International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(3), 2024, Pages 140 - 153
https://doi.org/10.5281/zenodo.13626097

A Stochastic Model for the Variation of Fourier Series Expansions with Time Delay Arising in Financial Market Price Changes

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Article Info
Received: 13 March 2024 Revised: 08 July 2024
Accepted: 22 July 2024 Available online: 30 July 2024

Abstract

In this paper, we derive a closed-form solution for the Stochastic Delay Differential Equation (SDDE). We formulated and proved theorems using Fourier series coefficients, which provided exact conditions for asset proce returns in three scenarios : linear, quadratic, and cubic functions. These price functions were utilized as the drift, representing the return rate in the SDDE solution, resulting in three distinct solutions. We
empirically evaluated these solutions to analyze the periodic impact of delay on each asset price function, revealing that an increase in the delay parameter reduces the value of time-varying asset investments.
Finally, our comparison of the asset values indicated that return rates following a linear trend offer the highest precision.

Keywords: Asset Pricing, Return rates, Fourier series expansions, Stochastic Analysis, Time Delay.
MSC2010: 91B70.

1 Introduction

Investment assessments rely on the rate of returns on assets, which can occur daily, weekly, monthly, or at other intervals. Assets are key instruments in the financial market that yield returns for investors.
Evaluating assets helps determine the market value of investments, enabling managers, investors, or governments to make informed decisions and avoid potential losses and negative impacts on their operations. The return on an asset investment provides an estimate that accurately examines the

turnover of an investment for the business's operational management over time.
A stochastic or random process is a mathematical process, usually defined as a sequence of random variables, where the sequence's index interprets time.

Fourier series, on the other hand, study processes and periodic functions related to time. Combining solutions from Stochastic Differential Equations (SDEs) and Fourier series with time delay offers meaningful mathematical insights into financial markets since both methods can approximate any finite and time-ordered series effectively. Fourier series analysis allows for comparisons based on amplitudes and identifying significant cycles with dominant amplitudes and their periods. Generally, following the basic features of the problem, an analytical solution is sought.

In mathematical modeling, a two-model approach involves developing a simplified "reduced" model that captures the essential features of a complex system and validating it against a more detailed "comprehensive" model. This technique uses the comprehensive model as a benchmark to test the accuracy of the reduced model, ensuring it captures the system's most important features. This approach has advantages: it allows for the development of a simplified model that is easily analyzed and understood while still capturing the system's essential features. The SDE part of the model can be refined, as differential equations are dynamic in modeling real-life systems. Time delay is significant in SDEs because it can affect the stability of the model, potentially leading to instability or oscillatory behavior, which is crucial in systems like feedback control systems where stability is vital.

Various methods have been used to measure stock market prices, with diverse results reported by scholars. For example, [1] conducted a stochastic analysis of a stock market price model using a proposed log-normal distribution model, which proved efficient for producing stock prices. [2] examined the stochastic analysis of stock market expected returns for investors, identifying the best stocks among various companies using the variances of four different stocks. This aligns with the findings of [3]. Additionally, [4] investigated the stochastic analysis of stock market expected returns and growth rates, while [5] focused on the stability analysis of the stochastic model for stock market prices, analyzing the unstable nature of market forces and applying a new differential equation model with stochastic volatility. [6] proposed analytical solutions for SDEs related to Martingale processes and found connections between some SDE solutions and other stochastic equations with diffusion parts. The second technique is to change SDE to ODE and omit the diffusion part of the stochastic equation by using Martingale processes.

Numerous authors have extensively studied SDEs in stock price modeling (see [7–12]). [13] explored stock price forecasting using Fourier series analysis, finding that level trading was less practical compared to momentum trading. [14] examined the effect of Fourier series expansion on SDE solutions, identifying sufficient conditions for price functions of return rates for capital investments periodically. This paper investigates the impact of delay on asset returns during periodic events by combining Fourier series expansions with SDE solutions for capital markets. The SDE was solved using Itô's theorem, resulting in closed-form analytical solutions. Theorems were developed and proved for three cases of asset returns: linear, quadratic, and cubic functions of prices over time.

This study extends the work of [15] by incorporating delay parameters in SDE solutions and Fourier series price functions for capital market price changes. The paper is structured as follows: Section 2.1 presents the mathematical preliminaries, results and discussions are provided in Section 3.1, and conclusions are drawn in Section 4.1.

2 Mathematical Preliminaries

We present below a few basic definitions which form the fundamentals of Financial Mathematics and are used in this work

Probability measure: Let Ω be a non-empty set and let F be a-algebra of subsets of Ω. Then a function that assigns every set$A \in F$ to a number in [0,1] is otherwise called probability measure $\wp$. In these
circumstances, it is denoted as $\wp(A)$ which is the probability of A such that the following condition holds:

$$\text{i. } \wp(\Omega) = 1 \tag{2.1}$$

$$\text{ii. if } A \in f, \text{ then } \wp(A) \geq 0 \tag{2.2}$$

iii. If $A_1, A_2, ...$ is a sequence of disjoint sets in F then

$$\wp(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \wp(A_n) \tag{2.3}$$

The pair (Ω, F) is called a measurable space while $(\Omega, \mathrm{F}, \wp)$ is called a probability space

Stochastic process: A stochastic process $X(t)$ is a relation of random variables $\{X_t(\gamma), t \in T, \gamma \in \Omega\}$, i.e., for each t in the index set T, $X(t)$ is a random variable. Now we understand t as time and call $X(t)$ the state of the procedure at time t. In view of the fact that a stochastic process is a relation of random variables, its requirement is similar to that for random vectors.

Stochastic Differential Equation (SDE): is integration of differential equation with stochastic terms. So, in considering the Geometric Brownian Motion (GBM) which govern price dynamics of a non-dividend paying stock as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dz(t) \quad , \tag{2.4}$$

Where S denotes the asset value, μ is the stock rate of return (drift) which is also known as the average rate of the growth of asset price and σ denotes the volatility otherwise called standard deviation of the returns. There is a Brownian motion or Wiener process which is defined on probability space $(\Omega, \mathrm{F}, \wp)$, [16]

Stochastic Differential Coefficient: Let the $(\Omega, \mathrm{F}, \wp)$ be a probability space with filtration $\{f_t : t \geq 0\}$ and $B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$, $t \geq 0$ is an m-dimensional Brownian Motion on that given probability space, stochastic differential equation coefficient functions f and g is in the form given as follows:

$$dX(t) = f(t, X(t))dt + g(t, X(t))B(t), 0 \leq t \leq T, X(0) = x_0, \tag{2.5}$$

Where $T > 0$, X_0 is an n-dimensional random variable and the coefficient functions are in the form of $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, *and* $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$.

Itô's lemma. Let $\mathrm{F}(S, t)$ be a twice continuous differential function on $[0, \infty) \times \mathrm{A}$ and let S_t denote an Itô's process

$$dS_t = a_t dt + b_t dz(t), \ t \geq 0,$$

Expanding F using Taylor series yields:

$$dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} (dS_t)^2 + \text{higher order terms } (h.o.t),$$

So, ignoring h.o.t and substituting for dS_t we obtain

$$dF_t = \frac{\partial F}{\partial S_t}(a_t dt + bdz(t)) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2}(a_t dt + bdz(t))^2 \tag{2.6}$$

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INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
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HTTPS://DOI.ORG/10.5281/ZENODO.13626097

$$= \frac{\partial F}{\partial S_t}(a_t dt + b dz(t)) + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}b_t^2 dt, \tag{2.7}$$

$$\left(\frac{\partial F}{\partial S_t}a_t + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}b_t^2\right)dt + \frac{\partial F}{\partial S_t}b_t dz(t) \tag{2.8}$$

More so, given that the variable $S(t)$ denotes stock price, then following GBM implies (2.4) hence, the function $F(S,t)$, Itô's lemma gives:

$$dF = \left(\mu S\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2}\right)dt + \sigma S\frac{\partial F}{\partial S}dz(t) \tag{2.9}$$

Thus, from initial stock price at time 0, S_0, adopting Itô's theorem in [17], we obtain the closed form solution as follows:

The derivation of SDE and its analytical solution

$$dS = \sigma S dx + \mu S dt \tag{2.10}$$

where S_0 is the initial price of the asset at time t. The asset price is completely deterministic and we can forecast the future worth of the asset with sureness. The expression dx, which contains the randomness that is certainly a characteristic of asset prices is called a Wiener process or Brownian motion.

$$dx^2 \to dt, \text{ as, } dt \to 0 \tag{2.11}$$

So, the smaller dt becomes, the more certainly dx^2 is equal to dt. Suppose that $f(S)$ is a smooth function of S and forget for the moment that S is stochastic. If we vary S by a small amount dS then clearly f also varied by a small amount provided we are not close to singularities of f. Applying Taylor series expansion we can write

$$df = \frac{df}{dS}dS + \frac{1}{2}\frac{d^2 f}{dS^2}dS^2 + ... \tag{2.12}$$

From (2.10) and squaring both sides

$$\begin{aligned} dS^2 &= (\sigma S dx + \mu S dt)^2 \\ &= \sigma^2 S^2 dx^2 + 2\sigma\mu S^2 dt dx + \mu^2 S^2 dt^2 \end{aligned}$$

$$dx = o(\sqrt{dt})$$

But

$$dS^2 = \sigma^2 S^2 dx^2 + ...$$

$$\text{Since } \; dx^2 \to dt, \;\; dS^2 \to \sigma^2 S^2 dt$$

Substituting (2.11) and retain only those terms which are at least as large as $o(dt)$ using the definition of dS, we find that

$$df = \frac{df}{dS}(\sigma S dx + \mu S dt) + \frac{1}{2}\sigma^2 S^2\frac{d^2 f}{dS^2}dt$$

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$$= \sigma S \frac{df}{dS} dx + \left(\mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} \right) dt \tag{2.13}$$

From (2.13) we can generalize by considering a function $f(S,t)$ of the random variable S and of time, t. There are two independent variable S and, t , hence it has to do with partial derivatives .Expansion of $f(S+dS, t+dt)$ in a Taylor series about (S,t) gives

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 + ...$$

Using expression for dS and dS^2 for the new expression for df is obtained as

$$df = \frac{\partial f}{\partial S} (\sigma S dx + \mu S dt) + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2$$

$$= \sigma S \frac{\partial f}{\partial S} dx + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt \tag{2.14}$$

Now we consider stochastic differential equation (2.10)

$$dS = \sigma S dx + \mu S dt$$

. Let $f(S) = \ln S$, partial derivatives are:

$$\frac{\partial f}{\partial S} = \frac{1}{S}, \ \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}, \ \frac{\partial f}{\partial t} = 0$$

Therefore, according to Itô yields,

$$df = \sigma S \frac{\partial f}{\partial S} dx + \left(\mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial S} \right) dt$$

$$= \sigma dx + \left(\mu - \frac{1}{2} \sigma^2 \right) dt$$

$$= \left(\mu - \frac{1}{2} \sigma^2 \right) dt + \sigma dx \tag{2.15}$$

Since the right hand side of the above equation is independent of , we compute the stochastic integral

$$f(s) = f_0 + \int_0^1 \left(\mu - \frac{1}{2} \right) dt + \int_0^1 \sigma dx$$

$$= f_0 + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma dx$$

Since $f(S) = \ln S$, a found solution for S becomes

$$\ln S = \ln S_0 + \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma dx$$

$$\ln S - \ln S_0 = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma dx$$

$$\ln \left(\frac{S}{S_0} \right) = \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma dx$$

$$S = S_0 \exp \left\{ \left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma dx \right\}$$

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https://doi.org/10.5281/zenodo.13626097

where dx is a standard Brownian Motion.

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma dx(t)}. \tag{2.16}$$

Stochastic delay differential equation : Suppose $(\Omega, \mathrm{F}, \wp)$ is a complete probability space with filtration $\{f_t : t \geq 0\}$ satisfying the conditions and $\{f_t : t \geq 0\}$ and
$B(t) = (B_1(t), B_2(t), ..., B_m(t))^T$, $t \geq 0$ is an m-dimensional Brownian Motion on that given probability space $(\Omega, \mathrm{F}, \wp)$; stochastic differential equations with a fixed time horizon $T > 0$ are in the following form:

$$dX(t) = f(t, X(t)), X(t-\tau)dt + g(t, X(t)), X(t-\tau)B(t),\ t \in [0, T], X(t) = \phi(t), t \in [-T, 0], \tag{2.17}$$

Where τ denotes delay parameter acting on time, t, and the initial path $\phi(t) : [-T, 0] \to \mathbb{R}^n$ is assumed to be continuous and f_0 - measurable random variable such that

$$\left[E\left(\underset{t \in [-1,0]}{\mathbf{Sup}} |\phi(t)|^p \right) \right]^{\frac{1}{p}} < \infty \tag{2.18}$$

Diffusion functions in the equations are as follows:

$$f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \ \text{ and } \ g : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$$

Fourier Series for Capital Market Investments

Let the function of an asset price be represented as $f(S)$ which is defined on the bounded interval $S \in [-1, 1]$ and outside of this interval we have $f(S + 2T) = f(S)$. That is to say $f(S)$ has a periodic influence on asset price which is $2T$. Therefore, the Fourier series expansion of $f(S)$ is given as follows.

$$\frac{a_0}{2} + \sum_{n=0}^{\infty} \left[a_n Cos\left(\frac{n\pi S}{T} + b_n Sin\left(\frac{n\pi S}{T} \right) \right) \right] \tag{2.19}$$

where a_n and b_n are the Fourier coefficients given as

$$a_n = \frac{1}{T} \int_{-T}^{7} f(S) Cos \frac{n\pi S}{T} dS \qquad n = 1, 2, ... \tag{2.20}$$

$$a_n = \frac{1}{T} \int_{C}^{C+27} f(S) Cos \frac{n\pi S}{T} dS \qquad n = 0, 1, 2, ... \tag{2.21}$$

$$b_n = \frac{1}{T} \int_{C}^{C+27} f(S) Sin \frac{n\pi S}{T} dS \qquad n = 1, 2, ... \tag{2.22}$$

C is the lower bound, which (by definition of $f(S)$) is equal to the interval; The details of this can be seen in the following references: [18–22].

Using (2.19) gives a_0; that is

$$a_0 = \frac{1}{T} \int_{-7}^{T} f(S) dS \tag{2.23}$$

In a situation where the lower limit C relates to $-T$, we have $C = -T$; such that
$C + 2T = -T + 2T = T$. So we have the following coefficients to determine the stock variables or quantities.

$$a_n = \frac{1}{T} \int_{-T}^{7} f(S) Cos \frac{n\pi S}{T} dS \quad n = 0, 1, 2, ... \tag{2.24}$$

$$b_n = \frac{1}{T} \int_{-T}^{7} f(S) Sin \frac{n\pi S}{T} dS \quad n = 1, 2, ... \tag{2.25}$$

3 Mathematical Formulation

The idea behind having an investment is primarily based on probability space $(\Omega, f, \wp)$ with well-equipped standard Brownian Motion such with a finite time horizon $T > 0$. Therefore, the considerable factors affecting price changes are very uncertain which are dependent on probability concepts over time. We considered [periodic events in the world market which causes] rate of return changes in different forms of price level with time delay, caused by periodic events in the world market. This price levels will be subjected to analysis, to realistically assess its price levels in each periodic event. Hence, we state and prove the following theorems:

Theorem 1 (Linear Property). Let there exist a solution of capital market investments whose rate of returns of asset price with time delay, is linear function of price, which follows Fourier series coefficients:

$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left[a_n Cos\left(\frac{n\pi S_1}{T} + b_n Sin\left(\frac{n\pi S_1}{T}\right)\right)\right]$ such that the periodic events lie on the open interval

$where \;\; f(S_1) = (t-\tau)S_1$

$0 < S_1 < 2\Pi$

Proof.

The rate of return as a linear function of price is being shown here, as we determine the coefficients of Fourier series as follows:

$$a_n = \frac{1}{T}\int_C^{C+2T} f(S_1)(t-\tau)Cos\frac{n\pi S_1}{T}dS_1 = \frac{1}{\pi}\int_C^{2T} f(S_1)(t-\tau)Cos\frac{n\pi S_1}{T}dS_1 = \frac{1}{\pi}\int_C^{2T}(t-\tau)S_1 CosnS_1 dS_1$$

Applying integration by parts using Nedu's Method

$$P_n(S_1) = (t-\tau)S_1 \quad and \quad f(S_1) = CosnS_1,$$

$$= \frac{1}{\pi}(t-\tau)\left[(S_1)\int CosnS_1 dS_1 - (1)\int CosnS_1 dS_1\right]_0^{2\pi} = \frac{1}{\pi}(t-\tau)\left[S_1\left(\frac{1}{n}SinnS_1\right) - 1\left(-\frac{CosnS_1}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}(t-\tau)\left[\left(S_1\frac{SinnS_1}{n}\right) - 1\left(-\frac{CosnS_1}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}(t-\tau)\left[\frac{Cos2n\pi}{n^2} - \frac{1}{n^2}\right] = (t-\tau)\frac{1}{n^2\pi}(1-1) = 0$$

$$if \quad n = 0, \quad a_0 = \frac{1}{\pi}\int_C^{27} f(S_1)(t-\tau)dS_1 = \frac{1}{\pi}\int_0^{2\pi}(t-\tau)S_1 dS = (t-\tau)\frac{1}{\pi}\left[\frac{S_1^2}{2}\right]_0^{2\pi} = (t-\tau)2\pi$$

Similarly

$$b_n = \frac{1}{T}\int_C^{C+2T} f(S_1)(t-\tau)Sin\frac{n\pi S_1}{T}dS_1 = \frac{1}{\pi}\int_C^{2T}(t-\tau)S_1 SinnS_1 dS_1$$

Applying integration by parts using Nedu's Method

$$P_n(S_1) = (t-\tau)S_1 \quad and \quad f(S_1) = SinnS_1,$$

$$= (t-\tau)\frac{1}{\pi}\left[(S_1)\int SinnS_1 dS_1 - (1)\int SinnS_1 dS_1\right]_0^{2\pi} = (t-\tau)\frac{1}{\pi}\left[S_1\left(-\frac{CosnS_1}{n}\right) - 1\left(-\frac{SinnS_1}{n^2}\right)\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[\frac{-2\pi Cos2n\pi}{n}\right] = -(t-\tau)\frac{2}{n}$$

$$\text{Hence} \quad f(S_1) = \sum_{n=1}^{\infty} -(t-\tau)\frac{2}{n}SinS_1$$

$$= (t-\tau)\pi - 2\left[SinS_1 + \frac{1}{2}Sin2S_1 + \frac{1}{3}Sin3S_1 + \frac{1}{4}Sin4S_1 + \frac{1}{5}Sin5S_1 + \frac{1}{6}Sin6S_1 + ...\right] \tag{3.1}$$

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This is the net gain or loss of an investment over a specified time period. Consequently, we set $f(S_1) = \mu$ of (2.9) which offer a complete solution of SDE with the effect of Fourier series expansion

$$S_1(t) = S_0 \exp\left(\left((t-\tau)\pi - 2[SinS_1 + \frac{1}{2}Sin2S_1 + \frac{1}{3}Sin3S_1 + \frac{1}{4}Sin4S_1 + ...] - \frac{\sigma^2}{2}\right)t + \sigma dz(t)\right) \quad (3.2)$$

Theorem 2 (Quadratic Property). There exists a solution of capital market investments whose rate of returns of asset price with time delay is a quadratic function of price which follows Fourier series coefficients:

$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left[a_n Cos\left(\frac{n\pi S_1}{T} + b_n Sin\left(\frac{n\pi S_1}{T}\right)\right)\right]$ such that the periodic events lie on the open interval
$where \;\; f(S_1) = (t-\tau)S_2^2$
$0 < S_1 < 2\Pi$

Proof.

The rate of return as a quadratic function of price is being shown here, as we determine the coefficients of Fourier series as follows:

$$a_n = \frac{1}{T}\int_C^{C+2T} f(S_2)Cos\frac{n\pi S_2}{T}dS_2 = \frac{1}{\pi}\int_C^{2T} f(S_2)Cos\frac{n\pi S_2}{T}dS_2 = \frac{1}{\pi}\int_C^{2T}(t-\tau)S_2^2 CosnS_2 dS_2$$

Applying integration by parts using Nedu's Method

$$P_n(S_2) = (t-\tau)S_2^2 \quad and \quad f(S_2) = CosnS_2,$$

$$= (t-\tau)\frac{1}{\pi}\left[(S_2^2)\int CosnS_2dS_2 - (2S_2)\int CosnS_2dS_2 + (2)\int CosnS_2dS_2\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[S_2^2\left(\frac{1}{n}SinnS_2\right) - 2S_2\left(-\frac{CosnS_2}{n^2} + 2\left(\frac{SinnS_2}{n^3}\right)\right)\right]_0^{2\pi}$$

$$=(t-\tau)\frac{1}{\pi}\left[\left(S_2^2\frac{SinnS_2}{n}\right) + 2S_2\left(\frac{CosnS_2}{n^2} - \frac{2SinnS_2}{n^3}\right)\right]_0^{2\pi} = (t-\tau)\frac{1}{\pi}\left[\frac{2S_2Cos2nS_2}{n^2}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{2S_2CosnS_2}{n^2}\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[\frac{4\pi 2\pi n}{n^2}\right] = (t-\tau)\frac{4\pi(-1)^{2n}}{n^2\pi} = (t-\tau)\frac{4}{n^2}, n \neq 0$$

$$if \;\; n = 0, \;\; a_0 = \frac{1}{\pi}\int_0^{2\pi}(t-\tau)S_2^2dS_2 = (t-\tau)\frac{1}{\pi}\left[\frac{S_2^3}{3}\right]_0^{2\pi} = (t-\tau)\frac{1}{n}\left(\frac{8\pi^3}{3\pi}\right) = (t-\tau)\frac{8\pi^2}{3}$$

$$b_n = \frac{1}{T}\int_C^{C+2T} f(S_2)Sin\frac{n\pi S_2}{T}dS_2 = \frac{1}{\pi}\int_C^{2T}(t-\tau)S_2^2 SinnS_2dS_2$$

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Applying integration by parts using Nedu's Method

$$P_n(S_2) = (t-\tau)S_2^2 \quad and \quad f(S_2) = SinnS_2,$$

$$= (t-\tau)\frac{1}{\pi}\left[(S_2^2)\int SinnS_2 dS_2 - (2S_2)\int SinnS_2 dS_2\ (2)\int SinnS_2 dS_2\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[S_2^2\left(-\frac{CosnS_2}{n}\right) - 2S_2\left(-\frac{SinnS_2}{n^2}\right) + 2\left(\frac{CosnS_2}{n^3}\right)\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[-S_2^2\frac{CosnS_2}{n} + 2S_2\frac{SinnS_2}{n^2} + 2\frac{CosnS_2}{n^3}\right]_0^{2\pi}$$

$$= (t-\tau)\frac{1}{\pi}\left[\frac{-4\pi^2 Cos2n\pi}{n} + \frac{2Cos2n\pi}{n^3}\right] = -(t-\tau)\left\{\frac{4\pi}{n} + \frac{2}{n^3}\right\}Cos2\pi n = -(t-\tau)\left\{\frac{4n^2\pi + 2}{n^3}\right\}$$

$$\text{Hence } f(S_2) = (t-\tau)\frac{8\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}CosnS_2 - \left(\frac{4n^2\pi+2}{n^3}\right)SinnS_2\right] = (t-\tau)\frac{8\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$= (t-\tau)\left(\frac{8\pi^2}{3} + 4CosS_2 + Cos2S_2 + \frac{4}{9}Cos3S_2 + \frac{1}{2}Cos4S_2 + \frac{4}{25}Cos5S_2 + \frac{1}{9}Cos6S_2 + ...\right) \quad (3.3)$$

Since, we are looking at rate of return that follows Fourier series with quadratic function of price. We therefore set $f(S_2) = \mu$ of (2.9) which results to a complete solution of SDE with the influence of Fourier series as follows.

$$S_2(t) = S_0 \exp\left(\left(\frac{8\pi^2}{3} + 4CosS_2 + Cos2S_2 + \frac{4}{9}Cos3S_2 + \frac{1}{2}Cos4S_2 + ... - \frac{\sigma^2}{2}\right)t + \sigma dz(t)\right) \quad (3.4)$$

Theorem 3 (Cubic Property). Let there be a solution of capital market investments whose rate of returns of asset price with time delay is a cubic function of price which follows Fourier series coefficients:

$\frac{a_0}{2} + \sum_{n=0}^{\infty}\left[a_n Cos\left(\frac{n\pi S_1}{T} + b_n Sin\left(\frac{n\pi S_1}{T}\right)\right)\right]$ such that the periodic events lie on the open interval

$where\ \ f(S_1) = (t-\tau)S_3^3$

$0 < S_1 < 2\Pi$

Proof.

We prove rate of return as a cubic function of price by defining the coefficients of Fourier series expansions following the steps:

$$a_n = \frac{1}{T}\int_C^{C+2T} f(S_3)Cos\frac{n\pi S_3}{T}dS_3 = \frac{1}{\pi}\int_C^{2T} f(S_3)Cos\frac{n\pi S_3}{T}dS_3 = \frac{1}{\pi}\int_C^{2T}(t-\tau)S_3^3 CosnS_3 dS_3$$

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$$P_n(S_3) = (t-\tau)S_3^3 \quad and \quad f(S_3) = CosnS_3,$$
$$= (t-\tau)\frac{1}{\pi}\left[(S_3^3)\int CosnS_3 dS_3 - (3S_3^2)\int CosnS_3 dS_3 + (6S_3)\int CosnS_3 dS_3 - (6)\int CosnS_3 dS_3\right]_0^{2\pi}$$
$$= (t-\tau)\frac{1}{\pi}\left[S_3^3\left(\frac{1}{n}SinnS_3\right) - 3S_3^2\left(-\frac{CosnS_3}{n^2}\right) + 6S_3\left(\frac{SinnS_3}{n^3}\right) - 6\left(\frac{CosnS_3}{n^4}\right)\right]_0^{2\pi}$$
$$= (t-\tau)\frac{1}{\pi}\left[\frac{3S_3^2CosnS_3}{n^2} - \frac{6CosnS_3}{n^4}\right]_0^{2\pi} = (t-\tau)\frac{1}{\pi}\left[\frac{12S_3Cos2n\pi}{n^2} - \frac{6}{n^4}Cos2n\pi + \frac{6}{n^4}\right]$$
$$= (t-\tau)\frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi + \frac{6}{n^4}(1 - Cos2n\pi)\right] = (t-\tau)\frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi + \frac{6}{n^4}\left(1-(-1)^{2n}\right)\right]$$
$$= (t-\tau)\frac{1}{\pi}\left[\frac{12\pi^2}{n^2}Cos2n\pi\right] = (t-\tau)\frac{12\pi^2(-1)^{2n}}{n^2\pi} = (t-\tau)\frac{12\pi^2}{n^2},\ n \neq 0$$
$$if\ \ n = 0,\ \ a_0 = (t-\tau)\frac{1}{\pi}\int_0^{2\pi} S^3 dS_3 = (t-\tau)\frac{1}{\pi}\left[\frac{S^4}{4}\right]_0^{2\pi} = (t-\tau)\frac{16\pi^4}{4\pi} = (t-\tau)4\pi^3$$

Similarly

$$b_n = \frac{1}{T}\int_C^{C+2T} f(S_3)Sin\frac{n\pi S_3}{T}dS_3 = (t-\tau)\frac{1}{\pi}\int_C^{2T} S_3^3 SinnS_3 dS_3$$

Applying integration by parts using Nedu's Method

$$P_n(S_3) = (t-\tau)S_3^3 \quad and \quad f(S_3) = SinnS_3,$$
$$= (t-\tau)\frac{1}{\pi}\left[(S_3^3)\int SinnS_3 dS_3 + (3S_3^2)\int SinnS_3 dS_3 - (6S_3)\int SinnS_3 dS_3 + (6)\int SinnS_3 dS_3\right]_0^{2\pi}$$
$$= (t-\tau)\frac{1}{\pi}\left[S_3^3\left(-\frac{CosnS_3}{n}\right) + 3S_3^2\left(-\frac{SinnS_3}{n^2}\right) - 6S_3\left(\frac{CosnS_3}{n^3}\right) + 6\left(\frac{SinnS_3}{n^4}\right)\right]_0^{2\pi}$$
$$= (t-\tau)\frac{1}{\pi}\left[\frac{-8\pi^3Cos2n\pi}{n} - \frac{12Cos2n\pi}{n^3}\right] = -(t-\tau)\left\{\frac{8\pi^2}{n} + \frac{12}{n^3}\right\}Cos2\pi n = -(t-\tau)\left\{\frac{8n^2\pi^2+12}{n^3}\right\}$$
$$\text{Hence } f(S_3) = (t-\tau)2\pi^3 + \sum_{n=1}^{\infty}\left[\frac{12\pi}{n^2}CosnS_3 - \left(\frac{8n^2\pi^2+12}{n^3}\right)SinnS_3\right] = (t-\tau)2\pi^3 + \sum_{n=1}^{\infty}\frac{12\pi CosnS_3}{n^2}$$
$$= ((t-\tau))2\pi^3 + 12\pi^2CosS_3 + 3\pi^2Cos2S_3 + \frac{4}{3}\pi^2Cos3S_3 + \frac{3}{4}\pi^2Cos4S_3 + \ldots \quad (3.5)$$

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https://doi.org/10.5281/zenodo.13626097

Therefore, this is rate of return that follows Fourier series with cubic function of price. Hence, we set $f(S_3) = \mu$ of (2.9) which yields a complete solution of SDE with the impact of Fourier series

$$S_3(t) = S_0 e\left(\left(2\pi^3 + 12\pi^2 Cos S_3 + 3\pi^2 Cos 2S_3 + \frac{4}{3}\pi^2 Cos 3S_3 - \frac{\sigma^2}{2}\right)t + \sigma dz(t)\right) \tag{3.6}$$

Table 1: **Time delay impact in assessing asset value when return is linear trend function**

Delay (τ) units	$S_1(t)$	**Delay (τ)**	$S_1(t)$	**Delay (τ)**	$S_1(t)$
0.1	774	0.15	764	0.45	705
0.2	754	0.25	744	0.55	685
0.3	734	0.35	724	0.65	665
0.4	715	0.45	705	0.75	646
0.5	695	0.55	685	0.85	626
0.6	675	0.65	665	0.95	606
0.7	655	0.75	646	1.05	587
0.8	636	0.85	626	1.15	577
0.9	616	0.95	616	1.25	547
1.0	596	1.05	587	1.35	527

Remark: Small values are considered better for prediction. Increase in delay of investment, increases value of asset. Rate of return is on the drift. Return rate is considered for linear, quadratic or cubic.

Table 2: **Time delay impact in assessing asset value when return is quadratic trend function**

Delay (τ)	$S_2(t)$	**Delay (τ)**	$S_2(t)$	**Delay (τ)**	$S_2(t)$
0.1	5263	0.15	5195	0.45	4790
0.2	5128	0.25	5060	0.55	4655
0.3	4993	0.35	4925	0.65	4521
0.4	4858	0.45	4790	0.75	4386
0.5	4723	0.55	4655	0.85	4251
0.6	4588	0.65	4521	0.95	4116
0.7	4453	0.75	4386	1.05	3981
0.8	4318	0.85	4251	1.15	3846
0.9	4183	0.95	4116	1.25	3711
1.0	4048	1.05	3981	1.35	3576

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https://doi.org/10.5281/zenodo.13626097

Table 3: **Time delay impact in assessing asset value when return is cubic trend function**

Delay (τ)	$S_3(t)$	**Delay** (τ)	$S_3(t)$	**Delay** (τ)	$S_3(t)$
0.1	5422	0.15	5352	0.45	4935
0.2	5283	0.25	5213	0.55	4796
0.3	5144	0.35	5074	0.65	4657
0.4	5005	0.45	4866	0.75	4518
0.5	4866	0.55	4796	0.85	4483
0.6	4727	0.65	4657	0.95	4341
0.7	4588	0.75	4518	1.05	4101
0.8	4449	0.85	4379	1.15	3953
0.9	4301	0.95	4240	1.25	3823
1.0	4171	1.05	4101	1.35	3684

Tables 1, 2, and 3 are common assessments of asset value over periodic events that follow linear, quadratic, and cubic price functions.

It can be seen that an increase in delay parameters causes a decrease in the value of the asset over time. This is obvious because, in every capital investment, any form of delay in making payments over an article bought eventually affects return rates in case of turnover; which is the total amount of sales a trader makes over a set of periods. This decrease in the value of assets could lead to the impairment of the company's goodwill and intangible assets within the periods.

However, in comparing the periodic events of the three Tables, it is very clear that there are significant price changes at all levels of asset values. Table 1 has the best value of asset pricing in terms of future price changes; which is an eye-opener for investors to enable them make useful decisions considering the investment level. Therefore, the entire trading activities which accrue under the linear trend function are profit indexed.

Table 2 describes the quadratic period of asset returns; it signifies uncertainty and inequality in the value of assets and their returns. This special remark attracts investors in the financial market because it is a trading business series around secular trend.

Table 3 demonstrates a cubic nature during the trading periods. In every periodic event of asset valuations, there are always minimum and maximum levels of asset returns; these indicate even and odd periods of investments. This remark informs investors of the actual periods to invest so as to make more profits.

4 Conclusion

The analysis of asset value and its return rates for periodic events has been readily well-known via Solution of SDE and Fourier series expansions with delay parameter in the model. The empirical studies which show increase in delay parameter decreases the value of assets in time varying investments. More so, in comparing each of the asset values it was discovered that return rates which follows linear trend is the best in terms of decision making. Thus, for further study, we will consider the algebraic analysis of these investment solutions and its further inferences in financial markets.

Acknowledgement

The authors are appreciative to all authors whose work have been cited in this paper as well as the peer reviewers for their prudent scrutiny of the result.

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International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(3), 2024, Pages 140 - 153
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