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On Rank of Semigroup of Order-Preserving Order-Decreasing Partial Contraction Mappings on a Finite Chain

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Abstract

Let $[n] = \{1, 2, \ldots, n\}$ be a finite chain, and $\mathcal{ODCP}_n$ be the semigroup of order-preserving and order-decreasing partial contraction mappings on $[n]$. In this paper, we study the rank properties of the two-sided ideals of $\mathcal{ODCP}_n$. We show that the rank of $K_p = \{\alpha \in \mathcal{ODCP}_n : |\text{im } \alpha| \leq p\}$, for $2 \leq p \leq n$, is

$$\sum_{k=p}^{n} \binom{n}{k}\binom{k-1}{p-1}$$

and hence, the rank of $\mathcal{ODCP}_n$ is $2n$.

Keywords: Transformation semigroup, Contraction mappings, Order-preserving, Generating sets, Ideals.
MSC2010: 20M20.

1 Introduction

Let $[n] = \{1, 2, \ldots, n\}$ be a finite chain and $\alpha : dom(\alpha) \subseteq [n] \to im(\alpha) \subseteq [n]$. α is said be *full* or *total* transformation if $dom(\alpha) = [n]$ otherwise is referred to as *strictly partial* transformation. The set of all strictly-partial and full transformation formed the *partial transformation semigroup* which is denoted by $\mathcal{P}_n$. It is worthy to note that, the empty map serves as zero in $\mathcal{P}_n$.

A transformation $\alpha \in \mathcal{P}_n$ is said to be *order preserving* (resp., *order reversing*) if (for all $x, y \in dom\ \alpha$) $x \leq y$ implies $x\alpha \leq y\alpha$ (resp. $x\alpha \geq y\alpha$); is *order decreasing* if (for all $x \in dom\ \alpha$) $x\alpha \leq x$; an *isometry* (i.e., *distance preserving*) if (for all $x, y \in dom\ \alpha$) $|x\alpha - y\alpha| = |x - y|$; a *contraction* if (for all $x, y \in dom\ \alpha$) $|x\alpha - y\alpha| \leq |x - y|$.

An element a of a semigroup S is called regular if there exists $x \in S$ such that $axa = a$. The semigroup S is called regular if all its elements are regular.

For non-empty subset A of a semigroup S, A is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and (*two-sided*) ideal if it is both a left and a right ideal. Ideals in semigroups possesses certain algebraic properties, closely related to the notion of normality in groups. They provide us with the powerful tool to analyze the internal structure of semigroups, revealing intricate pattern and capturing essential information about the elements behaviour under multiplication.

Let $x \in S$, then Sx, xS, and SxS are ideals generated by x in S. They are known as the *principal left, principal right*, and *principal two sided* ideal of S, respectively.

Green's relations are equivalence relations defined on a semigroup based on the principal ideals its elements generate. These relations are vital in understanding the algebraic structure of any semigroup more especially regular semigroups. There are five of these equivalences, namely, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$ relation, defined as follows: for $a, b \in S$,

$$\begin{aligned}
(a,b) \in \mathcal{L} &\iff S^1a = S^1b; \\
(a,b) \in \mathcal{R} &\iff aS^1 = bS^1; \\
(a,b) \in \mathcal{J} &\iff S^1aS^1 = S^1bS^1; \\
\mathcal{D} &= \mathcal{L} \circ \mathcal{R}; \\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R}.
\end{aligned}$$

The symbol S^1 denote a semigroup S with adjoint identity if S does not have one. It is worth mentioning that the relations $\mathcal{L}$ and $\mathcal{R}$ always commute (i.e., $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$). If a semigroup is non-regular, then, there is also need to understand its starred Green's relation (which is the generalization of the Green's relations) in order to classify such a semigroup or study its rank properties. The starred Green's equivalences are also five, they are $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{J}^*$, $\mathcal{D}^*$, and $\mathcal{H}^*$ defined as follows: Given any semigroup S, $a\mathcal{L}^*b$ (resp., $a\mathcal{R}^*b$) if and only if a and b are Green's $\mathcal{L}$-related (resp., Green's $\mathcal{R}$-related) in some over-semigroup of S. $\mathcal{D}^*$ is a meet of $\mathcal{L}^*$ and $\mathcal{R}^*$, while $\mathcal{H}^*$ is their joint. For more properties of Green's relation, starred Green's relation, or any other unexplained term we refer the reader to [8, 9, 16, 18, 19].

Let

$$\mathcal{PO}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{dom } \alpha)\ x \leq y \ \Rightarrow \ x\alpha \leq y\alpha\};$$

$$\mathcal{CP}_n = \{\alpha \in \mathcal{P}_n : (\text{for all } x, y \in \text{dom } \alpha)\ |x\alpha - y\alpha| \leq |x - y|\}$$

and

$$\mathcal{DCP}_n = \{\alpha \in \mathcal{CP}_n : (\text{for all } x \in \text{dom } \alpha)\ x\alpha \leq x\}.$$

Then, $\mathcal{ODCP}_n = \mathcal{PO}_n \cap \mathcal{DCP}_n$, called the *semigroup of order-preserving, order-decreasing partial contraction mappings.*

Let S be a semigroup, a non-empty subset G of S is said to be a *generating set* of S (denoted as $\langle G \rangle = S$) if for every $x \in S$, x can be written as a finite product of some elements in G. S is said to be a finitely generated semigroup if it is generated by a finite subset. The rank of a finitely generated semigroup measures the smallest number of elements required to generate the entire semigroup. That is,

$$\text{rank}(S) = \min\{|G| : G \subseteq S \text{ and } \langle G \rangle = S\}.$$

The notion of rank is of fundamental importance when studying semigroups. It helps us understand the structure of a given semigroup. It will also helps us to classify and characterized the complexity of a semigroup.

Several scholars have examined the combinatorial, algebraic and rank properties of various transformation semigroups, see for example, [5, 11, 13, 14, 17, 20, 23, 26]. On the other hand, the semigroup of partial contractions, which is relatively new has many interestingly open problems that are yet to be explored; see [27] for a comprehensive overview of such problems.

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Recently, Toker [24] investigated the ranks of the semigroup of order-preserving or order-reversing full contractions, $\mathcal{ORCT}_n$, and its subsemigroup $\mathcal{OCT}_n$, consisting of all order-preserving elements. Bugay [6] extended Toker's work by considering the ranks of ideals of the semigroups $\mathcal{ORCT}_n$ and $\mathcal{OCT}_n$, respectively. The study of nilpotent elements in $\mathcal{OCP}_n$ and its subsemigroup $\mathcal{OCI}_n$, consisting of all partial one-to-one transformations was presented in [2], and ranks of the subsemigroups generated by the nilpotents in the two semigroups were also determined. Alkharousi *et al.,* [3, 4] also study the rank and combinatorial properties of the semigroups of partial isometries on $[n]$. More studies on rank properties of isometries were further exploited in [1, 7].

Our aim in this paper is to investigate the rank properties of the two-sided ideal of the semigroup of order-preserving and order-decreasing partial contraction mappings on a finite chain.

2 Preliminaries

Let α be an element of $\mathcal{CP}_n$. Define dom α, im α, and $h(\alpha)$ as the domain, image, and $|\text{im }\alpha|$ of α, respectively. The kernel of α denoted by ker α, is defined as

$$\ker\,\alpha = \{(x, y) \in \text{dom }\alpha \times \text{dom }\alpha : x\alpha = y\alpha\}.$$

Additionally, for $\alpha, \beta \in \mathcal{CP}_n$, the composition of α and β is defined as $x(\alpha \circ \beta) = ((x)\alpha)\beta$ for all x in dom α. To avoid ambiguity, we use the notation $\alpha\beta$ for $\alpha \circ \beta$, and $x\alpha$ represents the image of x under α instead of the notation $\alpha(x)$. This ensures our composition of maps reads from left to right, i.e., $x\alpha\beta$ is equivalent to $\beta(\alpha(x))$.

A non-empty subset A of $[n]$ is said to be a *convex* subset if for every $x, y \in A$ with $x < y$ and if there exists $z \in [n]$ such that $x < z < y$ then $z \in A$. We use 1_A to denote a partial identity map of $A \subseteq [n]$ on A.

Given any transformation α in $\mathcal{ODCP}_n$, α can be expressed as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_p \\ x_1 & x_2 & \dots & x_p \end{pmatrix} (1 \leq p \leq n). \tag{2.1}$$

The blocks A_i $(1 \leq i \leq p)$ are equivalence classes under the relation ker α, and the collection of all the equivalence classes of the relation ker α partitioned the domain of α called the *kernel partition* of α, denoted by kp(α). Moreover, kp(α) is ordered under the usual ordering, that is, $A_1 < A_2 < \cdots < A_p$, where $A_i < A_j$ means $a < b$ for all $a \in A_i$ and $b \in A_j$. Furthermore, by order-preserving and order-decreasing properties of α, we have $x_i \leq x_{i+1}$ (for all $1 \leq i \leq p-1$) and $x_i \leq \min A_i$ (for all $1 \leq i \leq p$), respectively. Through out the paper, we refer to α as defined in (2.1) unless otherwise specified.

We begin by quoting the following results from related literature which shall be needed in subsequent discussions.

The first results is from [28] concerning the starred Green's relation of some semigroup of contraction mappings.

Theorem 2.1. *[28, Theorem 2.1] Let $S \in \{\mathcal{CP}_n, \mathcal{OCP}_n, \mathcal{ORCP}_n\}$ for $\alpha, \beta \in S$*

(i) $\alpha\mathcal{L}^*\beta$ *if and only if* im α = im β*;*

(ii) $\alpha\mathcal{R}^*\beta$ *if and only if* ker α = ker β*;*

(iii) $\alpha\mathcal{H}^*\beta$ *if and only if* im α = im β *and* ker α = ker β*;*

(iv) $\alpha\mathcal{D}^*\beta$ *if and only if* $|\text{im }\alpha| = |\text{im }\beta|$.

Remark 2.2. *It is an easy exercise to show that if $S = \mathcal{ODCP}_n$, then Theorem 2.1 above holds.*

The next result is derived from Umar and Zubair [27]. It was established in [27] among many other results that, the semigroup $\mathcal{ODCP}_n$ has a cardinal number of $\frac{(2+\sqrt{2})^n+(2-\sqrt{2})^n}{2}$, while the set of its nilpotents and that of idempotent elements has cardinalities of $|\mathcal{ODCP}_{n-1}|$ and $(1+n2^{n-1})$, respectively. Now using some algebraic simplifications we deduce the following lemma:

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Lemma 2.3. *Let $S = \mathcal{ODCP}_n$, $N(S)$ be the set of all nilpotents in S and $E(S)$ be the set of all idempotents in S. Then:*

(i) $|S| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} 2^{n-k}$

(ii) $|N(S)| = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} 2^{n-k-1}$

(iii) $|E(S)| = 1 + n2^{n-1}$.

Proof. (i) When the two expressions $(2+\sqrt{2})^n$ and $(2-\sqrt{2})^n$ are expanded using Binomial expansion and added together, the term with $\sqrt{2}$ will cancel out due to negative sign. The resulting expression will contain only terms with powers of two. Thus, $\left(2+\sqrt{2}\right)^n + \left(2-\sqrt{2}\right)^n$

$$\begin{aligned}
&= 2\left[\binom{n}{0}2^n + \binom{n}{2}2^{n-2}\left(\sqrt{2}\right)^2 + \binom{n}{4}2^{n-4}\left(\sqrt{2}\right)^4 + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}2^{(n-2\lfloor \frac{n}{2} \rfloor)}\left(\sqrt{2}\right)^{2\lfloor \frac{n}{2} \rfloor}\right] \\
&= 2\left[\binom{n}{2(0)}2^n + \binom{n}{2(1)}2^{n-2}2 + \binom{n}{2(2)}2^{n-4}2^2 + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}2^{(n-2\lfloor \frac{n}{2} \rfloor)}2^{(\lfloor \frac{n}{2} \rfloor)}\right] \\
&= 2\left[\binom{n}{2(0)}2^n + \binom{n}{2(1)}2^{n-1} + \binom{n}{2(2)}2^{n-2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}2^{(n-\lfloor \frac{n}{2} \rfloor)}\right] \\
&= 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}2^{n-k}.
\end{aligned}$$

Therefore,

$$\frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{2} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}2^{n-k}.$$

(ii) By substituting n with $n-1$ in (i), we have

$$|\mathcal{ODCP}_{n-1}| = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}2^{n-k-1},$$

and the result follows.

□

Let α be in $\mathcal{PO}_n$, for $2 \leq i \leq p-1$, define $h_i^\alpha = \min A_{i+1} - \max A_i$ and $d_i^\alpha = b_{i+1} - b_i$. Then

Lemma 2.4. *[1, Lemma 1] For $2 \leq p \leq n$, let α be in $\mathcal{PO}_n$. Then, α is a contraction if and only if $\min A_{i+1} - \max A_i \leq b_{i+1} - b_i$ (i.e., $d_i^\alpha \leq h_i^\alpha$) for all $i \in \{1, 2, \ldots, p-1\}$.*

We also defined the following terms which we used in describing the generating set of the two-sided ideals of $\mathcal{ODCP}_n$.

Definition 2.5 (Relations)**.** *Let $\alpha, \beta \in \mathcal{ODCP}_n$ such that $\alpha\mathcal{R}^*\beta$. Suppose im $\alpha = \{x_1, x_2, \ldots, x_p\}$ and im $\beta = \{y_1, y_2, \ldots, y_p\}$. We say α* exceeds *β if $x_i \geq y_i$ for all $1 \leq i \leq p$.*

Definition 2.6 (Comparability)**.** *Let α and β be $\mathcal{R}^*-$related. Then α and β are said to be* comparable *if either α exceeds β or β exceeds α.*

Definition 2.7 (Maximality)**.** *An element α is called a* maximum *element in its R^*-class if it exceed every elements in that R^*-class.*

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For the purpose of illustration, consider the following elements of $\mathcal{ODCP}_8$.

$$\alpha_1 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 2 & 3 \end{pmatrix}, \alpha_2 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 2 & 4 \end{pmatrix}, \alpha_3 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 2 & 5 \end{pmatrix},$$
$$\alpha_4 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 3 & 4 \end{pmatrix}, \alpha_5 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 3 & 5 \end{pmatrix}, \alpha_6 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 3 & 6 \end{pmatrix},$$
$$\alpha_7 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 4 & 5 \end{pmatrix}, \alpha_8 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 4 & 6 \end{pmatrix}, \alpha_9 = \begin{pmatrix} \{1,2\} & 5 & 8 \\ 1 & 4 & 7 \end{pmatrix}.$$

Then we have the following observations:

- all the elements belongs to the same R^*-class;
- α_2 exceeds α_1, while α_3 exceeds both α_1 and α_2;
- α_4 exceeds α_2 and α_1 but non-comparable to α_3;
- α_9 is a maximum element since is comparable to all the elements and exceeds each and every of them.

The following remark is also immediate.

Remark 2.8. *Observe that*

i If $\alpha \neq \beta$ and α exceeds β, then there exists $i \in \{1, 2, \ldots, p\}$ such that $x_i > y_i$.

ii if α and β are non-comparable, then there exists $i, j \in \{1, 2, \ldots, p\}$ such that $x_{i+1} - x_i > y_{i+1} - y_i$ and $y_{j+1} - y_j > x_{j+1} - x_j$.

We now have the following theorem.

Theorem 2.9. *Let $\alpha \in \mathcal{ODCP}_n$ $(n \geq 2)$ be as expressed in (2.1). Then α is a maximum element in its R^*-class if and only if $x_1 = \min A_1$ and $x_{i+1} = x_i + h_i^\alpha$.*

Proof. Let $\alpha \in \mathcal{ODCP}_n$ be as expressed in (2.1). Suppose that $x_1 = \min A_1$ and $x_{i+1} = x_i + h_i^\alpha$ (i.e., $x_{i+1} - x_i = \min A_{i+1} - \max A_i$). Our goal here is to show that α is a maximum element. Let

$$\gamma = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ z_1 & z_2 & \cdots & z_p \end{pmatrix}$$

be an arbitrary element in $\mathcal{R}^*_\alpha$. We show that α exceeds γ. By order-decreasing property of γ, we have $z_1 \leq \min A_1 = x_1$, which implies that $z_1 \leq x_1$. Now using induction hypothesis, suppose (for some $1 \leq k < p$) $z_k \leq x_k$, then by Lemma 2.4, we have

$$z_{k+1} - z_k \leq \min A_{k+1} - \max A_k = x_{k+1} - x_k$$

which implies that $z_{k+1} - x_k \leq x_{k+1} - x_k$. Therefore $z_{k+1} \leq x_{k+1}$, and hence $z_i \leq x_i$ (for all $i = 1, 2, \ldots, p$), as required.

Conversely, suppose by way of contradiction that there exists a maximum element

$$\beta = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ y_1 & y_2 & \cdots & y_p \end{pmatrix} \neq \alpha$$

in $\mathcal{R}^*_\alpha$. Then by Remark 2.8, there exists $i \in \{1, 2, \ldots, p\}$ such that $y_i > x_i$. If $i = 1$, then $y_1 > x_1 = \min A_1$ contradicting the order-decreasing property of β. Suppose $y_j = x_j$ (for some $1 \leq j \leq n-1$) and $y_{j+1} > x_{j+1}$, then, from the definition of α, we have $x_{j+1} - x_j = \min A_{j+1} - \max A_j$, which implies $y_{j+1} - x_j > \min A_{j+1} - \max A_j$ which implies $y_{j+1} - y_j > \min A_{j+1} - \max A_j$. This (by Lemma 2.4) contradicts the fact that α is a contraction, and the result follows.

□

Theorem 2.10. *In each $\mathcal{R}^*$ class of K_p, there exists a maximum element.*

Proof. It is clear that for any given kernel partition, one can define an element of $\mathcal{ODCP}_n$ using the criteria outlined in Theorem 2.9, and this element is the maximum element. □

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3 Rank of Ideals in $\mathcal{ODCP}_n$

In this section, we compute the rank of the two-sided ideals of the semigroup $\mathcal{ODCP}_n$. However, before we begin the investigation, we first of all define the ideals in $\mathcal{ODCP}_n$ as follows:

For $1 \leq p \leq n$, let

$$J_p = \{\alpha \in \mathcal{OCP}_n : |\text{im } \alpha| = p\} \tag{3.1}$$

and

$$K_p = \{\alpha \in \mathcal{ODCP}_n : |\text{im } \alpha| \leq p\} \tag{3.2}$$

be the two-sided ideal of $\mathcal{ODCP}_n$ for all $1 \leq p \leq n$. Observe that when $p = n$, we have $K_n = \mathcal{ODCP}_n$.

The following proposition is needed in finding the rank of K_p.

Proposition 3.1. *For $n \geq 4$, and $p \leq n-2$, $\alpha \in J_p$ can be written as product of elements in $\langle J_{p+1} \rangle$.*

Proof. Let α be in J_p. We proceed the proof using a case-by-case analysis.
Case I: Suppose im α is convex. Then α is of the form

$$\begin{pmatrix} A_1 & A_2 & \dots & A_p \\ k & k+1 & \dots & k+p-1 \end{pmatrix} \tag{3.3}$$

for some $1 \leq k \leq n-p+1$.

(i) If $\text{dom}(\alpha) = [n]$, then α is a full transformation. Moreover, given that $p \leq n-2$, there must exist a block say A_j $(1 \leq j \leq p)$ such that $|A_j| \geq 2$. Notice that, due to the order-preserving and order-decreasing properties of α, $\min A_1$ must be 1. This condition results in k being equal to 1 in Equation (3.3), as such α can be reexpressed as:

$$\begin{pmatrix} A_1 & A_2 & \dots & A_p \\ 1 & 2 & \dots & p \end{pmatrix}.$$

Define the mappings:

$$\gamma = \begin{pmatrix} A_1 & \dots & A_j \backslash \{x\} & \{x\} & A_{j+1} & \dots & A_p \\ 1 & \dots & j & j+1 & j+2 & \dots & p+1 \end{pmatrix}$$

and

$$\tau = \begin{pmatrix} 1 & \dots & \{j, j+1\} & j+2 & \dots & p+1 & p+2 \\ 1 & \dots & j & j+1 & \dots & p & p+1 \end{pmatrix},$$

respectively, where $x = \max A_j$. Then clearly, $\gamma, \tau \in J_{p+1}$ and $\gamma\tau = \alpha$.

(ii) Suppose dom $\alpha \subset [n]$, then there exists an element $c \in [n]$ such that $c \notin$ dom α.

If $c < \min A_1$ and $k \leq 2$, then α can be expressed as:

$$\begin{pmatrix} \min A_1 - 1 & A_1 & \dots & A_p \\ 1 & 2 & \dots & p+1 \end{pmatrix} \begin{pmatrix} 2 & 3 & \dots & p+1 & p+2 \\ k & k+1 & \dots & k+p-1 & k+p \end{pmatrix}$$

which is obviously a product of two elements in J_{p+1}.

If $c < \min A_1$ and $k > 2$, then by order-decreasing property of α we must have $\min A_1 > 2$; therefore, α can be expressed as:

$$\begin{pmatrix} \min A_1 - 2 & A_1 & \dots & A_p \\ k-2 & k & \dots & k+p-1 \end{pmatrix} \begin{pmatrix} k-1 & k & \dots & k+p-1 \\ k-1 & k & \dots & k+p-1 \end{pmatrix} \tag{3.4}$$

a product of two elements in J_{p+1}.

If $c > \max A_p$ and $k = 1$, then since $p+2 \leq n$, we can express α as:

$$\begin{pmatrix} A_1 & A_2 & \dots & A_p & \max A_p + 1 \\ 1 & 2 & \dots & p & p+1 \end{pmatrix} \begin{pmatrix} 1 & \dots & p & p+2 \\ 1 & \dots & p & p+1 \end{pmatrix}$$

a product of two elements in J_{p+1}.

If $c > \max A_p$ and $k \geq 2$, then α can be expressed as:

$$\begin{pmatrix} A_1 & A_2 & \dots & A_p & \max A_p + 1 \\ k & k+1 & \dots & k+p-1 & k+p \end{pmatrix} \begin{pmatrix} k-1 & k & \dots & k+p-1 \\ k-1 & k & \dots & k+p-1 \end{pmatrix}$$

a product of two elements in J_{p+1}.

If $\max A_j < c < \min A_{j+1}$ for some $j \in \{1, \dots, p-1\}$ and $k = 1$, then α can be written as:

$$\begin{pmatrix} A_1 & \cdots & A_j & c & A_{j+1} & \cdots & A_p \\ 1 & \cdots & j & j+1 & j+2 & \cdots & p+1 \end{pmatrix} \begin{pmatrix} 1 & \cdots & j & j+2 & \cdots & p+1 & p+2 \\ 1 & \cdots & j & j+1 & \cdots & p & p+1 \end{pmatrix}$$

a product of two elements in J_{p+1}.

If $\max A_j < c < \min A_{j+1}$ for some $j \in \{1, \dots, p-1\}$, and $k > 1$, then α can be written as a product of

$$\begin{pmatrix} A_1 & \cdots & A_j & c & A_{j+1} & \cdots & A_p \\ k & \cdots & k+j-1 & k+j & k+j+1 & \cdots & k+p \end{pmatrix}$$

and

$$\begin{pmatrix} k-1 & k & \cdots & k+j-1 & k+j+1 & \cdots & k+p \\ k-1 & k & \cdots & k+j-1 & k+j & \cdots & k+p-1 \end{pmatrix}.$$

If $\min A_j < c < \max A_j$ for some $j \in \{1, \dots, p\}$ and $k \leq 2$, then $k+p+1 \leq n$; therefore, α can be written as a product of

$$\begin{pmatrix} A_1 & \cdots & \{\min A_j, \dots, c-1\} & \{c+1, \dots, \max A_j\} & A_{j+1} & \cdots & A_p \\ k & \cdots & k+j-1 & k+j & k+j+1 & \cdots & k+p \end{pmatrix}$$

and

$$\begin{pmatrix} k & \cdots & \{k+j-1, k+j\} & k+j+1 & \cdots & k+p & k+p+1 \\ k & \cdots & k+j-1 & k+j & \cdots & k+p-1 & k+p \end{pmatrix}.$$

If $\min A_j < c < \max A_j$ for some $j \in \{1, \dots, p\}$ and $k > 2$, then we express α as in Equation (3.4).

Case II: Suppose Im α is not convex, i.e., there exists j such that $x_{j+1} - x_j \geq 2$. Then by property of contraction we must have $\min A_{j+1} - \max A_j \geq 2$. Define a mapping γ as:

$$\gamma = \begin{pmatrix} A_1 & \cdots & A_j & \max A_j + 1 & A_{j+1} & \cdots & A_p \\ x_1 & \cdots & x_j & x_j + 1 & x_{j+1} & \cdots & x_p \end{pmatrix}.$$

Notice that since $p \leq n-2$, then there exists $c \in [n]$ such that $c \notin \text{im } \gamma$. Define τ as 1_A, where $A = \text{im}(\alpha) \cup \{c\}$. Then, γ and τ are in J_{p+1} and $\alpha = \gamma\tau$. Hence, the proof.

□

Corollary 3.2. *$\langle J_p \rangle = K_p$ for all $2 \leq p \leq n-1$*

Proof. Since $\langle K_1 \rangle \subseteq \langle K_2 \rangle \subseteq \cdots \subseteq \langle K_p \rangle$, then $\bigcup_{i=1}^{p} \langle K_i \rangle \subseteq \langle K_p \rangle$ (i.e., $\mathcal{ODCP}_{n,p} \subseteq \langle K_p \rangle$). However, $K_p \subseteq \mathcal{ODCP}_{n,p}$ (by definition). Therefore, $\mathcal{ODCP}_{n,p} = \langle K_p \rangle$. □

In the next theorem, we show that the maximum elements generates $\langle J_p \rangle$.

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Theorem 3.3. *Let G be the set of all maximum elements in $\mathcal{R}^*$ classes of J_p. Then, G is the minimal generating set of $\langle J_p \rangle$.*

Proof. We proof the theorem in the following steps: Firstly, we show that necessarily, any generating set of $\langle J_p \rangle$ must contain G. Then, we show that G is sufficient in generating $\langle J_p \rangle$.

Let W be any generating set of $\langle J_p \rangle$. By way of contradiction, suppose there exists a maximum element say

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ x_1 & x_2 & \cdots & x_p \end{pmatrix}$$

which is not in W. Then W being a generating set of $\langle J_p \rangle$ means α can be written as $\alpha = \lambda_1 \lambda_2 \cdots \lambda_k$ (for some $k \in \mathbb{N}$). Let $\beta_1 = \lambda_1 \cdots \lambda_{k-1}$ and $\beta_2 = \lambda_k$. Notice that since $|\text{im } \alpha| = |\text{im } \beta_i|$ $(i = 1, 2)$, we must have $\ker(\alpha) = \ker(\beta_1)$ and $\text{im}(\alpha) = \text{im}(\beta_2)$. Therefore, β_1 is $\mathcal{R}^*$-related to α. Without lost of generality, we may assume that

$$\beta_1 = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix} \text{ and } \beta_2 = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ x_1 & x_2 & \cdots & x_p \end{pmatrix}.$$

Since α is a maximum in $\mathcal{R}^*_{\beta_1}$, then by Remark 2.8, we must have $b_j < x_j$ for some $1 \leq j \leq p$.

Now, $b_j < x_j = b_j\beta_2$ implies $b_j\beta_2 > b_j$. This contradicts the order-decreasing property of β_2.

To show the sufficient condition, let

$$\tau = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}$$

be an arbitrary element in J_p, and let

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ x_1 & x_2 & \cdots & x_p \end{pmatrix}$$

be a maximum element in $\mathcal{R}^*_\tau$.

Let $\eta_i = d_i^\alpha - d_i^\tau$. Observe that since α exceed τ, then

$$\eta_i \geq 0, \text{ for all } i \in \{1, 2, \ldots, p-1\} \tag{3.5}$$

Define β as

$$\begin{pmatrix} \{a_1, \ldots, x_1\} & \{x_2 - \eta_1, \ldots, x_2\} & \cdots & \{x_p - \eta_{p-1}, \ldots, x_p\} \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}.$$

Clearly, $\alpha\beta = \tau$. So, we show that β is a maximum element. But before then, we need to show that $\beta \in \mathcal{ODCP}_n$.

Now, for $1 \leq i \leq p-1$,

$$\begin{aligned} (x_{i+1} - \eta_i) - x_i &= x_{i+1} - [d_i^\alpha - d_i^\tau] - x_i \\ &= x_{i+1} - [(x_{i+1} - x_i) - (a_{i+1} - a_i)] - x_i \\ &= x_{i+1} - [x_{i+1} - x_i - a_{i+1} + a_i] - x_i \\ &= a_{i+1} - a_i. \end{aligned}$$

Therefore,

$$(x_{i+1} - \eta_i) - x_i = a_{i+1} - a_i \text{ (for all } 1 \leq i \leq p-1) \tag{3.6}$$

which implies (by Lemma 2.4) that β is an order-preserving partial contraction mapping. For the order-decreasing property, notice that a_1 is the minimum of the first block of β. Moreover, using Equation (3.6), we have $(x_2 - \eta_1) - x_1 = a_2 - a_1$ which implies $(x_2 - \eta_1) - a_1 \geq a_2 - a_1$, which implies by Equation (3.5) that $x_2 \geq a_2$. In general, if $a_k \leq x_k$, then $(x_{k+1} - \eta_k) - x_i = a_{k+1} - a_k$ will imply $(x_{k+1} - \eta_k) - a_k \leq a_{k+1} - a_k$. Thus, $a_{k+1} \leq x_{k+1}$, and therefore, $\beta \in \mathcal{ODCP}_n$. Finally, using Theorem 2.9, Equation (3.6) and the fact that a_1 equals to the minimum of the first block of β, we obtain that β is a maximum element, as required. ∎

Theorem 3.4. *The rank of* $\langle J_p \rangle$ *is*

$$\sum_{k=p}^{n} \binom{n}{k} \binom{k-1}{p-1}$$

Proof. Since the maximum elements cover all the $\mathcal{R}^*-$classes of J_p, by Theorem 3.3 we only need to count the number of $\mathcal{R}^*-$classes in J_p. Furthermore, we observe that the number of R^*-classes in J_p is the number of all possible ordered partition of $n-$element set into p classes, which is $\sum_{k=p}^{n} \binom{n}{k}\binom{k-1}{p-1}$. This number also coincide with the number of $R-$classes of the semigroup $\mathcal{PO}_n$ see [22, Lemma 4.1]. The result now follows. □

Corollary 3.5. *The rank of* K_p *is*

$$\sum_{k=p}^{n} \binom{n}{k} \binom{k-1}{p-1}.$$

Proof. It follows from Corollary 3.1 and Theorem 3.3. □

By substituting $n-1$ for p, we readily have the following corollary.

Corollary 3.6. $Rank(K_{n-1}) = 2n - 1$.

It should be noted that, $K_{n-1} \cup \{1_{[n]}\} = \mathcal{ODCP}_n$. Therefore, The rank $\mathcal{ODCP}_n$ is rank$(K_{n-1})+1$. Consequently, we have proved the following result.

Theorem 3.7. *The rank of* $\mathcal{ODCP}_n$ *is* $2n$.

4 Conclusion

The paper examines the rank properties of two-sided ideals of the semigroup of order-preserving order-decreasing partial contraction mappings on a finite chain $[n]$. The rank of such an ideal is a measure the size of its generating set in a combinatorial sense, and the paper's derivation of these ranks is based on combinatorial methods.

The paper's findings significantly advance the understanding of the algebraic structure of semigroups by leveraging Green's relations and their starred counterparts. The conditions outlined in Theorem 2.1, regarding the characterization of Green's relations offer a comprehensive framework for classifying elements within the semigroup. Additionally, the exploration of two-sided ideals and their rank properties provides valuable insights into the combinatorial structure of these subsets.

The lemmas and theorems presented throughout the paper are critical for establishing the main results. They help clarify the conditions under which elements can be considered maximum within their respective $\mathcal{R}$-classes, and how these maximal elements contribute to generating the entire ideal or even the entire semigroup. The thorough analysis presented in the paper thus not only enhances the theoretical understanding of semigroups but also contributes to the broader field of algebra through its detailed exploration of ideals and Green's relations.

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