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# Isotopism Classes of 2-dimensional Leibniz Algebras

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**Abstract**

This paper gives the classification of two-dimensional Leibniz algebras into isotopism classes. The matrix structure constants of the algebras were used to determine the isotopism between them. An algorithm was developed for n-dimensional Leibniz to achieve this objective. The algorithm was tested on the Leibniz algebras of dimension two over $\mathbb{Z}_2$. From the result obtained, it was observed that there is no isotopism between the algebras. Consequently, we conclude that isotopism and isomorphism are equivalent in this case.

**Keywords:** Isomorphism, Isotopism, Leibniz algebra, Structure constant.
**MSC2010: 08A05.**

## 1 Introduction

Leibniz algebras are non-commutative analog of Lie algebras and were first introduced by Bloh [1] and were called *D*-algebras. Loday in 1990 re-introduced the algebras and called it Leibniz algebra [2], which was actively studied by several researchers. The theory of these algebras and many results on Lie algebras were also extended to Leibniz algebras. [3–5, 22].

Generally, in the classification of algebras, isomorphism criterion is basically used to give it classification into isomorphism classes without considering other properties that might be shared by the non-isomorphic algebras [10]. The study of isotopism helps us describe some common properties of non-isomorphic algebras not captured by the isomorphism criterion.

The problem of classifications of Leibniz algebras into isotopism classes can be reduced to that of gathering together isomorphism classes with isotopic equivalence class representatives [6].

The classification of Leibniz algebras into isotopism classes is very interesting because it allows the combination of isomorphic and non-isomorphic algebras that share some properties. Isotopisms of Leibniz algebras constitute the generalization of isomorphisms of such algebras.

Isotopism of algebras was first introduced in 1942 by Albert in [6] as a generalization of isomorphism classes of algebras.

Isotopism has been used in the classification of some varieties of algebras; among them are Jordan algebras [10, 11], division algebras [9], alternative algebras [12, 15], absolute value algebras

[7,8], and more recently, classes of some family $F_n^p$ of Lie algebras over finite fields were considered by Falcón [14]. The descriptions into isotopism classes filiform Lie algebras up to dimension seven [13].This paper addresses the classification up to isotopism of 2-dimensional Leibniz algebras over $\mathbb{Z}_2$.

## 2 Preliminaries

One of the algebras of great importance in study of Loday algebras is the Lie algebra $L$, which is defined as follows:

**Definition 2.1.**
A Lie algebra $L$ is a vector space over a field $\mathbb{K}$ equipped with bilinear map, $[\cdot,\cdot] : L \times L \to L$ satisfying the following conditions:

$$[x, x] = 0,$$
$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0,$$

for all $x, y, z \in L$.

There is a notion of non-commutative Lie algebra called the Leibniz algebra. These algebras are defined by the following property:

**Definition 2.2.**
An algebra $L$ over a field $\mathbb{K}$ is called a Leibniz algebra if its bilinear operation $[\cdot,\cdot]$ satisfies the following Leibniz identity:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] \text{ for all } x, y, z \in L.$$

In fact, this is a right Leibniz algebra. For left Leibniz algebras, the identity has the following form:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \text{ for all } x, y, z \in L.$$

However, in this paper we consider the right Leibniz algebras calling them just Leibniz algebras.

Let $n$ be the dimension of Leibniz algebra $L$ and $\{e_1, e_2, ...e_n\}$ be its basis. Let define:

$$x = \sum_{i=1}^{n} \alpha_i e_i, \quad y = \sum_{j=1}^{n} \beta_j e_j.$$

The multiplication $[x, y]$ of two elements $x$ and $y$ in $L$, is completely determined by the multiplication $[e_i, e_j]$ by the pairs of basis elements, that is

$$[x, y] = \sum_{i,j=1}^{n} \alpha_i \beta_j [e_i, e_j].$$

To know $[x, y]$ it is enough to find the product $[e_i, e_j]$. Since $[e_i, e_j] \in L$, we write $[e_i, e_j]$ as a linear combination of $\{e_1, e_2, ...e_n\}$. That is

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k \qquad \forall\ i, j = 1, 2, ..., n.$$

in which $(c_{ij}^k) \in \mathbb{K}^{n^3}$. The numbers $c_{ij}^k \in \mathbb{K}$ are called the structure constants with respect to the basis $\{e_1, e_2, ...e_n\}$, see [16, 17] for more details on structure constants and multiplication of algebras.

Let $L$ be two dimensional Leibniz algebras over $\mathbb{F}$ with multiplication given by its bilinear map. Once a basis is fixed, we can identify the law $[\cdot,\cdot]$ with the structure constants. Then one can represent the bilinear map by matrix structure constant as follows:

$$L = c_{i,j}^{k} = \begin{pmatrix} c_{1,1}^{l} & c_{1,2}^{l} & c_{2,1}^{l} & c_{2,2}^{l} \\ c_{1,1}^{2} & c_{1,2}^{2} & c_{2,1}^{2} & c_{2,2}^{2} \end{pmatrix} \in Mat\,(2\times 4; F)\,, \quad \forall\ i,j = 1,2.$$

See [23] for more details on matrix structure constants.

**Definition 2.3.**
The annihilator of a Leibniz algebra $L$ is the set

$$ann\,(L) = \{x \in L : [x,y] = [y,x] = 0, \quad \text{for all} \ \ y \in L\}$$

**Definition 2.4.**
Two algebras $(L,[\cdot,\cdot])$ and $(L',[\cdot,\cdot])$ are said to be isotopic if there exist three non-singular linear transformations $\alpha$, $\beta$ and $\gamma$ from $L$ to $L'$ such that

$$[\alpha(x),\beta(y)] = \gamma([x,y]) \qquad \forall\ x,y \in L. \tag{2.1}$$

It is denoted by $L \simeq L'$ and the triple $(\alpha,\beta,\gamma)$ is said to be an isotopism between the algebras $L$ and $L'$ [18, 19].

**Remark 2.5.**

1. If $\gamma$ is the identity transformation, then isotopism is called principal isotopism.
2. If $\alpha = \beta$, then the isotopism $(\alpha,\alpha,\gamma)$ is a strong isotopism between $L \simeq L'$.
3. If $\alpha = \beta = \gamma$, then the isotopism $(\alpha,\alpha,\alpha)$ is an isomorphism, and is denoted by $L \cong L'$.

The list of 2-dimensional leibniz algebras upto isomorphism were presented Theorem 2.1 [21]:

**Theorem 2.1.**
Let $L$ be a two-dimensional Leibniz algebras over $\mathbb{Z}_2$. Then it is isomorphic to one of the following pairwise non-isomorphic Leibniz algebras.

$L_{2,1}$ : Abelian;

$L_{2,2} : [e_1,e_2] = e_1,\ [e_2,e_1] = e_1;$

$L_{2,3} : [e_2,e_2] = e_1;$

$L_{2,4} : [e_1,e_2] = e_1,\ [e_2,e_2] = e_1.$

Theorem 2.2 below represents the matrix structure constants of the algebras in Theorem 2.1. For further details, see [20, 23] on matrix representation of algebras.

**Theorem 2.2.**
Any non-trivial 2-dimensional Leibniz algebra over $\mathbb{Z}_2$ is isomorphic to any one of the following listed, by their matrices of structure constants, such algebras:

$$L_{2,1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$L_{2,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$L_{2,3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

$$L_{2,4} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

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# 3 Leibniz Algebra's Isotopism

In this section, we will introduce some important theorems offering insight into the nature of isotopism in Leibniz algebras. The aim is to foster a profound understanding of these algebraic structures, ultimately paving the way for broader applications and insight into its fundamentals.

**Theorem 3.1.**
Let $(\alpha, \alpha, \gamma)$ be an isotopism between two algebras $L$ and $L'$, then the image by $\alpha$ of the annihilator, $Ann(L)$ is equal to the annihilator, $Ann(L')$, i.e.,

$$\alpha(Ann(L)) = Ann(L')$$

*Proof.* Since $\alpha$ is a non-singular linear transformation, it follows that $\alpha(L) = L'$.
Considering $x \in Ann(L)$, then

$$[\alpha(x), y] = \gamma([x, \alpha^{-1}(y)]) = 0 \qquad \forall\ y \in L'.$$

Similarly,

$$[y, \alpha(x)] = \gamma([\alpha^{-1}(y), x]) = 0 \qquad \forall\ y \in L'.$$

Hence,

$$\alpha(Ann(L)) \subseteq Ann(L'). \tag{3.1}$$

Taking the inverse isotopism $(\alpha^{-1}, \alpha^{-1}, \gamma^{-1})$, between $L'$ and $L$ it can be inferred that

$$\alpha(Ann(L')) \subseteq Ann(L'). \tag{3.2}$$

From equations (3.1) and (3.2) equality holds. □

**Theorem 3.2.**
Isotopism of Leibniz algebras is an equivalent relation.

*Proof.* 1. . Let $(L, [\cdot,\cdot])$ be a Leibniz algebra and $\alpha$, $\beta$ and $\gamma$ be nonsingular linear transformations from $L$ to $L$ such that $L \simeq L$. Let $\alpha = \beta = \gamma = Id_L$. Then for all $x, y \in L$ we have

$$[Id_L(x), Id_L(y)] = Id_L[x, y] = [x, y] = Id_L[x, y] = \gamma[x, y].$$

Therefore, the relation $\simeq$ is reflexive.

2. Let $(L, [\cdot,\cdot])$ and $(L', [\cdot,\cdot])$ be two Leibniz algebras. Suppose that $L \simeq L'$, then there exist a non-singular linear transformation $\alpha, \beta$ and $\gamma$ from $L$ to $L'$ such that $(\alpha, \beta, \gamma)$ is an isotopism, i.e,

$$[\alpha(x_1), \beta(y_1)] = \gamma[x_1, y_1] \qquad x_1, y_1 \in L. \tag{3.3}$$

We want to prove that there exist an isotopism $(\alpha', \beta', \gamma')$ from $L'$ to $L$, such that

$$[\alpha'(x_2), \beta'(y_2)] = \gamma'[x_2, y_2] \qquad x_2, y_2 \in L'.$$

Let

$$\alpha' = \alpha^{-1} : L' \mapsto L,$$
$$\beta' = \beta^{-1} : L' \mapsto L,$$
$$\gamma' = \gamma^{-1} : L' \mapsto L.$$

Put,

$$\alpha(x_1) = y_1 \qquad \beta(x_2) = y_2 \qquad \forall\ x_1, x_2 \in L,\ y_1, y_2 \in L'.$$

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in equation 3.3, we have

$$\begin{aligned}[\alpha'(y_1), \beta'(y_2)] &= [\alpha'(\alpha(x_1), \beta'((\beta y_1))] = [\alpha^{-1}(\alpha(x_1), \beta^{-1}((\beta y_1))] \\ &= [x_1, x_2] = [\gamma^{-1}(\gamma(x_1, x_2))] = [\gamma'(\gamma(x_1, x_2))] \\ &= \gamma'[(\alpha(x_1), \beta(x_2))] = \gamma'[y_1, y_2]\end{aligned}$$

Therefore, $L' \simeq L$. Hence the relation $\simeq$ is symmetric.

3. Suppose that $(L, [\cdot, \cdot])$, $(L', [\cdot, \cdot])$ and $(L'', [\cdot, \cdot])$ be Leibniz algebras. Suppose that $L \simeq L'$ and $L' \simeq L''$ then there exist an isotopism $\alpha_1, \beta_1$ and $\gamma_1$ from $L$ to $L'$ and $\alpha_2, \beta_2$ and $\gamma_2$ from $L'$ to $L''$. That is

$$[\alpha_1(x_1), \beta_1(x_2)] = \gamma_1[x_1, x_2] \qquad \forall\ x_1, x_2 \in L$$

$$[\alpha_2(y_1), \beta_2(y_2)] = \gamma_2[y_1, y_2] \qquad \forall\ y_1, y_2 \in L'$$

To prove that $L \simeq L''$, we need need to prove that

$$[\alpha_3(z_1), \beta_3(z_2)] = \gamma_3[z_1, z_2] \qquad \forall\ z_1, z_2 \in L_1 \tag{3.4}$$

for some nonsingular linear transformations $(\alpha_3, \beta_3, \gamma_3)$ from $L_1$ to $L_3$. Let

$$\alpha_3 = \alpha_2 \circ \alpha_1 : L \mapsto L'',$$

$$\beta_3 = \alpha_2 \circ \alpha_1 : L \mapsto L'',$$

$$\gamma_3 = \alpha_2 \circ \alpha_1 : L \mapsto L''.$$

Taking the $RHS$ of equation (3.4) we obtained

$$\begin{aligned}\gamma_3[z_1, z_2] =&(\gamma_2 \circ \gamma_1)[z_1, z_2] \\ =&\gamma_2(\gamma_1[z_1, z_2]) = \gamma_2[\alpha_1(z_1), \beta_1(z_2)] \\ =&[\alpha_2(\alpha_1(z_1)), \beta_2(\beta_1(z_2)] \\ =&[\alpha_2 \circ \alpha_1(z_1)), (\beta_2 \circ \beta_1(z_2)] \\ =&[\alpha_3(z_1), \beta_3(z_2)].\end{aligned}$$

Thus, $L \simeq L''$ which implies that it is transitive. Consequently, the relation $\simeq$ is an equivalent relation.

□

**Theorem 3.3.**
Let $L$ and $L'$ be two $n$-dimensional Leibniz algebras on a vector space $V = Span\{e_1, ..., e_n\}$. Then $L$ and $L'$ are isotopic if and only if there exists nonsingular linear transformations $\alpha, \beta$ and $\gamma$ from $L$ to $L'$ such that

$$[\alpha(e_i), \beta(e_j)] = \gamma([e_i, e_j]). \tag{3.5}$$

*Proof.* Suppose that $L$ and $L'$ be two isotopic Leibniz algebras, we need to show that (3.5) is an isotopism. Defining:

$$x = \sum_{i=1}^{n} c_i e_i \ \text{ and } \ y = \sum_{j=1}^{n} d_j e_j \ \text{ where } c_i,\ d_j \in \mathbb{K}.$$

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We get that

$$\begin{aligned}
[\alpha(x),\beta(y)] =& \Big[\alpha\sum_{i=1}^{n} c_ie_i,\ \beta\sum_{j=1}^{n} d_je_j\Big] = \Big[\sum_{i=1}^{n} c_i\alpha(e_i),\ \sum_{j=1}^{n} d_j\beta(e_j)\Big]\\
=&\sum_{i,j=1}^{n}\Big[c_i\alpha(e_i),\ \beta(e_j)\Big] = \sum_{i,j=1}^{n} c_id_j\Big[\alpha(e_i),\ \beta(e_j)\Big]\\
=&\sum_{i,j=1}^{n} c_id_j\gamma\Big(\Big[e_i,\ e_j\Big]\Big) = \sum_{i,j=1}^{n}\gamma\Big(\Big[c_ie_i,\ d_je_j\Big]\Big)\\
=&\gamma\Big(\sum_{i,j=1}^{n}\Big[c_ie_i, d_j,\ e_j\Big]\Big)\\
=&\gamma\Big(\Big[\sum_{i=1}^{n} c_ie_i,\ \sum_{j=1}^{n} d_je_j\Big]\Big)\\
=&\gamma([x,\ y]).
\end{aligned}$$

Conversely,

$$\begin{aligned}
\gamma([x,y]_1) =& \gamma\Big(\Big[(\sum_{i=1}^{n} c_ie_i),\ (\sum_{j=1}^{n} d_je_j)\Big]\Big) = \sum_{i,j=1}^{n}\gamma\Big(\Big[c_ie_i,\ d_je_j\Big]\Big)\\
=& \sum_{i,j=1}^{n} c_id_j\gamma\Big(\Big[e_i,\ e_j\Big]\Big) = \sum_{i,j=1}^{n} c_id_j\Big[\alpha(e_i),\ \beta(e_j)\Big]\\
=& \Big[\sum_{i=1}^{n} c_i\alpha(e_i),\ \sum_{j=1}^{n} d_j\beta(e_j)\Big]\\
=& \Big[\alpha\sum_{i=1}^{n} c_ie_i,\ \beta\sum_{j=1}^{n} d_je_j\Big]\\
=& [\alpha(x),\ \beta(y)].
\end{aligned}$$

□

# 4 Main Result

In realm of algebraic structures, the concept of isotopism plays a crucial role in revealing hidden connections between seemingly disparate algebras. Let's delve into the main results that formalizes this concepts, shedding light on the nature of isotopism and its impact on algebraic structures.

## 4.1 Classification algorithm

In this section we present an algorithm for the classification of $n$-dimensional Leibniz algebras into isotopism classes. The algorithm is presented in the following

**Proposition 4.1.**
Let $L$ and $L^{'}$ be two isotopic Leibniz algebras on a vector space $V$ with matrices structure constant $\{d_{rs}^{t}\}$ and $\{c_{ij}^{k}\}$ respectively. Suppose that $(\alpha,\beta,\gamma)$ is an isotopism between both algebras related, respectively, to the matrices $\alpha=\alpha_{ri}$, $\beta=\beta_{sj}$ and $\gamma=\gamma_{tk}$. Then,

$$\sum_{r,s,t=1}^{n}\alpha_{ri}\beta_{sj}d_{rs}^{t} = \sum_{t,k=1}^{n} c_{ij}^{k}\gamma_{tk} \qquad \forall\ i,j=1,\cdots n.$$

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*Proof.* From definition (2.4) above, we have

$$[\alpha(e'_i),\ \beta(e'_j)] = \gamma([e'_i, e'_j]) \tag{4.1}$$

Suppose that

$$\alpha(e'_i) = \sum_{r=1}^{n} \alpha_{ri} e_r, \qquad \beta(e'_j) = \sum_{s=1}^{n} \beta_{sj} e_s, \qquad \gamma(e'_k) = \sum_{t=1}^{n} \gamma_{tk} e_k.$$

and considering the RHS of equation (4.1), it follows that:

$$\begin{aligned}
\gamma([e'_i, e'_j]) &= \gamma(\sum_{k=1}^{n} c_{ij}^k e'_k) \\
&= \gamma(c_{ij}^1 e'_1 + c_{ij}^2 e'_2 + \cdots + c_{ij}^n e'_n) \\
&= c_{ij}^1 \gamma(e'_1) + c_{ij}^2 \gamma(e'_2) + \cdots + c_{ij}^n \gamma(e'_n) \\
&= c_{ij}^1 (\gamma_{11} e_1 + \gamma_{21} e_2 \cdots + \gamma_{n1} e_n) \\
&+ c_{ij}^2 (\gamma_{12} e_1 + \gamma_{22} e_2 \cdots + \gamma_{n2} e_n) \\
&+ \cdots + c_{ij}^n (\gamma_{1n} e_1 + \gamma_{22} + \cdots + \gamma_{nn} e_n) \\
&= (c_{ij}^1 \gamma_{11} + c_{ij}^2 \gamma_{12} + \cdots + c_{ij}^n \gamma_{1n}) e_1 \\
&+ (c_{ij}^1 \gamma_{21} + c_{ij}^2 \gamma_{22} + \cdots + c_{ij} \gamma_{2n}) e_2 \\
&+ \cdots + (c_{ij}^1 \gamma_{n1} + c_{ij}^2 \gamma_{n2} + \cdots + c_{ij} \gamma_{nn}) e_n \\
&= \mathcal{A}_1 e_1 + \mathcal{A}_2 e_2 + \cdots + \mathcal{A}_n e_n.
\end{aligned} \tag{4.2}$$

From the LHS of the equation (4.1), we get that:

$$\begin{aligned}
[\alpha(e'_i),\ \beta(e'_j)] &= [\sum_{r=1}^{n} \alpha_{ri} e_r, \sum_{s=1}^{n} \beta_{sj} e_s] = \sum_{r,s=1}^{n} \alpha_{ri} \beta_{sj} [e_r, e_s] \\
&= \alpha_{11}\beta_{11}([e_1, e_1]) + \cdots + \alpha_{11}\beta_{n1}[e_1, e_n] \\
&+ \alpha_{21}\beta_{11}[e_2, e_1] + \cdots + \alpha_{21}\beta_{n1}[e_2, e_n] \\
&+ \cdots + \alpha_{n1}\beta_{11}[e_n, e_1] + \cdots + \alpha_{nn}\beta_{nn}([e_n, e_n]) \\
&= \alpha_{11}\beta_{11}(d_{11}^1 e_1 + d_{11}^2 e_2 + \cdots + d_{11}^n e_n) \\
&+ \cdots + \alpha_{11}\beta_{n1}(d_{1n}^1 e_1 + d_{1n}^2 + \cdots + d_{1n}^n e_n) \\
&+ \alpha_{21}\beta_{11}(d_{21}^1 e_1 + d_{21}^2 + \cdots + d_{21}^n e_n) \\
&+ \cdots + \alpha_{21}\beta_{n1}(d_{2n}^1 e_1 + d_{2n}^2 + \cdots + d_{2n}^n e_n) \\
&+ \cdots + \alpha_{n1}\beta_{nn}(d_{nn}^1 e_1 + d_{nn}^2 e_2 + \cdots + d_{nn}^n e_n) \\
&= (\alpha_{11}\beta_{11} d_{11}^1 + \cdots + \alpha_{11}\beta_{n1} d_{1n}^1 + \alpha_{21}\beta_{11} d_{21}^1 \\
&+ \cdots + \alpha_{21}\beta_{n1} d_{2n}^1 + \cdots + \alpha_{n1}\beta_{11} d_{n1}^1 + \cdots + \alpha_{n1}\beta_{nn} d_{nn}^1) e_1 \\
&+ (\alpha_{11}\beta_{11} d_{11}^n + \cdots + \alpha_{11}\beta_{n1} d_{1n}^n + \alpha_{21}\beta_{11} d_{21}^n \\
&+ \cdots + \alpha_{21}\beta_{n1} d_{2n}^n + \cdots + \alpha_{n1}\beta_{11} d_{n1}^n \\
&+ \cdots + \alpha_{n1}\beta_{nn} d_{nn}^n) e_n \\
&= \mathcal{A}'_1 e_1 + \mathcal{A}' e_2 + \cdots + \mathcal{A}'_n e_n.
\end{aligned} \tag{4.3}$$

Equating the coefficients of like terms, in equations (4.2) and (4.3), we obtain:

$$\mathcal{A}_1 = \mathcal{A}'_1,\ \mathcal{A}_2 = \mathcal{A}'_3, \cdots,\ \mathcal{A}_n = \mathcal{A}'_n.$$

□

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From Proposition 4.1, if $n = 2$, we have

$$\sum_{r,s,t=1}^{n} \alpha_{ri}\beta_{sj}d_{rs}^{t} = \sum_{t,k=1}^{n} c_{ij}^{k}\gamma_{tk} \qquad \forall\ i,j = 1,2. \tag{4.4}$$

## 4.2 Application

In this subsection, we explore some application of our results, demonstrating how it can have a meaningful impact in describing some common properties of non-isomorphic algebras not captured by isomorphism criterion. The algorithm is tested using $n = 2$ and is presented as follows:

**Proposition 4.2.**

There are no isotopism classes in the set of 2-dimensional Leibniz algebra over $\mathbb{Z}_2$.

*Proof.* Suppose that the structure constants of $L$ is represented as:

$$L : \begin{pmatrix} d_{11}^1 & d_{12}^1 & d_{21}^1 & d_{22}^1 \\ d_{11}^2 & d_{12}^2 & d_{21}^2 & d_{22}^2 \end{pmatrix}.$$

The linear transformations $(\alpha,\ \beta,\ \gamma)$ need to be non-singular, i.e.,

$$\alpha_{11}\alpha_{22} \neq\ \alpha_{12}\alpha_{21},\ \beta_{11}\beta_{22} \neq\ \beta_{12}\beta_{21},\ \gamma_{11}\gamma_{22} \neq\ \gamma_{12}\gamma_{21} \tag{4.5}$$

for the algebras to be isotopic.
From equation 4.4, we substitute the values of $\alpha_{ri}, \beta_{sj}$ and $\gamma_{tk}$ yields the following system equations:

$$\begin{cases} \alpha_{11}\beta_{11}d_{11}^1 + \alpha_{11}\beta_{21}d_{12}^1 + \alpha_{21}\beta_{11}d_{21}^1 + \alpha_{21}\beta_{21}d_{22}^1 = c_{11}^1\gamma_{11} + c_{11}^2\gamma_{12}, \\ \alpha_{11}\beta_{11}d_{12}^2 + \alpha_{11}\beta_{21}d_{12}^2 + \alpha_{21}\beta_{11}d_{21}^2 + \alpha_{21}\beta_{21}d_{22}^2 = c_{11}^1\gamma_{21} + c_{11}^2\gamma_{22}, \\ \alpha_{11}\beta_{12}d_{11}^1 + \alpha_{11}\beta_{22}d_{12}^1 + \alpha_{21}\beta_{12}d_{21}^1 + \alpha_{21}\beta_{22}d_{22}^1 = c_{12}^1\gamma_{11} + c_{12}^2\gamma_{12}, \\ \alpha_{11}\beta_{12}d_{11}^2 + \alpha_{11}\beta_{22}d_{12}^2 + \alpha_{21}\beta_{12}d_{21}^2 + \alpha_{21}\beta_{22}d_{22}^2 = c_{12}^1\gamma_{21} + c_{12}^2\gamma_{22}, \\ \alpha_{12}\beta_{11}d_{11}^1 + \alpha_{12}\beta_{21}d_{12}^1 + \alpha_{22}\beta_{11}d_{21}^1 + \alpha_{22}\beta_{21}d_{22}^1 = c_{21}^1\gamma_{11} + c_{21}^2\gamma_{12}, \\ \alpha_{12}\beta_{11}d_{11}^2 + \alpha_{12}\beta_{21}d_{12}^2 + \alpha_{22}\beta_{11}d_{21}^2 + \alpha_{22}\beta_{21}d_{22}^2 = c_{21}^1\gamma_{21} + c_{21}^2\gamma_{22}, \\ \alpha_{12}\beta_{12}d_{11}^1 + \alpha_{12}\beta_{22}d_{12}^1 + \alpha_{22}\beta_{12}d_{21}^1 + \alpha_{22}\beta_{22}d_{22}^1 = c_{22}^1\gamma_{11} + c_{22}^2\gamma_{12}, \\ \alpha_{12}\beta_{12}d_{11}^2 + \alpha_{12}\beta_{22}d_{12}^2 + \alpha_{22}\beta_{12}d_{21}^2 + \alpha_{22}\beta_{22}d_{22}^2 = c_{22}^1\gamma_{21} + c_{22}^2\gamma_{22}. \end{cases}$$

We now substitute the values of $c_{ij}^k$ and $d_{rs}^t$ from our algebras in the above equation to determine the isotopism between the algebras.
We now consider the isotopism between the algebras $L_{2,1},\ L_{2,2},\ L_{2,3}$ and $L_{2,4}$ in Theorem 2.1, as follows:
**Case: 1** Algebra $L_{2,1}$ is abelian algebra and therefore it is isotopic to itself.
**Case: 2**
Consider the isotopism between the algebras $L_{2,2}$ and $L_{2,3}$. Suppose that

$$L_{2,2} : \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the matrix structure constants of algebra $L_{2,2}$. Then

$$d_{12}^1 = d_{21}^1 = 1,\ d_{11}^1 = d_{11}^2 = d_{12}^2 = d_{21}^2 = d_{22}^1 = d_{22}^2 = 0.$$

Similarly, $L_{2,3}$ is given by

$$L_{2,3} : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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such that

$$c_{22}^1 = 1,\ c_{11}^1 = c_{11}^1 = c_{12}^1 = c_{12}^2 = c_{21}^1 = c_{21}^2 = c_{22}^2 = 0.$$

Substituting the values of ($d_{12}^1$, $d_{21}^1$, and $c_{22}^1$), in the system above we get the following equations.

$$\left.\begin{aligned} \alpha_{11}\beta_{21} + \alpha_{21}\beta_{11} &= 0, \\ \alpha_{11}\beta_{22} + \alpha_{21}\beta_{12} &= 0, \\ \alpha_{12}\beta_{21} + \alpha_{22}\beta_{11} &= 0, \\ \alpha_{12}\beta_{22} + \alpha_{22}\beta_{12} &= \gamma_{11}, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.6}$$

We now solve the system of equation considering condition (4.5) as follows:

**Case 2.1**:

Let $\alpha_{11} = 0 \implies \alpha_{12}\alpha_{21} \neq 0$. This means

$$(\alpha_{12}, \alpha_{21}) \neq (0, 0).$$

From $\alpha_{21}\beta_{11} = 0$ and $\alpha_{21}\beta_{12} = 0$ we get that $\beta_{11} = \beta_{12} = 0$. But this contradict the condition of non-singularity of $\beta$, i.e., $\beta_{11}\beta_{22} \neq \beta_{12}\beta_{21}$.

**Case 2.2:**

Suppose that $\alpha_{11} \neq 0$. From (4.6), we obtain

$$\beta_{21} = -\frac{\alpha_{21}\beta_{11}}{\alpha_{11}} \quad \text{and} \quad \beta_{22} = -\frac{\alpha_{21}\beta_{12}}{\alpha_{11}}.$$

Substituting the value of $\beta_{21}$ in equation (4.7)

$$\alpha_{12}\beta_{21} + \alpha_{22}\beta_{11} = 0 \tag{4.7}$$

$$\alpha_{12}\beta_{22} + \alpha_{22}\beta_{12} = \gamma_{11} \tag{4.8}$$

we get

$$\beta_{11}(\alpha_{22} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}}) = 0,$$

since

$$(\alpha_{22} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}}) \neq 0, \implies \beta_{11} = 0.$$

Similarly, substituting the value of $\beta_{22}$ in equation (4.8), we get

$$\beta_{12}(\alpha_{22} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}}) = \gamma_{11}.$$

Since $\gamma_{12} = 0, \implies \gamma_{11} \neq 0$ and $\left(\alpha_{22} - \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}}\right) \neq 0 \implies \beta_{12} \neq 0$. But from the condition, $\beta_{11}\beta_{22} \neq \beta_{12}\beta_{21}$, $\beta_{11} = 0 \implies \beta_{12}\beta_{21} \neq 0$. This mean that $\beta_{12} \neq 0$.

$$\beta : \begin{pmatrix} 0 & \beta_{12} \\ -\frac{\alpha_{21}\beta_{11}}{\alpha_{11}} & -\frac{\alpha_{21}\beta_{12}}{\alpha_{11}} \end{pmatrix} = \begin{pmatrix} 0 & \beta_{12} \\ 0 & -\frac{\alpha_{21}\beta_{12}}{\alpha_{11}} \end{pmatrix}.$$

Therefore, $\beta$ is singular and hence there is no solution. Consequently, the algebras $L_{2,2}$ and $L_{2,3}$ are not isotopic.

**Case: 3**

Consider the isotopism between algebras $L_{2,2}$ and $L_{2,4}$. The structure constants of algebra $L_{2,2}$ is given by

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$$L_{2,2} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which implies that

$$d_{12}^1 = d_{21}^1 = 1,\ d_{11}^1 = d_{11}^2 = d_{12}^2 = d_{21}^2 = d_{22}^1 = d_{22}^2 = 0.$$

Similarly,

$$L_{2,4} := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we have that

$$c_{12}^1 = c_{22}^1 = 1,\ c_{11}^1 = c_{11}^1 = c_{12}^2 = c_{21}^1 = c_{21}^2 = c_{22}^2 = 0.$$

Substituting the values of $(d_{11}^1,\ d_{21}^1$ and $c_{12}^1,\ c_{22}^1)$, we get the following system of equations:

$$\left.\begin{aligned} \alpha_{11}\beta_{21} + \alpha_{21}\beta_{11} &= 0, \\ \alpha_{11}\beta_{22} + \alpha_{21}\beta_{12} &= \gamma_{11}, \\ \alpha_{12}\beta_{21} + \alpha_{22}\beta_{11} &= 0, \\ \alpha_{12}\beta_{22} + \alpha_{22}\beta_{12} &= \gamma_{11}, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.9}$$

From the relation

$$\alpha_{11}\beta_{22} + \alpha_{21}\beta_{12} = \gamma_{11} \text{ and } \alpha_{12}\beta_{22} + \alpha_{22}\beta_{12} = \gamma_{11}$$

we have

$$\beta_{12}(\alpha_{21} - \alpha_{22}) + \beta_{22}(\alpha_{11} - \alpha_{12}) = 0.$$

Equation (4.9) can be written as follows:

$$\left.\begin{aligned} \alpha_{11}\beta_{21} + \alpha_{21}\beta_{11} &= 0, \\ \beta_{12}(\alpha_{21} - \alpha_{22}) + \beta_{22}(\alpha_{11} - \alpha_{12}) &= 0, \\ \alpha_{12}\beta_{21} + \alpha_{22}\beta_{11} &= 0, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.10}$$

**Case 3.1:**
Let $\beta_{11} = 0$. Then equation (4.10) becomes

$$\left.\begin{aligned} \alpha_{11}\beta_{21} &= 0, \\ \beta_{12}(\alpha_{21} - \alpha_{22}) + \beta_{22}(\alpha_{11} - \alpha_{12}) &= 0, \\ \alpha_{12}\beta_{21} &= 0, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.11}$$

Since $\beta_{11} = 0, \implies \beta_{12}\beta_{21} \neq 0$. That means that, $(\beta_{12}, \beta_{21}) \neq (0, 0)$. From $\alpha_{11}\beta_{21} = 0$ and $\alpha_{12}\beta_{21} = 0$ we get $\alpha_{11} = \alpha_{12} = 0$. This is a contradiction since $\alpha_{11}\alpha_{22} \neq \alpha_{12}\alpha_{21}$.
Suppose that $\beta_{11} \neq 0$. From equation (4.10) we have

$$\alpha_{21} = -\frac{\alpha_{11}\beta_{21}}{\beta_{11}} \text{ and } \alpha_{22} = -\frac{\alpha_{12}\beta_{21}}{\beta_{11}}.$$

Substituting the values of $\alpha_{21},\ \alpha_{22}$ in the relation

$$\beta_{12}(\alpha_{21} - \alpha_{12}) + \beta_{22}(\alpha_{11} - \alpha_{22}) = 0$$

we get

$$\beta_{12}\Big(-\alpha_{12} - \frac{\alpha_{11}\beta_{21}}{\beta_{11}}\Big) + \beta_{22}\Big(\alpha_{11} + \frac{\alpha_{12}\beta_{21}}{\beta_{11}}\Big) = 0.$$

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But $(\beta_{12}, \beta_{22}) \neq (0,0)$. Therefore,

$$\Big(-\alpha_{12} - \frac{\alpha_{11}\beta_{21}}{\beta_{11}}\Big) \neq 0 \text{ and } \Big(\alpha_{11} + \frac{\alpha_{12}\beta_{21}}{\beta_{11}}\Big) \neq 0.$$

This leads to contradiction since $\beta_{11}\beta_{22} \neq \beta_{12}\beta_{21}$. Consequently, there is no solution. Therefore there is no isotopism between the two algebras.

**Case: 4**

Let us consider,the isotopism between algebras $L_{2,3}$ and $L_{2,4}$. The matrix structure constants of algebra $L_{2,3}$ is given by

$$L_{2,3} := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then,

$$d_{12}^1 = d_{21}^1 = 1, \; d_{11}^1 = d_{11}^2 = d_{12}^2 = d_{21}^2 = d_{22}^1 = d_{22}^2 = 0.$$

Similarly,

$$L_{2,4} := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$c_{12}^1 = c_{22}^1 = 1, \; c_{11}^1 = c_{11}^1 = c_{12}^2 = c_{21}^1 = c_{21}^2 = c_{22}^2 = 0.$$

Substituting the values of $(d_{ij}^k, \; c_{ij}^t)$, we get the following equations: Solving the equations we get that:

$$\left.\begin{aligned} \alpha_{21}\beta_{21} &= 0, \\ \alpha_{21}\beta_{22} &= \gamma_{11}, \\ \alpha_{22}\beta_{21} &= 0, \\ \alpha_{22}\beta_{22} &= \gamma_{11}, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.12}$$

System (4.12) can be reduced to

$$\left.\begin{aligned} \alpha_{21}\beta_{21} &= 0, \\ \alpha_{21}\beta_{22} - \alpha_{22}\beta_{22} &= 0, \\ \alpha_{22}\beta_{21} &= 0, \\ \gamma_{21} &= 0. \end{aligned}\right\} \tag{4.13}$$

Suppose that $\beta_{22} = 0, \implies \beta_{12}\beta_{21} \neq 0$. This means that $(\beta_{12}, \beta_{21}) = (0,0)$. Now from $\alpha_{21}\beta_{21} = 0$ and $\alpha_{22}\beta_{21} = 0$. It implies that $\alpha_{21} = \alpha_{22} = 0$. This contradicts the claim $\alpha$ is non-singular.

When $\beta_{22} \neq 0$, it implies that $\alpha_{21} = \alpha_{22} = 0$. This also contradicts the condition $\alpha_{11}\alpha_{22} \neq \alpha_{12}\alpha_{21}$. Thus, there is no solution and hence no isotopism. □

## 5 Conclusion

Summarizing all the cases discussed above, we have the following table

Table 1: List of Isotopisms of two-dimensional Leibniz algebras over $\mathbb{Z}_2$

| Algebra | $L_{2,1}$ | $L_{2,2}$ | $L_{2,3}$ | $L_{2,4}$ |
|---|---|---|---|---|
| $L_{2,1}$ | Isotopic | Not Isotopic | Not Isotopic | Not Isotopic |
| $L_{2,2}$ | Not Isotopic | Isotopic | Not Isotopic | Not Isotopic |
| $L_{2,3}$ | Not Isotopic | Not Isotopic | Isotopic | Not Isotopic |
| $L_{2,4}$ | Not Isotopic | Not Isotopic | Not Isotopic | Isotopic |

From table 1 above, we conclude that there is no isotopism between algebras $L_{2,1}$, $L_{2,2}$, $L_{2,3}$, $L_{2,4}$ respectively. This implies that in 2−dimension leibniz algebra isotopism and isomorphism are equivalent.

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