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Common Fixed Point Results for Asymptotic Quasi-Contraction Mappings in Quasi-Metric Spaces

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Abstract

In this article, two new classes of pairs $\{T, f\}$ consisting of an asymptotic quasi-contraction T with respect to a map f are introduced. The existence and uniqueness of the common fixed point of such pairs $\{T, f\}$ is then established in the context of a quasi-metric space X, when T and f are weakly compatible. The theorems obtained rely on some conditions on the orbit $O_{T,f}(x, \infty)$ of T at points $x \in X$ with respect to f, and are in fact, improvements of previously known results in the literature of metric-types. An example is given to validate the results obtained.

Keywords: Asymptotic quasi-contraction, Meir-Keeler type mappings, Weakly compatible maps, Orbits, Jungck sequence.
MSC2010: 32H50, 47H10.

1 Introduction

There are many generalizations of the fundamental contraction mapping theorem established by Banach [1] which served as pivotal tool for solving problems in analysis and applied mathematics. For instance see [2-6]. One of the most general contraction mapping was introduced by Ciric [7] in 1971. In 2003, Kirk [8] introduced asymptotic contraction mapping which is a generalization of Banach contraction mapping and established the fixed point of the operator in a complete metric space under certain conditions. Asymptotic contraction map is for example, applied in dynamical system to solve analytical problems. Several works have been carried out in this regard, see [9,10]. Meir-keeler [11] extended Banach contraction principle by proving the following result:

> Let (X, d) be a complete metric space and f a self map of X.
> Assume that for every $\epsilon > 0$, there exists $\delta > 0$ such that:
> $\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(fx, fy) < \epsilon$ for all $x, y \in X$.
> Then f has a unique fixed point.

Suzuki [12] combined the works of Kirk [8] and Meir-Keeler [11] to introduce a class of map called asymptotic contraction of Meir-Keeler type (for short, ACMK). In 2008, Singh and Pant [13] extended the result of Suzuki [12] to more genaral maps. Our interest is in the result of Singh et al. [14] on maps that are called generalized asymptotic contraction of Meir-Keeler type (for short, GACMK).

Definition 1.1. *Let (X, d) be a metric space, Y a subset of X, and $T, f : Y \to X$ two maps. The map T is called a generalized asymptotic contraction of Meiler-Keeler ype (in short GACMK) with respect to f if there is a sequence $\{\psi_n\}$ of self-maps on $[0, \infty)$ such that:*

(P_1) $\limsup_{n\to\infty} \psi_n(\epsilon) \leq \epsilon$ *for all* $\epsilon \geq 0$;

(P_2) *for each* $\epsilon > 0$, *there exist* $\delta > 0$ *and* $\nu \in \mathbb{N}$ *such that* $\psi_\nu(t) \leq \epsilon$ *for all* $t \in [\epsilon, \epsilon + \delta]$;

(P_3) $d(T^n x, T^n y) < \psi_n(M(x, y))$ *for all* $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $M(x, y) > 0$,
where $M(x, y) = \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}$.

In the same reference, they stated the following result.

Theorem 1.2. *Let (X, d) be a metric space and $T, f : Y \to X$ such that $T(Y) \subseteq f(Y)$. Let T be a GACMK with respect to f. If $T(Y)$ or $f(Y)$ is a complete subspace of X, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.*

Recall that the following definitions of coincidence points and weakly compatible mappings:

Definition 1.3. [15] *Let X be a set and T and f be two self maps of X. A point x in X is called a coincidence point of T and f if and only if $Tx = fx$. We shall call $z := Tx = fx$ a point of coincidence of T and f.*

Definition 1.4. [16] *Two self maps T and f in a set X are said to be weakly compatible if they commute at their points of coincidence, that is, if $Tx = fx$ for some $x \in X$, then $Tfx = fTx$.*

It should be noted that Definition 1.1 and Theorem 1.2 due to Singh et al. [14] pose a few problems:

1. for $x \in X$ and $n \in \mathbb{N}$, the point $T^n x$ is not defined, except if $Y = X$;
2. the function f may not satisfy $f(x) = x$ for all $x \in Y$. Therefore, a sequence[1] $\{x_n\}$ such that $Tx_n = fx_{n+1}$ for all $n \geq 0$, may not satisfy $x_n = T^n x_0$ for all $n \geq 0$, an assumption which seem to have been used in several lines of their proof.

In this paper, our aim is to provide an ammendment of the maps of Singh et al. [14] and establish an ensing common fixed point theorem in the framework quasi-metric spaces. The following are needed definitions in the setting of quasi-metric spaces:

Definition 1.5. [17] *Let X be a non-empty set. A function $d : X \times X \to [0, \infty)$ is said to be a quasi-metric on X if for any $x, y, z \in X$ the following holds:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

In such case, (X, d) is said to be a quasi-metric space.

Definition 1.6. [17–20] Let (X, d) be a quasi-metric space. A sequence $\{x_n\}$ in X is said to be:

(i) forward-convergent to some $x \in X$ (called the forward limit) if $\forall \epsilon > 0$ $\exists N_\epsilon \in \mathbb{N}$: $d(x_n, x) < \epsilon$ $\forall n \geq N_\epsilon$; in other words, $\{x_n\}$ in X is forward-convergent to x if $\lim_{n\to\infty} d(x_n, x) = 0$.

[1] The existence of such a sequence is discussed in subsection 2.1.

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(ii) backward-convergent to some $x \in X$ (called the backward limit) if $\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N}$: $d(x, x_n) < \epsilon\ \forall n \geq N_\epsilon$; in other words, $\{x_n\}$ in X is backward-convergent to x if $\lim\limits_{n\to\infty} d(x, x_n) = 0$.

(iii) convergent to some $x \in X$ if it is both forward-convergent and backward-convergent to x.

(iv) forward Cauchy (or left K-Cauchy) if $\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N}$: $d(x_n, x_{n+k}) < \epsilon\ \forall n \geq N_\epsilon\ \forall k \in \mathbb{N}$.

(v) backward Cauchy (or right K-Cauchy) if $\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N}$: $d(x_{n+k}, x_n) < \epsilon\ \forall n \geq N_\epsilon\ \forall k \in \mathbb{N}$.

(vi) Cauchy (or p-Cauchy) if $\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N}$: $d(x_n, x_m) < \epsilon\ \forall n \geq N_\epsilon\ \forall k \in \mathbb{N}$. In other words, $\{x_n\}$ is Cauchy if it is forward Cauchy and backward Cauchy.

Definition 1.7. [21] *A quasi-metric space X is said to be complete if every Cauchy sequence in X is convergent in X.*

2 Generalized Asymptotic Contractions of Meir-Keeler type with respect to some functions

In the next section, we give ammend the definition of a generalized asymptotic contraction of Meir-Keeler type as proposed by Singh et al. [14], then prove a common fixed point theorem in complete quasi-metric space.

2.1 Jungck sequence and orbits with respect to f

Suppose X is a non-empty set, Y is a non-empty subset of X, and $T, f : Y \to X$ such that $T(Y) \subseteq f(Y)$.

Choosing $x_0 \in Y$, one can define a Jungck sequence $\{x_n\}$ of points in Y such that

$$Tx_n = fx_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

If f is the inclusion map, that is, such that $f(x) = x$ for any $x \in Y$, then by repetitive application of (2.1), the sequence $\{x_n\}$ is such that $x_n = T^n x_0$, or more generally, $x_{n+k} = T^n x_k$ for any $n, k \geq 0$, with the convention that T^0 is the identity function on Y.

If f is not necessarily the inclusion map, and $x \in X$, the sequence of sets $\{(Tf^{-1})^n(x)\}_{n\geq 0}$ is defined as follows:

$$(Tf^{-1})^n(x) = \begin{cases} \{x\} & \text{if } n = 0 \\ \{Tu :\ f(u) = x\} & \text{if } n = 1 \\ \{Tu :\ f(u) \in (Tf^{-1})^{n-1}(x)\} & \text{if } n > 1. \end{cases} \tag{2.2}$$

Given that $\emptyset \neq T(Y) \subseteq f(Y)$, each $(Tf^{-1})^n(x)$ is non-empty and for $n > 1$,

$$(Tf^{-1})^n(x) = \left\{ Tu \;\middle|\; \begin{array}{c} \exists u_1, u_2, \cdots, u_{n-1} \in Y,\ f(u) = Tu_1,\ f(u_{n-1}) = x \\ \text{and if } n > 2,\ f(u_1) = Tu_2, \ldots, f(u_{n-2}) = Tu_{n-1}, \end{array} \right\} \tag{2.3}$$

The Jungck sequence $\{x_n\}$ defined in (2.1) is such that for all $n, k \geq 0$, $Tx_{n+k} \in (Tf^{-1})^n(Tx_k)$, and in particular, $Tx_n \in (f^{-1}T)^n(Tx_0)$ and $Tx_{n+1} \in (Tf^{-1})(Tx_n)$.

Notice that when f is a bijection on Y, each $(Tf^{-1})^n(x)$ is the singleton. In the special case where $f(x) = x$ for all $x \in Y$, $(Tf^{-1})^n(x) = \{T^n x\}$ for each $n \geq 0$ and $x \in Y$.

In literature, orbits of self-maps and orbital completeness have been defined as follow:

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Definition 2.1. [22] *Let $T : X \to X$ be a self mapping on a metric space. For each $x \in X$ and for any positive whole number n, define the sets:*

$$O_T(x, n) = \{x, Tx, T^2x, T^3x, \cdots, T^nx\}$$

and

$$O_T(x, \infty) = \{x, Tx, T^2x, T^3x, \cdots, T^nx, \cdots\}.$$

The set $O_T(x, \infty)$ is called the orbit of T at x and the metric space X is called T- orbitally complete if every Cauchy sequence in $O_T(x, \infty)$ is convergent in X.

The following definition is an extension of the concepts of orbits and orbital completeness for maps for which Jungck sequences are defined:

Definition 2.2. *Let $T, f : Y \to X$ be a two mappings defined on a subset Y of a set X, for which $T(Y) \subseteq f(Y)$. For each $x \in X$ and for any positive whole number n, the set*

$$O_{T,f}(x, n) := \left\{z | \quad z \in (Tf^{-1})^i(x),\ 0 \le i \le n\right\} = \bigcup_{i=0}^{n} (Tf^{-1})^i(x)$$

will be called the n-orbit of T at x with respect to f, while the set

$$O_{T,f}(x, \infty) := \left\{z | \quad z \in (Tf^{-1})^i(x),\ i \ge 0\right\} = \bigcup_{i=0}^{\infty} (Tf^{-1})^i(x)$$

will be called the orbit of orbit of T at x with respect to f.

If X is endowed with a quasi-metric d, then X is called $\{T, f\}$-orbitally complete if every Cauchy sequence in $O_{T,f}(x, \infty)$ is convergent in X.

It immediately follows that the well-known concepts of orbits and orbital completeness of a map T are obtained when $f(x) = x$ for all $x \in Y$. Motre precisely, $O_{T,f}(x, n)$ becomes $O_T(x, n)$ and $O_{T,f}(x, \infty)$ becomes $O_T(x, \infty)$.

2.2 GACMK re-introduced

We can then re-introduce the definition of Generalized Asymptotic Contraction of Meir-Keeler type (GACMK) with respect to a function f:

Definition 2.3. *Let (X, d) be a quasi-metric space, Y a subset of X, and $T, f : Y \to X$ two mappings. The map T is called generalized asymptotic contraction of Meir - Keeler type (GACMK, simply) with respect to f if there exists a sequence $\{\psi_n\}$ of functions from $[0, \infty)$ into itself satisfying the following:*

(G_1) $\limsup_{n\to\infty} \psi_n(\epsilon) \le \epsilon$ *for all* $\epsilon > 0$;

(G_2) *for each $\epsilon > 0$ there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \le \epsilon$ for all $t \in [\epsilon, \epsilon + \delta]$;*

(G_3) *For all $n \in \mathbb{N}$, and $x, y \in Y$ such that $M(x, y) > 0$,*

$$d(u, v) < \psi_n(M(x, y)), \tag{2.4}$$

for all $u \in (Tf^{-1})^{n-1}(Tx)$, $v \in (Tf^{-1})^{n-1}(Ty)$, where:

$$M(x, y) = \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\right\}. \tag{2.5}$$

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Remark 2.4.

1. It should be noted that Definition 2.3 differs with Definition 1.1 proposed by Singh et al. [14] in that the inequality (2.4), $d(u,v) < \psi_n(M(x,y))$, holds for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$ rather than for $u = T^n x$ and $v = T^n y$ (which would only exist if $Y = X$).

2. Suppose $f(x) = x$ for all $x \in Y$, then for any $x, y \in Y$, we have that

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}$$

and $(u,v) = (T^n x, T^n y)$ for all $n \in \mathbb{N}$, $u \in (Tf^{-1})^{n-1}(Tx)$, and $v \in (Tf^{-1})^{n-1}(Ty)$. Under this condition, for any $n \in \mathbb{N}$, $d(T^n x, T^n y) < \psi_n(M(x,y))$ for $x, y \in Y$ such that $M(x,y) > 0$, and

(i) if $\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\} = d(x,y)$ in (iii) of Definition 2.3, T becomes an asymptotic contraction of Meir-Keeler type (or simply, ACMK) discussed in by Suzuki [12];

(ii) the case where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$ for all $x, y \in X$ is discussed by Singh and Pant [13].

2.3 A fixed point theorem

Now we prove the following theorem on the existence and uniqueness of the common fixed point of a pair $\{T, f\}$ where T is a GACMK with respect to f.

Theorem 2.5. *Let (X,d) be a quasi-metric space and $T, f : Y \to X$ such that $T(Y) \subseteq f(Y)$. Let T be a GACMK with respect to f, i.e., such that for all $n \in \mathbb{N}$ and $x, y \in Y$ such that $M(x,y) > 0$,*

$$d(u,v) < \psi_n(M(x,y))$$

for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$M(x,y) = \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty) + d(fy,Tx)}{2}\right\}.$$

and $\{\psi_n\}$ is a sequence of self-maps on $[0,\infty)$ such that:

(G$_1$) $\limsup_{n\to\infty} \psi_n(\epsilon) \le \epsilon$ *for all* $\epsilon \ge 0$*;*

(G$_2$) *for each $\epsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \le \epsilon$ for all $t \in [\epsilon, \epsilon + \delta]$.*

Suppose further that:

(G$_4$) *any two distinct elements z_1, z_2 in $O_{T,f}(x,\infty)$, $\frac{d(z_1,z_2)}{2} < d(z_2,z_1)$.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

Proof. Let $x_0 \in Y$ be arbitrary chosen. Since $T(Y) \subset f(Y)$, we can define a Jungck sequence $\{x_n\}$ of points in Y, and a sequence $\{y_n\}$ in X such that

$$y_n = Tx_n = fx_{n+1}, \quad n = 0, 1, 2, \ldots.$$

This means that

$$Tx_{n+k} \in (Tf^{-1})^n(Tx_k) \;\; \forall n, k \ge 0, \tag{2.6}$$

and that

$$d(Tx_{n+i}, Tx_{n+j}) < \psi_{n+1}(M(x_i, x_j)) \quad \text{for all } n \geq 0 \text{ and } i, j \text{ with } M(x_i, x_j) > 0. \tag{2.7}$$

First step: We prove by cases that T and f have a coincidence point if $T(Y)$ or $f(Y)$ is complete.

If there is $n \geq 0$ such that $Tx_n = Tx_{n+1}$, then x_{n+1} is a coincidence point of T and f, since $fx_{n+1} = Tx_{n+1}$. Suppose now that $Tx_n \neq Tx_{n+1}$ for all $n \geq 0$.

For any $n \geq 0$, we have $d(fx_{n+1}, Tx_{n+1}) = d(Tx_n, Tx_{n+1}) > 0$, hence $M(x_i, x_j) > 0$ for any distinct $i, j \geq 0$.

For any $n \geq 1$,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &< \psi_1(M(x_n, x_{n+1})) \\ &\leq M(x_n, x_{n+1}) \\ &= \max\left\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1}), \tfrac{d(Tx_{n-1}, Tx_{n+1})}{2}\right\} \\ &\leq \max\left\{d(Tx_{n-1}, Tx_n), d(Tx_n, Tx_{n+1})\right\}, \end{aligned} \tag{2.8}$$

hence for all $n \geq 1$,

$$\begin{cases} M(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ d(Tx_n, Tx_{n+1}) &< d(Tx_{n-1}, Tx_n). \end{cases} \tag{2.9}$$

Thus, the sequence $\{d(Tx_n, Tx_{n+1})\}_{n\geq 0}$ is decreasing and bounded below hence converges to some $\alpha \geq 0$, and $d(Tx_n, Tx_{n+1}) > \alpha$ for all $n \geq 0$.

Suppose $\alpha > 0$. Since $0 < \alpha < d(Tx_0, Tx_1)$, there are $\delta_1 > 0$ and $\mu_1 \in \mathbb{N}$ such that $\psi_{\mu_1}(t) \leq \alpha$ for all $t \in [\alpha, \alpha + \delta_1]$. By definition of α, there is $\mu_2 \in \mathbb{N}$ such that $M(x_{\mu_2}, x_{\mu_2+1}) = d(Tx_{\mu_2-1}, Tx_{\mu_2}) < \alpha + \delta_1$. Therefore, $d(Tx_{\mu_2+\mu_1-1}, Tx_{\mu_2+\mu_1}) < \psi_{\mu_1}(M(x_{\mu_2}, x_{\mu_2+1})) \leq \alpha$, a contradiction. Thus $\alpha := \lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$.
Next we show that $\{Tx_n\}$ is forward-Cauchy sequence. Assuming $\{y_n\}$ is not a forward-Cauchy sequence, there exists $\beta > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $k \in \mathbb{N}$, $m_k < n_k$, $d(Tx_{m_k}, Tx_{n_k}) \geq \beta$ and $d(Tx_{m_k}, Tx_{n_k-1}) < \beta$. By the triangle inequality,

$$d(Tx_{m_k}, Tx_{n_k}) \leq d(Tx_{m_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{n_k}),$$

and as $k \to \infty$, $d(Tx_{m_k}, Tx_{n_k}) \to \beta$. We also have that:

$$\begin{aligned} d(Tx_{m_k}, Tx_{n_k}) &< \psi_1(M(x_{m_k}, x_{n_k})) \\ &= \psi_1\left(\max\left\{\begin{array}{l} d(Tx_{m_k-1}, Tx_{n_k-1}), d(Tx_{m_k-1}, Tx_{m_k}) \\ d(Tx_{n_k-1}, Tx_{n_k}), \frac{d(Tx_{m_k-1}, Tx_{n_k}) + d(Tx_{n_k-1}, Tx_{m_k})}{2} \end{array}\right\}\right), \end{aligned}$$

hence as $k \to \infty$, $\beta \leq \psi_1(\beta) < \beta$, a contradiction. Thus $\{Tx_n\}$ is a forward-Cauchy sequence.

We may prove similarly that $\{Tx_n\}$ is a backward-Cauchy sequence. Indeed, for any $n \geq 0$, we have $d(Tx_{n+1}, fx_{n+1}) = d(Tx_{n+1}, Tx_n) > 0$, hence $M(x_i, x_j) > 0$ for any distinct $i, j \geq 0$. For any $n \geq 1$,

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &< \psi_1(M(x_{n+1}, x_n)) \\ &\leq M(x_{n+1}, x_n) \\ &= \max\left\{d(Tx_n, Tx_{n-1}), d(Tx_{n+1}, Tx_n), \tfrac{d(Tx_{n+1}, Tx_{n-1})}{2}\right\} \\ &\leq \max\left\{d(Tx_n, Tx_{n-1}), d(Tx_{n+1}, Tx_n)\right\}, \end{aligned} \tag{2.10}$$

hence for all $n \geq 1$,

$$\begin{cases} M(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ d(Tx_{n+1}, Tx_n) &< d(Tx_n, Tx_{n-1}). \end{cases} \tag{2.11}$$

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The sequence $\{d(Tx_{n+1}, Tx_n)\}_{n\geq 0}$ is decreasing and bounded below hence converges to some $\gamma \geq 0$, and $d(Tx_{n+1}, Tx_n) > \gamma$ for all $n \geq 0$. In fact, $\gamma = 0$ and $\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0$. To see this, assume $\gamma > 0$. Since $0 < \gamma < d(Tx_1, Tx_0)$, there are $\eta_1 > 0$ and $\zeta_1 \in \mathbb{N}$ such that $\psi_{\zeta_1}(t) \leq \gamma$ for all $t \in [\gamma, \gamma+\eta_1]$. By definition of γ, there is $\zeta_2 \in \mathbb{N}$ such that $M(x_{\zeta_2+1}, x_{\zeta_2}) = d(Tx_{\zeta_2}, Tx_{\zeta_2-1}) < \gamma+\eta_1$. Therefore, $d(Tx_{\zeta_2+\zeta_1}, Tx_{\zeta_2+\zeta_1-1}) < \psi_{\zeta_1}(M(x_{\zeta_2+1}, x_{\zeta_2})) \leq \gamma$, a contradiction.

If one assumes $\{Tx_n\}$ is not a backward-Cauchy sequence, there would exist $\epsilon > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that for all $k \in \mathbb{N}$, $m_k < n_k$, $d(Tx_{n_k}, Tx_{m_k}) \geq \epsilon$ and $d(Tx_{n_k-1}, Tx_{m_k}) < \epsilon$. By the triangle inequality,

$$d(Tx_{n_k}, Tx_{m_k}) \leq d(Tx_{n_k}, Tx_{n_k-1}) + d(Tx_{n_k-1}, Tx_{m_k}),$$

and as $k \to \infty$, $d(Tx_{n_k}, Tx_{m_k}) \to \epsilon$. Therefore, letting $k \to \infty$ in the following

$$\begin{aligned} d(Tx_{n_k}, Tx_{m_k}) &< \psi_1(M(x_{n_k}, x_{m_k})) \\ &= \psi_1\left(\max\left\{\begin{array}{l} d(Tx_{n_k-1}, Tx_{m_k-1}), d(Tx_{n_k-1}, Tx_{n_k}) \\ d(Tx_{m_k-1}, Tx_{m_k}), \frac{d(Tx_{n_k-1}, Tx_{m_k}) + d(Tx_{m_k-1}, Tx_{n_k})}{2} \end{array}\right\}\right), \end{aligned}$$

hence as $k \to \infty$, $\beta \leq \psi_1(\beta) < \beta$, a contradiction. Thus $\{Tx_n\}$ is a backward-Cauchy sequence.

The sequence $\{Tx_n\}$ is both forward and backward Cauchy hence it is a Cauchy sequence.

If $f(Y)$ is complete, then the Cauchy sequence $\{Tx_n\}$ as a Cauchy sequence in $f(Y)$ converges to some element $f(u)$, where $u \in Y$. If there is $n_0 \geq 0$ such that $M(x_{n_0}, u) = 0$, then $fu = Tu$. Otherwise, for each $n \in \mathbb{N}$,

$$\begin{aligned} d(Tx_n, Tu) &< \psi_1(M(x_n, u)) \\ &= \psi_1\left(\max\left\{d(fx_n, fu), d(fx_n, Tx_n), d(fu, Tu), \frac{d(fx_n, Tu) + d(fu, Tx_n)}{2}\right\}\right). \end{aligned}$$

Taking $n \to \infty$, $d(fu, Tu) \leq \psi_1(\max\{d(fu, Tu)\})$, hence $d(fu, Tu) = 0$ and $fu = Tu$. The same conclusion is easily obtained if $T(Y)$ is assumed to be complete.

Second step: We prove that if $Y = X$, then T and f have a unique common fixed point provided T and f are weakly compatible.

$Tu = fu$ and T and f are weakly compatible, hence $Tfu = fTu$ and in fact, $TTu = Tfu = fTu = ffu$.

Suppose $TTu \neq Tu$. Then $M(u, Tu) > 0$ and since $Tu \in (Tf^{-1})^0(Tu)$ and $T(Tu) \in (Tf^{-1})^0(T(Tu))$,

$$\begin{aligned} d(Tu, TTu) &\leq \psi_1(M(u, Tu)) \\ &= \psi_1\left(\max\left\{d(fu, fTu), d(fu, Tu), d(fTu, TTu), \frac{d(fu, TTu) + d(fTu, Tu)}{2}\right\}\right) \\ &= \psi\left(\max\{d(Tu, TTu)\}\right). \end{aligned}$$

Thus $d(Tu, TTu) = 0$, so $Tu = TTu = fTu$. Theorefore, $z := Tu$ is a common fixed point of T and f.

To prove the uniqueness of the common fixed point of T and f, we assume v to be another common fixed point of T and f such that $Tu \neq v$. Then:

$$\begin{aligned} d(z, v) = d(Tz, Tv) &\leq \psi_1(M(z, v)) \\ &\leq \psi_1\left(\max\left\{d(fz, fv), d(fz, Tz), d(fv, Tv), \frac{d(fv, Tz) + d(fz, Tv)}{2}\right\}\right) \\ &= \psi_1\left(\max\left\{d(z, v), \frac{d(v, z) + d(z, v)}{2}\right\}\right) \end{aligned}$$

which implies that

$$d(z,v) < \frac{d(v,z)+d(z,v)}{2}. \tag{2.12}$$

Similarly,

$$\begin{aligned} d(v,z) = d(Tv,Tz) &\leq \psi_1(M(v,z)) \\ &\leq \psi_1\left(\max\left\{d(fv,fz), d(Tv,fv), d(Tz,fz), \tfrac{d(Tv,fz)+d(Tz,fv)}{2}\right\}\right) \\ &\leq \psi_1\left(\max\left\{d(v,z), \tfrac{d(v,z)+d(z,v)}{2}\right\}\right) \end{aligned}$$

which implies that

$$d(v,z) < \frac{d(v,z)+d(z,v)}{2}. \tag{2.13}$$

Combining inequalities (2.12) and (2.13), $d(v,z)+d(z,v) < d(v,z)+d(z,v)$, a contradiction. Thus $z = v$: the common fixed point is unique. □

Note that Theorem 2.5 provides an erratum to the result of Singh et al. [14] in metric spaces:

Theorem 2.6. *Let (X,d) be a metric space and $T,f : Y \to X$ such that $T(Y) \subseteq f(Y)$. Let T be a GACMK with respect to f, i.e., such that for all $n \in \mathbb{N}$ and $x,y \in Y$ such that $M(x,y) > 0$,*

$$d(u,v) < \psi_n(M(x,y))$$

for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$M(x,y) = \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}.$$

and $\{\psi_n\}$ is a sequence of self-maps on $[0,\infty)$ such that:

(P_1) $\limsup_{n\to\infty} \psi_n(\epsilon) \leq \epsilon$ *for all* $\epsilon \geq 0$;

(P_2) *for each $\epsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \leq \epsilon$ for all $t \in [\epsilon, \epsilon+\delta]$.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

Proof. The Theorem follows from Theorem 2.5. The condition (G_4) is automatically satisfied in a metric space given the symmetry of a metric. □

A careful analysis of the proof of Theorem 2.5 reveals that condition (G_4) is not necessary if $M(x,y)$ is replaced with $\max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\}$. Therefore, we have the following result:

Theorem 2.7. *Let (X,d) be a quasi-metric space and $T,f : Y \to X$ such that $T(Y) \subseteq f(Y)$. Let T be a GACMK with respect to f, i.e., such that for all $n \in \mathbb{N}$ and $x,y \in Y$ such that $\{d(fx,fy), d(fx,Tx), d(fy,Ty)\} \neq \{0\}$,*

$$d(u,v) < \psi_n(\max\{d(fx,fy), d(fx,Tx), d(fy,Ty)\})$$

for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$M(x,y) = \max\left\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(fy,Tx)}{2}\right\}.$$

and $\{\psi_n\}$ is a sequence of self-maps on $[0,\infty)$ such that:

(G_1) $\limsup_{n\to\infty} \psi_n(\epsilon) \leq \epsilon$ *for all* $\epsilon \geq 0$;

(G_2) *for each $\epsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\psi_\nu(t) \leq \epsilon$ for all $t \in [\epsilon, \epsilon+\delta]$.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

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3 A fixed point theorem for asymptotic quasi-contractions

In this section, we consider the case where there is a real number $c \in (0,1)$ such that $\psi_n(t) = ct$ for all $t \in [0,\infty)$ and every $n \in \mathbb{N}$, albeit, $M(x,y)$ is further relaxed.

Theorem 3.1. *Let (X,d) be a quasi-metric space and $T, f : Y \to X$ such that $T(Y) \subseteq f(Y)$. Suppose also that there is $c \in (0,1)$ such that for all $n \in \mathbb{N}$ and $x, y \in Y$ with $M(x,y) > 0$,*

$$d(u,v) \leq cM(x,y) \tag{3.1}$$

for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$M(x,y) = \max\left\{ \begin{array}{l} d(fx,fy), d(fy,fx), d(fx,Tx), d(Tx,fx), d(fy,Ty), \\ d(Ty,fy), d(fx,Ty), d(Ty,fx), d(fy,Tx), d(Tx,fy) \end{array} \right\}. \tag{3.2}$$

If $T(Y)$ or $f(Y)$ is $\{T,f\}$-orbitally complete, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

Proof. Let $x_0 \in Y$ be arbitrary chosen. Since $T(Y) \subset f(Y)$, we can define a Jungck sequence $\{x_n\}$ of points in Y such that

$$Tx_n = fx_{n+1}, \quad n = 0,1,2,\ldots.$$

Step 1: Suppose $T(Y)$ or $f(Y)$ is $\{T,f\}$-orbitally complete. We want to show that T and f have a coincidence point. We distinguish two cases:

Case 1: If there is $n \geq 0$ such that $Tx_n = Tx_{n+1}$, then x_{n+1} is a coincidence point of T and f, since $fx_{n+1} = Tx_{n+1}$.

Case 2: Suppose now that $Tx_n \neq Tx_{n+1}$ for all $n \geq 0$. For any $n \geq 0$, we have $d(fx_{n+1}, Tx_{n+1}) = d(Tx_n, Tx_{n+1}) > 0$, hence $M(x_i, x_j) > 0$ for any distinct $i, j \geq 0$.

If we let $D(x,y) = \max\{d(x,y), d(y,x)\}$ for all $x, y \in X$, the pair (X,D) is a metric space and conditions (3.12) and (3.4) become for all $x, y \in X$ such that $N(x,y) > 0$:

$$D(u,v) \leq cN(x,y) \tag{3.3}$$

for all $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$N(x,y) = \max\{D(fx,fy), D(fx,Tx), D(fy,Ty), D(fx,Ty), D(fy,Tx)\}. \tag{3.4}$$

Let $n \geq 2$ and $i, j \geq 1$ such that $1 \leq i < j \leq n$.
Since $Tx_i \in (Tf^{-1})^{i-1}(Tx_1)$, $Tx_j \in (Tf^{-1})^{i-1}(Tx_{j-i+1})$, and $M(x_1, x_{j-i+1}) > 0$, we have that

$$\begin{aligned} D(Tx_i, Tx_j) &\leq cN(x_1, x_{j-i+1}) \\ &\leq c\max\left\{ \begin{array}{l} D(Tx_0, Tx_{j-i}), D(Tx_0, Tx_1), D(Tx_{j-i}, Tx_{j-i+1}), \\ D(Tx_0, Tx_{j-i+1}), D(Tx_{j-i}, Tx_1) \end{array} \right\} \\ &\leq c\,\mathbf{diam}(O_{T,f}(Tx_0, n)), \end{aligned} \tag{3.5}$$

where

$$\mathbf{diam}(O_{T,f}(Tx_0,n)) := \sup\{D(z_1,z_2) : \; z_1, z_2 \in O_{T,f}(Tx_0,n)\}.$$

In particular, for all $n \geq 2$,

$$D(Tx_1, Tx_n) \leq c\,\mathbf{diam}(O_{T,f}(Tx_0,n)).$$

Given $n \geq 1$, $O_{T,f}(Tx_0,n))$ is finite hence, there is $m \in \{2,3,\cdots,n\}$ such that

$$D(Tx_0, Tx_m) = \mathbf{diam}(O_{T,f}(Tx_0,n)). \tag{3.6}$$

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Now, for $n \geq 3$,

$$\begin{array}{rcl} D(Tx_1, Tx_m) & \leq & cN(x_1, x_m) \\ & = & c\max\left\{ \begin{array}{l} D(Tx_0, Tx_{m-1}), D(Tx_0, Tx_1), D(Tx_{m-1}, Tx_m), \\ D(Tx_0, Tx_m), D(Tx_{m-1}, Tx_1) \end{array} \right\} \end{array}$$

hence $m > 2$,

$$\max\left\{ \begin{array}{l} D(Tx_0, Tx_{m-1}), D(Tx_0, Tx_1), D(Tx_{m-1}, Tx_m), \\ D(Tx_0, Tx_m), D(Tx_{m-1}, Tx_1) \end{array} \right\} = D(Tx_0, Tx_m),$$

so $D(Tx_1, Tx_m) \leq cD(Tx_0, Tx_m)$ and

$$\begin{array}{rcl} D(Tx_0, Tx_m) & \leq & D(Tx_0, Tx_1) + D(Tx_1, Tx_m) \\ & \leq & D(Tx_0, Tx_1) + cD(Tx_0, Tx_m). \end{array}$$

Therefore,

$$D(Tx_0, Tx_m) \leq \frac{1}{1-c} D(Tx_0, Tx_1). \tag{3.7}$$

From (3.6) and (3.7), the sequence $\{\mathbf{diam}(O_{T,f}(Tx_0, n))\}_{n\geq 3}$ is bounded above; since it is also increasing, it converges to the positive number $\mathbf{diam}(O_{T,f}(Tx_0, \infty))$.

To show that $\{Tx_n\}$ is a Cauchy sequence, let $n, m \in \mathbb{N}$ with $n < m$. Since $Tx_n \in (Tf^{-1})^0(Tx_n)$, $Tx_m \in (Tf^{-1})^0(Tx_m)$, and $M(x_n, x_m) > 0$, we have that

$$\begin{array}{rcl} D(Tx_n, Tx_m) & \leq & cN(x_n, x_m) \\ & \leq & c\max\left\{ \begin{array}{l} D(Tx_{n-1}, Tx_{m-1}), D(Tx_{n-1}, Tx_n), \\ D(Tx_{m-1}, Tx_m), D(Tx_{n-1}, Tx_m), D(Tx_{m-1}, Tx_n) \end{array} \right\} \\ & \leq & c\ \mathbf{diam}(O_{T,f}(Tx_{n-1}, m-n+1)). \end{array} \tag{3.8}$$

Repeating the argument that led to (3.6), there is $m_1 \in \{2, 3, \ldots, m-n+1\}$ such that

$$D(Tx_{n-1}, Tx_{n-1+m_1}) = \mathbf{diam}(O_{T,f}(Tx_{n-1}, m-n+1)). \tag{3.9}$$

Thus, from (3.8),

$$\begin{array}{rcl} D(Tx_n, Tx_m) & \leq & c\ \mathbf{diam}(O_{T,f}(Tx_{n-1}, m-n+1)) \\ & = & cD(Tx_{n-1}, Tx_{n-1+m_1}) \\ & \leq & c^2\mathbf{diam}(O_{T,f}(Tx_{n-2}, m_1+1)) \\ & \leq & c^2\mathbf{diam}(O_{T,f}(Tx_{n-2}, m-n+2)) \end{array} \tag{3.10}$$

Repeating the process, one obtains:

$$\begin{array}{rcl} D(Tx_n, Tx_m) & \leq & c^n\mathbf{diam}(O_{T,f}(Tx_0, m)) \\ & \leq & c^n\mathbf{diam}(O_{T,f}(Tx_0, \infty)). \end{array}$$

Thus, as $n, m \to \infty$, $D(Tx_n, Tx_m) \to 0$, hence $\{Tx_n\}$ is a Cauchy sequence in X. $f(Y)$ or $T(Y)$ being $\{T, f\}$-orbitally complete, $\{Tx_n\}$ has a limit, say fu, where $u \in X$. Suppose that $fu \neq Tu$; then $N(u, x_n) > 0$ for all $n \geq 1$, and

$$\begin{array}{rcl} D(Tu, Tx_n) & \leq & cN(u, x_n) \\ & = & c\max\left\{ \begin{array}{l} D(fu, Tx_{n-1}), D(fu, Tu), D(Tx_{n-1}, Tx_n), \\ D(Tx_{n-1}, Tu), D(fu, Tx_n) \end{array} \right\}. \end{array} \tag{3.11}$$

Thus, as $n \to \infty$, $D(Tu, fu) \leq cD(fu, Tu)$, a contradiction. Thus $D(Tu, fu) = 0$ and $Tu = fu$. u is a coincidence point of T and f.

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Step 2: Now, we show that if $Y = X$, then T and f have a unique common fixed point.

$Tu = fu$, and T and f commute at their conicidence point u, hence $TTu = Tfu = fTu = ffu$. Therefore, if one supposes that $Tu \neq TTu$, then $N(u, Tu) > 0$ and

$$D(Tu, TTu) \leq cN(u, Tu) = cD(Tu, TTu),$$

a contradiction. Therefore $D(Tu, TTu) = 0$ and $TTu = Tu = fTu$: Tu is a common fixed point of T and f.

Step 3: To show that $z := Tu$ is the only common fixed point of T and f, suppose v is another a common fixed point of the pair $\{T, f\}$. We have that $z \neq u$ so $N(z, v) > 0$ and

$$D(z, v) = D(Tz, Tv) \leq cN(z, v) = cD(z, v),$$

a contradiction. Thus $D(z, v) = 0$ and $z = v$. □

The following corollary holds in metric spaces:

Corollary 3.2. *Let (X, d) be a metric space and $T, f : Y \to X$ mappings such that $T(Y) \subseteq f(Y)$. Suppose also that there is $c \in (0, 1)$ such that for all $n \in \mathbb{N}$ and $x, y \in Y$ with $M(x, y) > 0$,*

$$d(u, v) \leq cM(x, y) \tag{3.12}$$

for any $u \in (Tf^{-1})^{n-1}(Tx)$ and $v \in (Tf^{-1})^{n-1}(Ty)$, where

$$M(x, y) = \max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\}. \tag{3.13}$$

If $T(Y)$ or $f(Y)$ is $\{T, f\}$-orbitally complete, then T and f have a coincidence point. Further, if $Y = X$, then T and f have a unique common fixed point provided that T and f commute at a coincidence point.

4 Example

To validate the results obtained, we construct the following example of the applicability of Theorem 2.5.

Example 4.1. *Let $X = [1, 2]$ be equipped with the usual metric $d(x, y) = |x - y|$. Let $T, f : X \to X$ be defined by: $Tx = \dfrac{x + 2}{3}$ and $fx = \dfrac{x + 1}{2}$ for all $x \in [1, 2]$. Here, $T(X) = \left[1, \frac{4}{3}\right]$ and $f(X) = \left[1, \frac{3}{2}\right]$ hence $T(X) \subset f(X)$. Here, f is bijective, and $f^{-1}(x) = 2x - 1$ for all $x \in \left[1, \frac{3}{2}\right]$. Therefore, $(Tf^{-1})(x) = \dfrac{2x + 1}{3}$ for all $x \in \left[1, \frac{3}{2}\right]$ and for all $n \geq 0$, $(Tf^{-1})^n(x) = \left(\frac{2}{3}\right)^n (x - 1) + 1$ for all $x \in \left[1, \frac{3}{2}\right]$.*

$$(Tf^{-1})^n(x) = \begin{cases} 1 & \text{if } x = 1, \\ \left(\frac{2}{3}\right)^n (x - 1) + 1 & \text{if } x \in \left(1, \frac{3}{2}\right], \end{cases}$$

One can check that for all $n \geq 1$, and $x, y \in \left[1, \frac{3}{2}\right]$,

$$d((Tf^{-1})^n(x), (Tf^{-1})^n(y)) = \left(\frac{2}{3}\right)^n |x - y|.$$

In particular, for all $n \geq 0$ and $x, y \in [1, 2]$,

$$d((Tf^{-1})^{n-1}(Tx), (Tf^{-1})^{n-1}(Ty)) = \left(\frac{2}{3}\right)^{n-1} \left|\frac{x - y}{3}\right|.$$

Moreover, $d(fx, fy) = \frac{1}{2}|x-y|$ *for* $x \in [1,2]$*. Therefore, for all* $x, y \in [1,2]$ *such that* $M(x,y) > 0$,

$$d((Tf^{-1})^{n-1}(Tx), (Tf^{-1})^{n-1}(Ty)) = \left(\frac{2}{3}\right)^n d(fx, fy) < \psi_n(M(x,y)),$$

where

$$\psi_n(t) = 2\left(\frac{2}{3}\right)^n t \ \ \forall t \geq 0.$$

One can check that $\{\psi_n\}$ *satisfies conditions* (G_1) *and* (G_2) *of Theorem 2.5 since for all* $\epsilon \geq 0$, $\limsup_{n\to\infty} \psi_n(\epsilon) = 0 \leq \epsilon$*, and if* $\epsilon > 0$, $\delta = \frac{\epsilon}{8}$*, and* $\nu = 2$*, then* $\psi_\nu(t) \leq \epsilon$ *for all* $t \in [\epsilon, \epsilon+\delta]$.

In addition, T *and* f *commute at their only coincidence point* $x = 1$ *hence all the conditions of Theorem 2.5 hold. Therefore,* T *and* f *have a unique common fixed point. It is easily seen that* $x = 1$ *is the only common fixed point of* T *and* f*. Given* $x_0 \in [1,2]$*, consider the Jungck sequence* $Tx_n = fx_{n+1}$*. In the uninteresting case that* $x_0 = 1$, $x_n = 1$ *for any* $n \geq 1$*. Now, let* $x_0 \in (1,2]$*. Then* $3x_{n+1} = 2x_n + 1$ *for all* $n \geq 0$*. A quick computation gives the following formula for any* $n \geq 1$*:*

$$x_n = \left(\tfrac{2}{3}\right)^n (x_0 - 1) + 1$$

$$Tx_n = \tfrac{1}{3}\left(\tfrac{2}{3}\right)^n (x_0 - 1) + 1.$$

Thus $\{Tx_n\}$ *converges to* $T(1) = 1$*, the common fixed point of the pair* $\{T, f\}$.

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