7], Huang & Xunxiang [8]. Jumps inclusion in stochastic volatility model is another focus of studies in financial markets. Cheng & Zhihong [9] studied vulnerable options pricing in the Bifractional Brownian Environment.

The uncertainties nature of risky assets' return has set various financial experts on the move in examining other related concepts to asset price valuation. Each of the stochastic volatility model seek to bring an improvement in option valuation as a representative of the real life behaviour of risky assets in various financial markets. For instance, Christoffersen et al. [10] incorporated double stochastic volatility term structure as an improvement on the univariate version of Heston model (1993) because it was noticed that the univariate Heston model is not able to fit financial market data very well. Various stochastic models formulation have been tailored to enhancing the existing ones for better performance while applying the models in computation of option price subject to an underlying asset value and the scenario surrounding the timeline to the option maturity.

In recent times, Bankole and Ugbebor [11] introduced the concept of economic recession-induced volatility in an option pricing model. Their studies iterates the uncertainties effect of economic recession on assets' returns in financial markets. They further justified the need for an elaborate studies on the concept to strengthens investors' decision on their investment in a recessed economy. Bankole and Adinya [12] introduced a control variable in option valuation model with stochastic interest rate and recession induced volatility model with jumps. The importance of volatility control was investigated and the outcome of the financial instrument traded by adapting the model was visualised in sample paths. The uncertainties events warranting stock price variation has been one of the concerns of investors. It was observed that uncertainties information inflow in the economy contributes to stock price fluctuation in the Stock market. Economic recession has been one of the uncertainty events that pose challenges on investor's asset price returns forecast. An intuition on asset price dynamics prediction under economic recession was studied in [13] via recurrence relation.

However, our attention is given to option pricing in a recessed economy with respect to recession-induced stochastic volatility concept. The parent model worked upon for an improvement is the Heston model. The paper is structured as follows: The preliminaries, Heston model, Model assumptions, Economic recession-induced volatility Heston model, Main results, fast Fourier transform of the option price with respect to Trapezoidal rule and Simpson's rule, Tables of Results, and Conclusion.

2 Preliminary

2.1 The Heston model

The univariate stochastic volatility Heston model [3] is given as:

$$\begin{cases} dS(t) = rS(t)dt + \sqrt{v(t)}S(t)dW_1(t), \quad S(0) = S_0 > 0 \\ \\ dv(t) = \kappa\big(\theta - v(t)\big)dt + \sigma\sqrt{v(t)}dW_2(t), \quad v(0) = v_0 > 0 \end{cases} \tag{2.1}$$

where $S(t)$ is the asset value at time t; r is the riskless interest rate, κ is the mean reverting rate, θ is the mean reversion level, σ as volatility of volatility term, v_0, initial variance, and W_1 and W_2 are two Brownian motions driving the system, and are correlated.

Our attention is to give a modified version of Heston model, which incorporates economic recession-induced volatility term.

2.2 The Basic Definitions and Assumptions

Assumption 1: Consider an economy in regime switching setting where the economy is either in recession or recession-free state. Let $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbf{Q})$ be a filtered probability space. Assume there exists economic risk factors on the underlying stock asset such that the price of asset, X, evolves under the influence of two sources of volatilities arising from economic recession and other risk factors in the stock market. Suppose *economic recession filtration* $\mathcal{F}_t^{rec}$ and *filtration from other sources* $\mathcal{F}_t^{os}$ are defined such that the respective volatility components $v^{rec}(t,X_t) = \Big(v^{rec}(s,X_s)\Big|\mathcal{F}_t^{rec}\Big)$ and $v^{os}(t,X_t) = \Big(v^{os}(s,X_s)\Big|\mathcal{F}_t^{os}\Big)$ where $s<t$ and $\mathcal{F}_t^{rec}, \mathcal{F}_t^{os} \subseteq \mathcal{F}$. For simplicity purpose, henceforth, we denote $v^{rec}(t,X_t) := v^{rec}$ and $v^{os}(t,X_t) := v^{os}$ respectively.

Assumption 2: Suppose further that the economy state is unique. Then, the stock asset volatility is 'economic state dependent'.

Definition 2.1. *Let the assumptions above hold. Then, the total stock volatility is defined as:*

$$\widehat{v}(t) \;=\; \begin{cases} v^{rec} + v^{os}, & \text{if the economy is in recession state,} \\ v^{os}, & \text{if the economy is in recession-free state.} \end{cases} \tag{2.2}$$

In the next subsection, we incorporated the volatility induced recession $\widehat{v}(t)$ based on (2.2) in recession state into the Heston model (2.1). The model is hereby referred to as "Economic Recession-induced Volatility Heston Model (ERVHM)" in what follows.

2.3 The Model

Let X be stock asset defined on a filtered probability space $\Big(\Omega, \mathcal{F}_t, \mathbf{Q}, \mathbb{F}\Big)$. Suppose the filtration of the market is generated by standard Brownian motion in a specified time, $t \in [0,T]$ subject to stochastic volatility, $\widehat{v}(t)$, defined in (2.2). Taking $\mathbf{Q}$ to be a risk-neutral probability measure. Suppose further that the stock asset is domiciled in a recessed economy. Then the options payoff on the underlying stocks are state dependent. According to Bankole & Ugbebor [11], there exists uncertainties in options price on an underlying assets under the exposure of economy recession. A concept of economic recession induced volatility uncertainties was introduced and incorporated into a form of stochastic volatility model.

In this study, we incorporate recession-induced stochastic volatility term structure in the univariate Heston model. The proposed model for stock asset price $S(t)$ computation is given as:

$$\begin{cases} dS(t) = \big(r-q\big)S(t)dt + \sqrt{\widehat{v}(t)}S(t)dW^s(t), \qquad S(0) = S_0 > 0 \\[2ex] d\widehat{v}(t) = \kappa\big(\zeta^{rec} + \zeta^{os} - \widehat{v}(t)\big)dt + \sigma_{\hat{v}}\sqrt{\widehat{v}(t)}dW^{\hat{v}}(t), \quad \widehat{v}(0) = \widehat{v}_0 > 0 \end{cases} \tag{2.3}$$

where where the symbols are as earlier defined.

The parameters ζ^{rec} and ζ^{os} are respectively the economic recession-induced long term volatility constant and the long term volatility constant from other sources. The notation: r, is the riskless interest rate, q is the dividend rate for an option paying dividend, $\sigma_{\hat{v}}$ is the volatility of volatility term, and $\kappa_{\hat{v}}$ is the the long-run price variance. The stock asset $S(t)$, is correlated with the volatility process $\widehat{v}(t)$ which are uniquely driven by Wiener processes W^s and $W^{\hat{v}}$. The correlation is given as: $\langle dW^s, dW^{\hat{v}}\rangle_t = \rho dt$.

3 Main Results

Theorem 3.1. *Let a stock asset price, $S(t)$, evolves by the model given in (2.3), the characteristic function for stock price forecast under recession is of the form:*

$$f(i\varphi) = \exp\Big(C(T-t) + D(T-t)x_t + E(T-t)\widehat{v}_t + i\varphi x_t\Big) \tag{3.1}$$

where $C(T-t), D(T-t), E(T-t)$, are deterministic constants for the stochastic processes, $x = \ln S(t)$ and $\widehat{v}_t$, where $\widehat{v}_t = v^{rec} + v^{os}$ is well-defined in (2.2) considering recession state regime.

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Proof. Following the authors ([14], [15], [16]), the characteristic function of the related Partial Integro-Differential Equation (PIDE) evolves in the form given in (3.1). Let the logarithm stock price, $x = \ln S(t)$, satisfy (3.1), applying Itô Lemma, the drift term of the model characteristic function is given as:

$$\frac{\partial f}{\partial t}+\left(r-q-\frac{\widehat{v}(t)}{2}\right)\frac{\partial f}{\partial x}+\kappa\left(\zeta^{rec}+\zeta^{os}-\widehat{v}(t)\right)\frac{\partial f}{\partial \widehat{v}}+\frac{1}{2}\widehat{v}(t)\frac{\partial^2 f}{\partial x^2}+\frac{1}{2}\sigma_{\widehat{v}}^2\widehat{v}(t)\frac{\partial^2 f}{\partial \widehat{v}^2}+\rho\sigma_{\widehat{v}}\widehat{v}(t)\frac{\partial^2 f}{\partial x\partial \widehat{v}}. \tag{3.2}$$

By the fundamental asset valuation theorem ([14], [15]), the drift term given in (3.2) is set to zero, and the partial derivatives are substituted to obtain the following:

$$\begin{aligned}0=f\Bigg[&\left(r-q-\frac{\widehat{v}(t)}{2}\right)(D(T-t)+i\varphi)+\kappa\left(\zeta^{rec}+\zeta^{os}-\widehat{v}(t)\right)\times\left(E(T-t)-\frac{\partial C(T-t)}{\partial t}-\frac{\partial D(T-t)}{\partial t}x(t)\right)\\&-\frac{\partial E(T-t)}{\partial t}\widehat{v}(t)+\frac{1}{2}\widehat{v}(t)(D(T-t)+i\varphi)^2+\frac{1}{2}\sigma_{\widehat{v}}^2\widehat{v}(t)(E(T-t))^2+\rho\sigma_{\widehat{v}}\widehat{v}(t)\Big(D(T-t)+i\varphi\Big)E(T-t)\Bigg].\end{aligned} \tag{3.3}$$

Simplifying and arranging in terms of the stochastic processes $x(t)$, $\widehat{v}(t)$, and the constant term leads to:

$$\begin{aligned}0=&\Big[-\frac{\partial D(T-t)}{\partial t}\Big]x(t)+\Big[-\frac{1}{2}D(T-t)-\frac{1}{2}i\varphi-\kappa E(T-t)-\frac{\partial E(T-t)}{\partial t}+\frac{1}{2}(D(T-t))^2\\&+i\varphi(D(T-t))-\frac{1}{2}\varphi^2+\frac{1}{2}\sigma_{\widehat{v}}^2(E(T-t)^2)+\rho\sigma_{\widehat{v}}\Big(D(T-t)\Big)\Big(E(T-t)\Big)+\rho\sigma i\varphi(C(T-t))\Big]\widehat{v}(t)\\&+\Big[(r-q)D(T-t)+(r-q)i\varphi+\kappa(\zeta^{rec}+\zeta^{os})E(T-t)-\frac{\partial C(T-t)}{\partial t}\Big].\end{aligned} \tag{3.4}$$

Since the stochastic processes, $x(t)$ and $\widehat{v}(t)$, cannot be zero, we equate the coefficients to be zero as well as the constant term in (3.4). The following system of ordinary differential equations emerged:

$$\frac{\partial D(T-t)}{\partial t} = 0, \tag{3.5}$$

$$\frac{\partial C(T-t)}{\partial t} = (r-q)i\varphi+\kappa\big(\zeta^{rec}+\zeta^{os}\big)E(T-t)+(r-q)D(T-t). \tag{3.6}$$

$$\begin{aligned}\frac{\partial E(T-t)}{\partial t} =& -\frac{1}{2}D(T-t)-\frac{1}{2}i\varphi-\kappa E(T-t)+\frac{1}{2}\big(D(T-t)\big)^2+i\varphi\big(D(T-t)\big)\\&-\frac{1}{2}\varphi^2+\frac{1}{2}\sigma_{\widehat{v}}^2\big(E(T-t)\big)^2+\rho\sigma_{\widehat{v}}\big(D(T-t)\big)\big(E(T-t)\big)+\rho\sigma_{\widehat{v}}i\varphi\big(E(T-t)\big),\end{aligned} \tag{3.7}$$

To this end, at the option maturity time, $t = T$, results to $f(i\varphi) = \exp\big(i\varphi x(T)\big)$, and the following initial conditions holds: $C(0) = 0,\quad D(0) = 0,\quad E(0) = 0$. Also, from (3.5),

$$\frac{\partial D(T-t)}{\partial t} = 0 \quad \text{and } D(0) = 0 \;\Rightarrow\; D(T-t) = 0. \tag{3.8}$$

Using (3.8) in (3.7), we have:

$$\frac{\partial E(T-t)}{\partial t} = -\frac{1}{2}i\varphi-\kappa E(T-t)-\frac{1}{2}\varphi^2+\frac{1}{2}\sigma_{\widehat{v}}^2\big(E(T-t)\big)^2+\rho\sigma_{\widehat{v}}i\varphi\big(E(T-t)\big). \tag{3.9}$$

Using the initial condition, $C(0) = 0$, we have:

$$\frac{\partial E(T-t)}{\partial t} = -\frac{1}{2}\sigma_{\widehat{v}}^2\left[E^2(T-t)+\left(\frac{2\rho i\varphi}{\sigma_{\widehat{v}}^2}-\frac{2\kappa}{\sigma_{\widehat{v}}^2}\right)E(T-t)-\frac{i\varphi}{\sigma_{\widehat{v}}^2}-\frac{\varphi^2}{\sigma_{\widehat{v}}^2}\right]. \tag{3.10}$$

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The (3.10) is a form of Ricatti differential equation. The reader can see the following references for a related solution of Ricatti differential equation ([11], [15], [16]). We give the solution to (3.10) as:

$$E(T-t)=\frac{\left(e^{d_j(T-t)}-1\right)\left(\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j\right)}{\sigma_{\hat{v}}^2\left(1-g_je^{d_j(T-t)}\right)}, \tag{3.11}$$

where

$$\begin{aligned} d_j &= \sqrt{\left(\rho\sigma_{\hat{v}}i\varphi-\kappa\right)^2+\sigma_{\hat{v}}^2\left(i\varphi+\varphi^2\right)} \\ g_j &= \frac{\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j}{\rho\sigma_{\hat{v}}i\varphi-\kappa+d_j}. \end{aligned}$$

Setting the time to maturity of the option $\tau=T-t$, the solution (3.10) is written as:

$$E(\tau)=\frac{\left(\kappa-\rho\sigma_{\hat{v}}i\varphi+d_j\right)\left(1-e^{d_j\tau}\right)}{\sigma_{\hat{v}}^2\left(1-g_je^{d_j\tau}\right)}, \tag{3.12}$$

Next, we solve for $C(T-t)$ in (3.5) in what follows.

$$\frac{\partial C(T-t)}{\partial t}=(r-q)i\varphi+\kappa\big(\zeta^{rec}+\zeta^{os}\big)E(T-t)+(r-q)D(T-t). \tag{3.13}$$

Substituting (3.8) into (3.13), it reduces to:

$$\frac{\partial C(T-t)}{\partial t}=(r-q)i\varphi+\kappa\big(\zeta^{rec}+\zeta^{os}\big)E(T-t). \tag{3.14}$$

We integrate the both sides of (3.14) as follows:

$$\int_{s=t}^{s=T}\partial C(T-s)ds=(r-q)i\varphi ds\int_{s=t}^{s=T}\kappa\big(\zeta^{rec}+\zeta^{os}\big)E(T-s)ds. \tag{3.15}$$

$$\begin{aligned} \Rightarrow \quad -C(T-s)\Big|_{s=t}^{s=T} &= (r-q)i\varphi s\Big|_{s=t}^{s=T}+\int_{s=t}^{s=T}\kappa(\zeta^{rec}+\zeta^{os})E(T-s)ds, && (3.16)\\ \Rightarrow \quad -C(0)+C(T-t) &= (r-q)i\varphi(T-t)+\kappa(\zeta^{rec}+\zeta^{os})\int_{s=t}^{s=T}E(T-s)ds. && (3.17) \end{aligned}$$

Using (3.12) in (3.16), we have:

$$\begin{aligned} C(T-t) &= (r-q)i\varphi(T-t)+\kappa(\zeta^{rec}+\zeta^{os})\int_{s=t}^{s=T}\frac{\left(e^{d_j(T-s)}-1\right)\left(\rho\sigma_v i\varphi-\kappa-d_j\right)}{\sigma_{\hat{v}}^2\left(1-g_je^{d_j(T-s)}\right)}ds, \\ &= (r-q)i\varphi(T-t)+\frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left(\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j\right)\int_{s=t}^{s=T}\frac{\left(e^{d_j(T-s)}-1\right)}{\left(1-g_je^{d_j(T-s)}\right)}ds, \\ &= (r-q)i\varphi(T-t)+\frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left(\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j\right)\times\left[-\ln\frac{\left(e^{d_j(T-s)}-1\right)}{d_j}+\frac{\ln\left(g_je^{d_j(T-s)}-1\right)}{d_j\cdot g_j}+\frac{\ln\left(e^{d_j(T-s)}\right)}{d_j}d_j\right]_{s=t}^{s=T}. \end{aligned}$$

Simplifying further yields:

$$\begin{aligned} C(T-t) &= (r-q)i\varphi(T-t)+\frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left(\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j\right)\times\left[\left(\frac{-\ln(g_j-1)}{d_j}+\frac{\ln(g_j-1)}{d_jg_j}+\frac{\ln 1}{d_j}\right)\right. \\ &\quad \left.-\left(-\frac{ln\left(g_je^{d_j(T-t)}-1\right)}{d_j}+\frac{\ln\left(g_je^{d_j(T-t)}-1\right)}{d_j\cdot g_j}+\frac{\ln\left(e^{d_j(T-t)}\right)}{d_j}\right)\right], \end{aligned}$$

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Setting the time to maturity $T - t := \tau$, the simplification continue as follows:

$$\begin{aligned} C(\tau) &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\left[\frac{-\ln(g_j-1)}{d_j} + \frac{\ln(g_j-1)}{d_j g_j} + \frac{ln\left(g_j e^{d_j\tau}-1\right)}{d_j}\right. \\ &\left. -\frac{\ln\left(g_j e^{d_j\tau}-1\right)}{d_j\cdot g_j} - \frac{\ln\left(e^{d_j\tau}\right)}{d_j}\right], \\ &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\left[\frac{1}{d_j}\left(ln\left(g_j e^{d_j(T-t)}-1\right) - \ln(g_j-1)\right)\right. \\ &\left. + \frac{1}{d_j\cdot g_j}\left(\ln(g_j-1) - \ln\left(g_j e^{d_j\tau}-1\right)\right) - \frac{d_j\tau}{d_j}\right], \end{aligned}$$

$$\begin{aligned} C(\tau) &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}(\rho\sigma_v i\varphi - \kappa - d_j)\left[\frac{1}{d_j}\ln\left(\frac{g_j e^{d_j\tau}-1}{g_j-1}\right)\right. \\ &\left. + \frac{1}{d_j g_j}\ln\left(\frac{g_j-1}{g_j e^{d_j\tau}-1}\right) - \tau\right], \\ &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\left[\frac{1}{d_j}\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right. \\ &\left. + \frac{1}{d_j g_j}\ln\left(\frac{1-g_j}{1-g_j e^{d_j\tau}}\right) - \tau\right], \\ &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\left[\frac{g_j\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right) + \ln\left(\frac{1-g_j}{1-g_j e^{d_j\tau}}\right)}{d_j\cdot g_j} - \tau\right], \end{aligned}$$

$$\begin{aligned} C(\tau) &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left[-(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\tau + (\rho\sigma_{\hat{v}}i\varphi - \kappa + d_j)\cdot\left(\frac{g_j-1}{d_j}\right)\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right], \\ &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left[-(\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j)\tau + (\rho\sigma_{\hat{v}}i\varphi - \kappa + d_j)\cdot\left(\frac{\frac{\rho\sigma_{\hat{v}}i\varphi-\kappa-d_j}{\rho\sigma_{\hat{v}}i\varphi-\kappa+d_j}-1}{d_j}\right)\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right], \\ &= (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left[(\kappa - \rho\sigma_{\hat{v}}i\varphi + d_j)\tau + \left(\frac{\rho\sigma_{\hat{v}}i\varphi - \kappa - d_j - \rho\sigma_{\hat{v}}i\varphi + \kappa - d_j}{d_j}\right)\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right], \end{aligned}$$

This is finally given as:

$$C(\tau) = (r-q)i\varphi\tau + \frac{\kappa(\zeta^{rec}+\zeta^{os})}{\sigma_{\hat{v}}^2}\left[(\kappa - \rho\sigma_{\hat{v}}i\varphi + d_j)\tau - 2\ln\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]. \tag{3.18}$$

We have been able to determine the coefficient terms $C(\tau), D(\tau)$, and $E(\tau)$, for the characteristic function, $f(i\varphi)$, (3.1) for the model proposed here. The characteristic function derived will be applied in the simulation studies of the option price.

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3.1 Fast Fourier Transform of the option price with respect to Trapezoidal rule and Simpson's rule

Let $C(k)$ denote the call price. By Trapezoidal rule integral approximate application, we write

$$\begin{aligned} C(k) &\approx \frac{1}{\pi}\exp(-\alpha k)\Re\Bigg[e^{-iv_2k}\varphi(v_1)+\cdots+e^{-iv_{N-1}k}\varphi(v_{N-1})+\frac{1}{2}e^{-iv_1k}\varphi(v_1) \\ &\quad +\frac{1}{2}e^{-iv_Nk}\varphi(v_Nk)\Bigg] \\ &= \xi\frac{\exp(-\alpha k)}{\pi}\sum_{j=1}^{N}\Re\left[e^{-iv_jk}\varphi(v_j)\right]w_j, \end{aligned} \tag{3.19}$$

where w_j denotes the weights, ξ is the incremental value for, N, at uniformly intermediate points given by

$$v_j = (j-1)\,\xi, \quad j = 1, 2, \cdots, N. \tag{3.20}$$

With reference to Carr and Madan [17], we express the call price as

$$C(k) = \frac{\exp(-\alpha k)}{\pi}\int_0^\infty \Re\left[e^{-ivk}\varphi(v)\right]dv \tag{3.21}$$

where $\varphi(v) = \exp\big(-r(T-t)\big)\dfrac{f\,(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+iv(2\alpha+1)}$, f is the model derived characteristic function. Under the Trapezoidal rule considered here, the weights, w_j, is set such that $w_1 = w_N = \dfrac{1}{2}$ and $w_j = 1; \quad j = 2, \cdots, .$

For Simpson's rule, the weight function is chosen such that $w_1 = w_N = \dfrac{1}{3}$ while

$$w_j = \begin{cases} \frac{4}{3}, \text{when} \quad \text{j is even;} \\ \frac{2}{3}, \text{when} \quad \text{j is odd;} \end{cases} \tag{3.22}$$

3.2 Table of Result

Parameters used for simulation:
$N = 2^{11}$; $S_0 = 100$; $r = 0.08$; $q = 0.05$; $\tau = 0.5$; $\kappa = 0.2$; $\theta = 0.05$; $\sigma = 0.3$; $\lambda = 0$; $\rho = -0.8$; $\hat{v}_0 = 0.05$; $\alpha = 1.5$.

The simulation results for the modified Heston model subject to the parameters values highlighted above are reported in Table 1.

Table 1: Options prices comparison with respect to integration technique

Strike price	Exact prices	FFT Trapezoidal prices	FFT Simpson's prices	% Error
82.8204	19.0690	19.0708	19.0708	0.009
88.1911	14.7038	14.7041	14.7041	0.002
93.9101	10.5069	10.5009	10.5009	−0.0571
100.0000	6.6832	6.6879	6.6879	0.0703

In Table 1, the Trapezoidal and Simpson's integral approximates adapted to the model simulation returned comparable results. It is however shows that the accuracy of the two approaches cannot be overemphasised. They are both applicable. Rouah [18] provided more illustration on simulation studies in a related problem. However, the model we present here encompasses economic

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recession parameters. The effect of such parameters is seen in the options output in Table 2.

Model parameters used to obtain the option prices in Table 2 are:
$S = 100$; $K = 100$; $T = 0.5$; $r = 0.10$; $q = 0.07$; $\kappa = 2$; $\theta = 0.06$; $\zeta^{rec} = 0.09$, $\zeta^{os} = 0.0737$, $\sigma = 0.1$; $\hat{v}_0 = 0.06$; $\rho = -0.7$; $\alpha = 1.75$;

Table 2: Options prices comparison based on method Integrand

Method	Call prices
Heston Integrand	7.3460
Carr and Madan Integrand	7.3461
Modified Heston model Integrand (ERVHM)	7.3466

4 Conclusion

We investigate European option price computation under recession-induced volatility. We propose a model incorporating recession parameter which is an extension of the univariate Heston stochastic volatility model. The model characteristic function was derived in affine form and was used in the setting of Fast Fourier transform of Carr and Madan [17]. We compared the option prices with the Heston [3] Integrand, Carr-Madan [17] integrand, and the modified Heston integrand studied here. The options values obtained with the modified Heston-type model integrand shows an improvement on the popular Heston univariate model call prices. The major achievement in this study is the inclusion of recession parameters in the univariate Heston model and computational procedures shown.

Declaration of competing interests

The authors declare that there is no any form of competing interest.

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