IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
10(2), 2024, PAGES 1 - 8
HTTPS://DOI.ORG/10.5281/ZENODO.10936738

On Collapse Of Order-Preserving and Idempotent of Order-Preserving Full Contraction Transformation Semigroup

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Article Info
Received: 23 December 2023 Revised: 08 March 2024
Accepted: 11 March 2024 Available online: 11 April 2024

Abstract

Let T_n be the set of full transformation semigroup on $X_n = \{1, 2, 3, \cdots, n\}$, $C^+(\alpha) =| OCT_n |= q$ be it's subsemigroup on collapse of order-preserving full contraction transformation, $C^+(\alpha_E) =| E(OCT_n) |= q_E$ be the collapse on idempotent of order-preserving full contraction transformation and $C^+(\alpha) =| C^+(\alpha) |= \cup_{t \in Im\alpha} | \{t\alpha^{-1} \geq 2\} |$ be the formula for total number of collapsible element.In this paper, we investigate the collapse element on order-preserving and idempotent of order-preserving full contraction transformation semigroup.

Keywords: Collapse,Contraction, Full Transformation semigroup, Idempotent, Order preserving.
MSC2010: 06F20.

1 Introduction

A semigroup is an algebraic structure conforming of a set together with an associative binary operation. The binary operation of a semigroup is most constantly denoted multiplicative x.y, or exclusively xy, denotes the result of applying the semigroup operation to the ranged ordered pair(x, y). Associativity is formally ventilated as that(xy) z = x(yz) for all x, y and z in the semigroup. The name " semigroup " originates in the fact that a semigroup generalizes a group by husbanding only associativity and check under the binary operation from the axioms defining a group. From the contrary point of prospect(of adding preferably than removing axioms), a semigroup is an associative magma. As in the case of groups or magmas, the semigroup operation need not be commutative, consequently xy is not inevitably equal to yx; a true illustration of associative but non-commutative operation is matrix multiplication. However, such semigroup is called a commutative semigroup or(less constantly than in the analogous case of groups)If the semigroup operation is commutative.it is called an abelian semigroup

A transformation $\alpha \in T_n$ is said to be full contraction transformation semigroup if $\mid x\alpha - y\alpha \mid \leq \mid x - y \mid$ for all $x, y \in X_n$. The set of all order- preserving full contraction transformation semigroup is denoted by OCT_n and it is the subsemigroup of T_n.

An element $e \in T_n$ is said to be an idempotent in full transformation if and only if $e^2 = e$. Interestingly, such idempotent that satisfy the contraction transformation condition is said to be an idempotent contraction. Various enumerative problems have been considered for certain classes of semigroups. For example, it is well known that P_n has order $(n+1)^n$. Also the number of idempotents in P_n is given by

$$|E(P_n)| = \sum_{r=0}^{n} \begin{pmatrix} n \\ r \end{pmatrix} (r+1)^{n-r},$$

as obtained by Garba [1], and the number of nilpotent for P_n is given by

$$|N(P_n)| = (n+1)^{n-1}$$

which is deduced from [2,3]. The collapsible element for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ for all $n \geq 2$ $(n \in \mathbb{N})$ in T_n was studied by [4], while [5] studied the collapsible element for $|t\alpha^{-1}| = 2$ and $|t\alpha^{-1}| = 3$ for all $n \geq 2$ $(n \in \mathbb{N})$ in P_n.

2 Preliminaries and Literature Review

Contraction transformation semigroup has been truthful over the years see [4–16] another important class of semigroup has also arouse interest and this class of semigroup is order-preserving and order-decreasing see [2,3,7,12,16,17], also many researchers study the idempotent of these classes of semigroup see [1,12,18,19], the collapse of transformation semigroup was also studied by [20,21].

The study of Umar [22] showed the combinatorial problems in the theory of symmetric inverse semigroup and some relevant results from their his work are:
Proposition 2.1 [22]
Let $S = I_n$, then $F(n;\ p,\ k) = (n, p)(k-1, p-1)p! \forall\ n \geq k \geq p \geq 0$
Corollary 2.2 [22]
Let $S = I_n$, then $F(n,\ p) = (n, p)^2 p!$ for all $n \geq p \geq 0$.
The algebraic and combinatorial properties of DP_n(Subsemigroup of partial Isometries), ODP_n(Subsemigroup of order-preserving partial Isometries), and $ODDP_n$(Subsemigroup of order-preserving order-decreasing partial Isometries) was studied by [10] and some of the results obtained are:
$DP_n = \frac{2(2n-p+1)}{(p+1)}(n, p)$where $p \geq 1,\ n^2(p) = 1,\ 1(p) = 0$
$ODP_n = \frac{2(2n-p+1)}{(p+1)} \begin{pmatrix} n \\ p \end{pmatrix}$
$ODDP_n = \begin{pmatrix} n+1 \\ p+1 \end{pmatrix}$, if $p \geq 1$ and $ODDP_n = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ if $p = 1$
Some combinatorial results obtained by [11] on $ORCT_n$(Order reversing full contraction transformation) and $ODCT_n$(order decreasing full contraction transformations) are presented below:
Corollary 2.3 [11]
Let $S = ORCT_n$, then $|S| = |ORCT_n| = (n+1)2^{(n-1)} - n$, for $n \geq 1$
Corollary 2.4 [11]
Let $S = ODCT_n$, then
$F(n,\ k) = \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} for\ k \geq 1$
$F(n,\ m) = 2^{(n-m-1)}, for\ n \geq m \geq 1$

$F(n,\ p) = \begin{pmatrix} n-1 \\ p-1 \end{pmatrix}, for\ p \geq 1.$

[11] worked on a research of combinatorial result for certain semigroups of order-preserving full contraction mappings of a finite chain where some result were also generated.
The study of [23] showed that full transformation semigroup is metricizable. Suppose $(X,\ d)$ is a metric space for

$$D(a,\ b) = \begin{cases} 0 & a = b \\ n \in \mathbb{N} & a \neq b \end{cases}$$

for all $a, b \in S$. Then the distance between a point x, and itself is zero:
$D(a, b) = 0$ iff $a = b$ shows that
If $a - b = 0$ then $a = b$
If $a - b\ \neq 0$ then $a \neq b$ shows that
$0 \leq D(a,\ b) \leq n.$
The property of alternating semigroup was investigated by [21]. Some of their result which are relevant to this dissertation work are:

1. Let $S = A_n^c$, then $F(n, p_{n-1}) = \frac{n^2(n-1)!}{2}$

2. Let $S = A_n^c$, then $F(n,p) = \begin{cases} \frac{n!}{2} & p = n \\ \begin{pmatrix} n \\ p \end{pmatrix}^2 p! & 0 \leq p \leq n-2 \end{cases}$

Some results obtained on some signed semigroup of order preserving transformation by [18] which are relevant to the study are listed. Let T_n be the set of full transformation and P_n be the set of partial transformations. A transformation T_n is said to be order-preserving if for all $i, j \in \{1, 2, 3, ..., n\}$; $i \leq j \implies x_i \leq x_j$

Definition 1.1: Collapse In Transformation Semigroup S, an element α in S is collapsible, $c(\alpha)$ if there exists a number $C^+(\alpha) =\mid t\alpha^{-1} \mid \geqslant 2$ where t is an element in the image of α. [21]

Definition 1.2: Idempotent An element $e \in S$ is idempotent if $e^2 = e$, a full transformation e is idempotent if and only $Ime = F(e)$, such that $F(e)$ is the set of all fixed point of the full transformation and $Im(e)$ is the image set of the Transformation. [8]

Definition 1.3: Full Transformation semigroup A transformation $\alpha : Dom(\alpha) \subset X_n \longrightarrow Im(\alpha) \subset X_n$ is said to be full or total if $Dom\alpha = X_n$; otherwise it is called strictly partial. [16]

Definition 1.4: Order-preservingA transformation $\alpha \in T_n$ is said to be order-preserving if $(\forall x, y \in Dom\alpha) x \leq y \Rightarrow x\alpha \leq y\alpha (x\alpha \geq y\alpha)$. [17]

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3 Main Results

Table 1: Collapse in order-preserving full contraction transformation Semigroup $C^{+}(\alpha) =\mid OCT_n \mid$

n/q	0	1	2	3	4	5	$\sum F(n;q) =\mid OCT_n \mid = q$
1	1						1
2	1	0	2				3
3	1	0	4	3			8
4	1	0	6	6	7		20
5	1	0	8	8	15	13	45

Theorem 3.1. *Let $S =\mid OCT_n \mid$ then $S = 2^{n-2}(n+1)$ for $n = 1, 2, 3, 4$*

Proof. Let $n = m + 1 \Rightarrow m = n - 1$ and let $2^m \leq m^2$
We proof by mathematical induction we are to show that

$$2^{n-2}(n+1) = 2^{m-1}(m+2) \tag{3.1}$$

now, by multiplying $\log_2$ on both side of eqn (3.1) we have

$$\begin{aligned}
(n-2)(n+1) =&(m-1)(m+2) \\
=&m^2 + m - 2 \\
<&2^m - 2 + m \quad since \quad 2^m \leq m^2 \\
=&2^m - 4 + m + 2 \\
=&2^m - 2^2 + m + 2 \\
<&2^m - 2^m(m+2) \quad since \quad 2^m \leq m^2 \Rightarrow 2^m(m+2) \leq m^2(m+2) \\
=&2^{m-1}(m+2) \\
=&2^{n-2}(n+1)
\end{aligned} \tag{3.2}$$

by replacing the value of m=n-1
hence the proof is complete by eqn (3.1) □

International Journal of Mathematical Sciences and Optimization: Theory and Applications
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Table 2: Collapse on idempotent of order-preserving full contraction transformation Semigroup $C^{+}(\alpha_E) =\mid E(OCT_n) \mid$

n/q_E	0	1	2	3	4	5	$\sum F(n;q_E) =\mid E(OCT_n) \mid = q_E$
1	1						1
2	1	0	2				3
3	1	0	2	3			6
4	1	0	2	2	5		10
5	1	0	2	2	4	7	16

Theorem 3.2. *Let* $S =\mid E(OCT_n) \mid$ *then* $S = \binom{n+1}{2}$ *for* $n = 1, 2, 3, 4$

Proof. by pascal identity for positive natural number n and k

$$\binom{n}{m} + \binom{n}{m-1} = \binom{n+1}{m} \tag{3.3}$$

if $m \neq n+1$ then $m-1 \neq n$ thus

$$(x+y)^{n+1} = \sum_{m=0}^{n+1} \binom{n+1}{m} x^{n+1-m} y^m \tag{3.4}$$

for x=1 and y=1 we have

$$(1+1)^{n+1} \sum_{m=0}^{n+1} \binom{n+1}{m} 1^{n+1-m} 1^m \tag{3.5}$$

$$2^{n+1} \sum_{m=0}^{n+1} \binom{n+1}{m} = \binom{n+1}{2} \tag{3.6}$$

□

Lemma 3.3. *Let* $S =\mid OCT_n \mid + \mid E(OCT_n) \mid$ *for all* $n = 5$ *then*

$$S = 2^{n-1}\left[\frac{(n+1)}{2} + 1\right] - 3$$

Proof. The proof follows from the consequence of table 1 and 2 for all $n = 5$ □

Table 3: Formulars obtained on Collapse of order-preserving and idempotent of order-preserving full contraction transformation semigroup

$C^{+}(\alpha)$	formular
$\mid OCT_n \mid$	$2^{n-2}(n+1) \quad \forall n \leq 4$
$\mid E(OCT_n) \mid$	$\binom{n+1}{2} \; \forall n \leq 4$
$\mid OCT_n \mid + \mid E(OCT_n) \mid$	$2^{n-1}\left[\frac{(n+1)}{2}+1\right]-3 \quad \forall n=5$

Note The formulars obtained for $|OCT_n|$ and $|E(OCT_n)|$ are similar with the result of $ODDT_n$ on the work of [15] and OCP_n(subsemigroup of order-preserving partial contraction mapping) on the work of [10] respectively.

4 Discussion and Conclusion

The focus of this paper is about some combinatorial and algebraic properties of collapse on order-preserving and idempotent of order-preserving full contraction transformation semigroup. The paper defines some concepts such as semigroup, transformation, contraction, order-preserving, idempotent, and collapse, and gives some examples and formulas for them.The results obtained in this studies unify existing and gives new results in combinatorials which are presented in two theorems, one Lemma and their proofs, one for the number of elements in the semigroup of order-preserving full contraction transformation, and one for the number of elements in the subsemigroup of idempotent of order-preserving full contraction transformation. The results conclude that order-preserving and idempotent of order-preserving full contraction transformation semigroup is an area of research study in the theory of transformation semigroup and more useful research can be carried out in this area of study.The paper contains many references to previous works on related topics, such as partial transformation semigroups, order-decreasing transformations, alternating semigroups, metricization, and nildempotency.

5 Acknowledgments

Gp Capt. ISA DANJUMA ADAMU and NUHU DANTALA ADAMU (First Author's parents) are duly acknowledge . And to also thank the editors and reviewers for their valuable suggestion and comment that helped to improve this manuscript.

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