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On the Digraphic Decomposition of Stable Quasi-Idempotents within Finite Partial Transformation Semigroups

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Abstract

This study explores the universal classification of elements in the finite partial transformation semigroup P_n on the set $X_{n+1} = \{0, 1, 2, \dots, n\}$. The primary focus is on the interplay between the powers of transformations, equivalence relations, and their cyclic and quasi-idempotent structures. A key observation is that for any transformation $\alpha \in P_n$, repeated application stabilizes, leading to a periodic behavior characterized by the (m, r)-path cycle—a notation capturing both cyclic and linear components of α. To further analyze α, the concept of orbits is introduced, defined as equivalence classes under the relation $x \sim y$ if $x\alpha^m = y\alpha^r$. These orbits provide a framework for understanding the dynamics of α. The study also examines specific elements like idempotents ($\varepsilon^2 = \varepsilon$) and quasi-idempotents ($\xi^2 \neq \xi$, $\xi^4 = \xi^2$), offering classifications based on the sizes of their cyclic portions within their orbits. A notable result is that stable quasi-idempotents generate the ideal $P_n \backslash S_n$, where S_n denotes the symmetric group. This work contributes a digraphic characterization of P_n, advancing the understanding of its algebraic structure. The findings have potential applications in semigroup theory, automata, and computational mathematics, particularly in analyzing transformation systems with finite domains.

Keywords: (m,r)-Path Cycle, Digraph, m-Potent, Transformation Semigroup.
MSC2010: 20M20.

1 Introduction

The study of finite partial transformation semigroups P_n has been central to understanding algebraic structures, particularly their applications in automata theory, combinatorics, and computational algebra. P_n consists of all partial mappings on the set $X_{n+1} = \{0, 1, 2, \ldots, n\}$, with composition as the primary operation. Historically, the term "product" has been used in semigroup theory to refer to this operation, as first popularized by Clifford and Preston [2] while translating the product of transpositions in the symmetric subgroup S_n of P_n. This terminology was later adopted by Howie [3], who noted its equivalence to "multiplication" in semigroup contexts, a perspective that gained traction following Rees's work in 1940 [4]. While "product" has a soothing and general appeal, it maintains its role as a stand-in for composition, whereas "decomposition" remains irreplaceable in describing semigroup structures.

In comparing the behavior of transpositions in S_n with the idempotents in P_n, Lipscomb [5] introduced the term "disjoint spans" to capture the sets $\text{Span}(\alpha) = \text{dom}(\alpha) \cup \text{im}(\alpha)$. Further advancements by Ganyushkin and Mazorchuk [6], using digraphic isomorphisms [7], formalized representations of elements in P_n as (m, r)-path cycles. A special case of these cycles is the m-path $[\alpha \mid \alpha^{(m+1)} = \alpha^m]$, whose simplest instance is the 3-path. Similarly, the m-potent $[\alpha \mid \alpha_{(m+1)} = \alpha_m]$, with idempotents as a subset, provides a unifying framework for understanding stable and quasi-idempotent elements.

Stable quasi-idempotents are characterized by their decomposability into 3-paths, 2-chains, and (m, r)-path cycles, with their digraphic representations providing a visual and structural framework for analysis. The order of the set $Z(\xi)$ of distinct elements in $\text{dom}(\xi)$ corresponds to the diameter of the digraph of ξ, a concept aligned with Madu's work [1], which linked quasi-nilpotency indices to digraph diameters. By definition, a quasi-idempotent stabilizer [9] identifies stable quasi-idempotents as those for which repeated compositions eventually stabilize both the domain and image.

The investigation into idempotent and quasi-idempotent structures has a long history. The study of idempotent products in transformation semigroups was pioneered by Howie [3], building on the foundational proofs of Clifford and Preston [2] that semigroups can be universally embedded in transformation semigroups. The prefix "quasi" was introduced later to describe elements that exhibit idempotent-like behavior under repeated application, with Ganyushkin and Mazorchuk [6] advancing this terminology. A stable quasi-idempotent, being both stable and quasi-idempotent, contrasts with general quasi-idempotents, as not all quasi-idempotents are stable. Furthermore, while the square of a quasi-idempotent is idempotent, stable quasi-idempotents unify these elements within the framework of functional digraphs.

Recently, Ibrahim et al. [16] explored the collapse of elements within the order-preserving and idempotent structures of the order-preserving full contraction transformation semigroup, T_n. They examined the subsemigroup of order-preserving transformations, $C^+(\alpha)$, and the idempotent subset, $C^+(\alpha_E)$, providing explicit formulas for the total number of collapsible elements and shedding light on their structural behaviors. Similarly, Imam [13] extended the study to quasi-idempotents in the semigroup of partial order-preserving transformations, $\mathcal{PO}_n$, demonstrating that $\mathcal{PO}_n$ is quasi-idempotent generated and providing an upper bound for its quasi-idempotent rank as $\lceil \frac{5n-4}{2} \rceil$.

Transpositions, such as $(12) \neq (12)^2 = (12)^4$, are quasi-idempotents but not stable quasi-idempotents. Since S_n is generated by products of transpositions, Howie [11] introduced $T_n \setminus S_n$ to study the behavior of partial transformations outside S_n. Similarly, $P_n \setminus S_n$ forms a regular semigroup generated by stable quasi-idempotents. A corollary of Howie's Theorem asserts that every finite semigroup can be embedded in a finite regular stable quasi-idempotent-generated semigroup, demonstrating the universal role of $P_n \setminus S_n$ in semigroup theory.

This study aims to establish a universal characterization of stable quasi-idempotents in P_n, examining their structural, digraphic, and algebraic properties in relation to idempotents and quasi-idempotents. By investigating their role within the ideal $P_n \setminus S_n$, the work highlights their foundational significance and offers a pathway to embedding finite semigroups into stable quasi-idempotent-generated structures.

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2 Preliminaries

The following preliminaries outline the essential definitions and notations needed to understand the results presented in this study.

Definition 2.1. *[15]*A set S is called a semigroup if it satisfies the following axioms:

- Closure: For any $a, b \in S$, there exists a unique $c \in S$ such that $c = ab$ (the product of a and b).
- Associativity: For all $a, b, c \in S$, $a(bc) = (ab)c$.

Definition 2.2. *[6]*Let $X_n = \{1, 2, \cdots, n\}$. A mapping α such that $\mathrm{dom}\alpha \subseteq \mathrm{X_n} \longrightarrow \mathrm{im}\alpha \subseteq \mathrm{X_n}$ is called *partial transformation* of X_n. The set of all partial transformations of X_n is denoted by $\mathcal{P}_n$ and as the composition of partial transformations is again a partial transformation, the set $\mathcal{P}_n$ is a semigroup under composition called the partial transformation semigroups.

Definition 2.3. *[15]*Let T be a subset of a semigroup S.

- T is a left ideal of S if $ST \subseteq T$.
- T is a right ideal of S if $TS \subseteq T$.
- T is an ideal of S if T is both a left and right ideal.

Definition 2.4. Let P_n be a partial transformation semigroup;

- An element $\alpha \in P_n$ is called an idempotent if $\alpha = \alpha^2$. [3]
- An element $\beta \in P_n$ is a quasi-idempotent if $\beta \neq \beta^2 = \beta^4$. [1]
- An element $\xi \in P_n$ is a stable quasi-idempotent if $\xi \neq \xi^2 = \xi^3$.
  For such ξ, $\xi^n = \xi^3$ for all $n \geq 3$.

Definition 2.5. Let S be a semigroup, and let $a \in S$.

- The index and period of a are the smallest integers $m \geq 1$ and $r \geq 1$ such that $a^{m+r} = a^m$.
- If $r = 1$, a is called an m-potent element.
- If $m = r = 1$, a is an idempotent.

In linear notation, this is represented as $\alpha = [x_1, x_2, \ldots, x_m \mid x_r]$. Special cases include:

- If $r = m$, α is an m-path.
- If $r = 1$ and $m \geq 2$, α is an m-cycle.
- If $m = r = 1$, α is a loop.
- If $m = r = 2$, α is an idempotent of defect one.

Definition 2.6. Let $X_n = \{1, 2, \ldots, n\}$, and let T_n be the full transformation semigroup on X_n. An element $\alpha \in T_n$ is called an (m, r)-path cycle if:

- There exists $\{x_1, x_2, \ldots, x_m\} \subseteq X_n$ such that $x_i\alpha = x_{i+1}$ for $1 \leq i < m$, and $x_m\alpha = x_r$ for $1 \leq r \leq m$.
- For all $x \in X_n \setminus \{x_1, x_2, \ldots, x_m\}$, $x\alpha = x$.

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Definition 2.7. A graph Γ is a pair (V, E) where V is a finite set and E is a set of 2-subsets of V. The set V is the set of vertices and the set E is the set of edges. Given a graph Γ we use $V(\Gamma)$ and $E(\Gamma)$ to denote the set of vertices and the set of edges, respectively, of the graph Γ. An edge $\{i, j\}$ is said to join the vertices i and j and this edge is denoted ij. The vertices i and j are called the *ends* of the edge ij. If $ij \in E(\Gamma)$ we say that the vertices i and j are *adjacent* in the graph Γ. We say that i and j are *incident* to the edge ij. We say that two edges are adjacent if they have a common incident vertex. [10]

We think of a graph as a collection of vertices, some of which are joined by edges, and as a result graphs are often represented as pictures. For example, the graph $\Gamma = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\})$ is given in Figure 1.

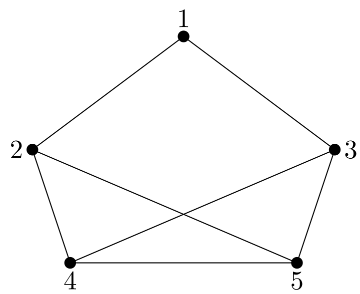
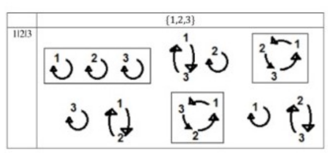
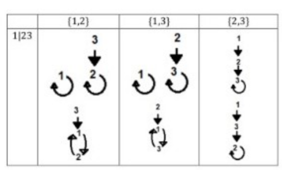


Figure 1: The graph $\Gamma = (V, E)$.

Definition 2.8. A *directed graph* or *digraph* $\overrightarrow{\Gamma}$ is a pair $(V, \overrightarrow{E})$ where V is a finite non-empty set of points, called *vertices*, and $\overrightarrow{E}$ is a collection of ordered pairs of points in V, called *directed edges* or *arcs*. A directed edge $(i, j) \in \overrightarrow{E}$ is denoted $i \to j$.

This section presents the distribution of the stable quasi-idempotents, quasi-idempotents and idempotents and other classified $m - potents$, as follows. For T_3 and P_2 we have;





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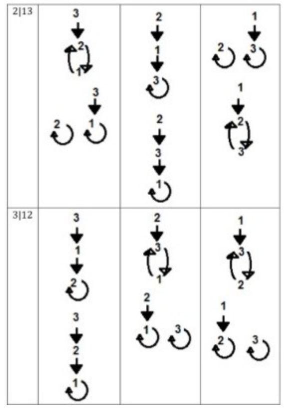

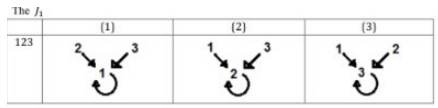

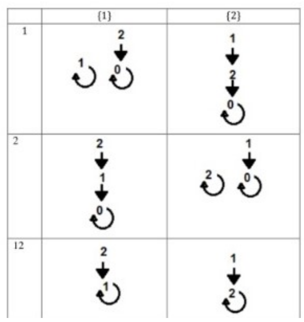

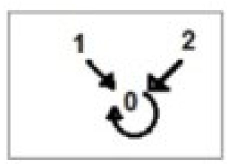

3 Main Results

Classification and Characterization Theorems

Since transformations are special cases of mappings, and functions and equations are special classes of mappings, they share boundaries. The popular fact $P_n \setminus S_n$ depicts that $S_n = \{y = f(x) : x, y \in X_n\}$ is a special class of $P_n = \{\alpha^{m+r} = \alpha^m \mid m, r \in \mathbb{Z}^+\}$, whence $x\alpha^{m+r} = y\alpha^m$ since $\alpha^{m+r} = \alpha^m$ implies that dom $\alpha^{m+r} =$ dom α^m and im $\alpha^{m+r} =$ im α^m.

When $m = 0$ and $r = 1$ in $x\alpha^{m+r} = y\alpha^m$, we get $y = x\alpha$ analog of $y = f(x)$. Thus, translations are on par broader than functions and equations. Thus, if the quasi-idempotent transformation generates the entire P_n, then the quasi-idempotent functions should generate S_n.

Since function $y = f(x)$ are monographical $G = \{(x, y) : y = f(x)\}$, then $y = f^{m+r}(x)$ and $x = f^m(y)$ are digraphical (see [6] for digraphs).

Thus, we take the following results (we use the words digraphical and di-graphical interchange-

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ably to be the analogous of graphical, and 'is digraphical' means 'has digraph'):

Theorem 3.1. Every element of a finite partial transformation semigroup is digraphical.

Proof. Since every element $\alpha \in T_n$ has the universal property $\alpha^{m+r} = \alpha^m$ [12] , and $T_n \subseteq P_n \subseteq T_{n+1}$ or through Vagner [14] , the P_n is isomorphic to P_n^*, a subsemigroup of T_{n+1}. Both T_n and T_{n+1} are digraphical, so P_n is digraphical. Since every element $\alpha \in T_n$ has $x\alpha^{m+r} = y\alpha^m$ such that $x, y \in X_n$, then every element $\alpha \in P_n$ has $x\alpha^{m+r} = y\alpha^m$ such that $x, y \in X_{n+1}$, where $X_{n+1} = X_n \cup \{0\} = \{0, 1, 2, \ldots, n\}$. Let $m = 0$ in $x\alpha^{m+r} = y\alpha^m$, we have $y = f^r(x)$, which is monographical. If $r = 0$ in $x\alpha^{m+r} = y\alpha^m$, then $f^m(x) = f^m(y)$, which is monographical. If $m \neq 0 \neq r$ in $x\alpha^{m+r} = y\alpha^m$, then there exists $z \in X_{n+1}$ such that $z = x\alpha^{m+r}$ and $z = y\alpha^m$, that is, $z = f^{m+r}(x)$ and $z = f^m(y)$, which are both monographable. Thus, every element $\alpha \in P_n$ has a digraph. □

Universal Property

In a finite partial transformation semigroup P_n, every element α satisfies the universal property:

$$\alpha^{m+r} = \alpha^m, \quad \text{for some integers } m \geq 1 \text{ and } r \geq 1.$$

This means that repeated applications of the transformation α stabilize after m applications, such that further applications yield no new results.

Example 3.2. Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \quad X = \{1, 2, 3\}.$$

- Compute iterated applications:

$$\begin{aligned} \alpha(1) = 2, \quad \alpha(2) = 3, \quad \alpha(3) = 3, \\ \alpha^2(1) = \alpha(\alpha(1)) = \alpha(2) = 3, \\ \alpha^2(2) = \alpha(\alpha(2)) = \alpha(3) = 3, \\ \alpha^2(3) = \alpha(\alpha(3)) = \alpha(3) = 3. \end{aligned}$$

- Verify the universal property: Stabilization occurs at $m = 2$, so $\alpha^{m+r} = \alpha^m$ for $r \geq 0$.
- Digraph representation:

$$1 \to 2 \to 3 \circlearrowright$$

Example 3.3. Let

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ \_ & 3 & 2 \end{pmatrix}, \quad X = \{1, 2, 3\}.$$

- Compute iterated applications:

$$\begin{aligned} \beta(1) \text{ is undefined}, \quad \beta(2) = 3, \quad \beta(3) = 2, \\ \beta^2(2) = \beta(\beta(2)) = \beta(3) = 2, \\ \beta^2(3) = \beta(\beta(3)) = \beta(2) = 3. \end{aligned}$$

- Verify the universal property: Stabilization occurs at $m = 2$, so $\beta^{m+r} = \beta^m$ for $r \geq 0$.
- Digraph representation:

$$2 \leftrightarrow 3 \quad \text{(isolated: 1)}.$$

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Example 3.4. Let

$$\gamma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 3 \end{pmatrix}, \quad X = \{0, 1, 2, 3\}.$$

- Compute iterated applications:

$$\begin{aligned} \gamma(0) = 1, \quad \gamma(1) = 2, \quad \gamma(2) = 3, \quad \gamma(3) = 3, \\ \gamma^2(0) = \gamma(\gamma(0)) = \gamma(1) = 2, \\ \gamma^2(1) = \gamma(\gamma(1)) = \gamma(2) = 3, \\ \gamma^2(2) = \gamma(\gamma(2)) = \gamma(3) = 3, \\ \gamma^2(3) = \gamma(\gamma(3)) = \gamma(3) = 3. \end{aligned}$$

- Verify the universal property: Stabilization occurs at $m = 2$, so $\gamma^{m+r} = \gamma^m$ for $r \geq 0$.
- Digraph representation:

$$0 \to 1 \to 2 \to 3 \circlearrowleft$$

Theorem 3.5. Each equivalence class $\{x : x\alpha^{m+qr} = y\alpha^m\}$ generated by the equivalence relation $x \sim y$ defined by $x\alpha^{m+qr} = y\alpha^m$ is digraphical, where $\alpha \in P_n$.

Proof. By Theorem 3.1, $\alpha^{m+r} = \alpha^m$ is digraphical, which implies that $\alpha^{m+r+r} = \alpha^{m+r}$ is digraphical. Since $\alpha^{m+r+r} = \alpha^{m+r}$ implies $\alpha^{m+qr} = \alpha^m$, then $\alpha^{m+qr} = \alpha^m$ is also digraphical.

Let $m + qr = p$. Then:

$$\begin{aligned} \frac{m-p}{q} \in \mathbb{Z} \implies m \equiv p \pmod{q}, \\ \frac{m-q}{p} \in \mathbb{Z} \implies m \equiv q \pmod{p}, \\ \frac{p-m}{r} \in \mathbb{Z} \implies p \equiv m \pmod{r}, \\ \frac{p-m}{q} \in \mathbb{Z} \implies p \equiv m \pmod{q}, \\ \frac{p-qr}{m} = 1 \in \mathbb{Z} \implies p \equiv qr \pmod{m}. \end{aligned}$$

Thus, we have mod p, mod q, mod r, and mod m generating various equivalence relations, each producing distinct equivalence classes.

Thus, every equivalence class generated by the equivalence relation is digraphical for all $p, q, r, m \in \mathbb{Z}$. The set of equivalence classes generated by $p \equiv qr \pmod{m}$ is $\{p : p \equiv qr \pmod{m}\}$. Defining a relationship of $x\alpha^{m+r} = y\alpha^m$ by $x \equiv y$ yields the equivalence classes $\{x : x \equiv y\}$ or $\{y : y \equiv x\}$ or $\{x : x\alpha^{m+r} = y\alpha^m\}$ or $\{y : x\alpha^{m+r} = y\alpha^m\}$.

Thus, $\alpha^{m+qr} = \alpha^m \implies \alpha^m\alpha^{qr} = \alpha^0\alpha^m \implies \alpha^m = 1$ (cycle) and $\alpha^{qr} = \alpha^m$ (standard and acyclic orbits). When $m = 2$ in $\alpha^m = 1$, we get the trivial orbit. The terminal orbit is when x or y is equal to 0 since $x, y \in X_{n+1} = \{0, 1, 2, \ldots, n\}$.

Since the orbits are the equivalence classes generated by the equivalence relation $\alpha^{m+qr} = \alpha^m$, the proof is complete. □

Theorem 3.6. Every digraph of $\alpha \in P_n$ is quasi-idempotent $\alpha \neq \alpha^2 = \alpha^4$ generated.

Proof. Since quasi-idempotents defined by $\alpha \neq \alpha^2 = \alpha^4$ are not just equations and functions but mappings, then $\alpha \neq \alpha^2 = \alpha^4$ may contain $\beta = \beta^2$ (idempotent) by setting $\beta = \alpha^2$. That is, $\alpha \neq \alpha^2 = \alpha^4$ implies that $\alpha \neq \beta = \beta^2$. Thus, both $\alpha \neq \alpha^2$ and $\alpha = \alpha^2$ are in $\alpha \neq \alpha^2 = \alpha^4$. Since equality '=' is an equivalence relation, being reflexive, symmetric, and transitive, then every element $\alpha \in P_n$ is digraphical since P_n is trivially a factor semigroup of partial maps. Thus,

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the digraph of $\alpha^{m+qr} = \alpha^m \in P_n$ is $\alpha \neq \alpha^2 = \alpha^4$ generated, whence $\alpha \neq \alpha^2 = \alpha^4$ implies $\alpha \neq \alpha^2 = \alpha^4 = \alpha^6 = \alpha^8 = \cdots$. Since $\beta = \beta^2 \in P_n$ (idempotents in P_n) and $\alpha \neq \alpha^2 \in P_n$ (non-idempotents in P_n) for arbitrary two elements $\alpha, \beta \in P_n$, then every digraph of $\alpha \in P_n$ is quasi-idempotent $\alpha \neq \alpha^2 = \alpha^4$ generated. □

To illustrate the theorem, we need an example of a partial transformation $\alpha \in P_n$ such that its digraph demonstrates the property $\alpha \neq \alpha^2 = \alpha^4$. Below is an example along with its explanation.

Example 3.7. Let $X_4 = \{1, 2, 3, 4\}$, and define a partial transformation $\alpha \in P_4$ as follows:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & - \end{pmatrix}$$

Applying α again, we get:

$$1\alpha^2 = 2\alpha = 3, \quad 2\alpha^2 = 3\alpha = 3, \quad 3\alpha^2 = 3\alpha = 3, \quad 4\alpha^2 \text{ is undefined.}$$

Thus,

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & - \end{pmatrix}$$

Continuing, $\alpha^4 = \alpha^2$, as further applications of α have no effect.

Theorem 3.8. Every digraph of $\alpha \in P_n$ is stable quasi-idempotent $\alpha \neq \alpha^2 = \alpha^3$ generated.

Proof. Since $\alpha \neq \alpha^2 = \alpha^3$ implies that $\alpha \neq \alpha^2 = \alpha^3 = \alpha^4 = \cdots$, then both $\beta = \beta^2 \in P_n$ (idempotents in P_n) and $\alpha \neq \alpha^2 \in P_n$ (non-idempotents in P_n) exist in $\alpha \neq \alpha^2$ and $\alpha^2 = \alpha^3$. Since $\alpha \in P_n$ has the universal digraphic characterization as $\alpha^{m+qr} = \alpha^m \implies \alpha^m\alpha^{qr} = \alpha^0\alpha^m \implies \alpha^m = 1$ (cycle) and $\alpha^{qr} = \alpha^m$ (standard and acyclic orbits), then every digraph of $\alpha \in P_n$ is stable quasi-idempotent $\alpha \neq \alpha^2 = \alpha^3$ generated. □

Lemma 3.9. Every element $\alpha \in P_n$ having $\alpha^{m+r} = \alpha^m$ has the division algorithm of polynomial functions embedded in it.

Proof. Since $\alpha^{m+r} = \alpha^m$ implies $\alpha^{m+r} = \alpha^{m+r+r} = \alpha^m$, then $\alpha^{m+r} = \alpha^m$ implies $\alpha^{m+qr} = \alpha^m = \alpha^{m+r}$. That is, $\alpha^{m+r} = \alpha^m$ if and only if $\alpha^{m+qr} = \alpha^{m+r}$.

Let $m + r = p$. Then $\alpha^{m+qr} = \alpha^p$. Since $\alpha^{m+qr} = \alpha^p$ does not imply that $p = m + qr = qr + m$, but the reverse may not be true. That is, $p = m + qr$ implies that $\alpha^{m+qr} = \alpha^p$.

When $p = m + qr$, we get reflexivity of $x \sim y$ defined by $x\alpha^{m+qr} = y\alpha^p$. Let p be a polynomial function or equation, then q is the quotient, r is the remainder, and d is the divisor. Since $p = qd + r$, then $\alpha^{m+r} = \alpha^m$ has the division algorithm embedded in it. □

Lemma 3.10. Every element $\alpha \in P_n$ having $\alpha^{m+qr} = \alpha^m$ is decomposable.

Proof. For p to be a polynomial function or equation, let q be the quotient, r the remainder, and d the divisor. Then $p = qd + r$, and $\alpha^{m+r} = \alpha^m$ has the division algorithm embedded in it as stated in Lemma 3.5.

Since the polynomial equation satisfying the division algorithm $p = qd + r$ is factorizable, every element $\alpha \in P_n$ having $\alpha^{m+qr} = \alpha^m$ is decomposable. □

Lemma 3.11. Let A_i denote disjoint subsets (blocks or partitions) of a domain X, and let $a_i \in A_i$ represent selected elements (images) under the transformation. The following hold:

Case 1: Idempotents (ε) An element ε is idempotent if it is characterized by the transformation matrix:

$$\varepsilon = \begin{pmatrix} A_1 & A_2 & \cdots & A_i \\ a_1 & a_2 & \cdots & a_i \end{pmatrix},$$

where $a_i \in A_i$ and each block A_i is mapped entirely to the single element a_i.

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Case 2: Stable Quasi-Idempotents (ξ) An element ξ is a stable quasi-idempotent if it is characterized by the matrix:

$$\xi = \begin{pmatrix} A_i & A_j \\ A_i\xi & a_i \end{pmatrix},$$

where $A_i \cap A_j = \emptyset$, A_i and A_j are disjoint blocks, and $A_i\xi$ is the image of A_i under ξ.

Case 3: Quasi-Idempotents (γ) An element γ is a quasi-idempotent if it is characterized by the matrix:

$$\gamma = \begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}.$$

Proof. **Case 1: Proof of Idempotency**

Let $\varepsilon_{aa} = \begin{pmatrix} A_1 & A_2 & \cdots & A_i \\ a_1 & a_2 & \cdots & a_i \end{pmatrix}$ and $\varepsilon_{bb} = \begin{pmatrix} B_1 & B_2 & \cdots & B_j \\ b_1 & b_2 & \cdots & b_j \end{pmatrix}$, where A_i and B_i are disjoint subsets, and $a_i \in A_i$, $b_i \in B_i$. Then:

$$\varepsilon_{aa}\varepsilon_{bb} = \begin{pmatrix} A_1 & A_2 & \cdots & A_i \\ a_1 & a_2 & \cdots & a_i \end{pmatrix}\begin{pmatrix} B_1 & B_2 & \cdots & B_j \\ b_1 & b_2 & \cdots & b_j \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & \cdots & A_i \\ b_1 & b_2 & \cdots & b_i \end{pmatrix} = \varepsilon_{ab}.$$

If $\varepsilon_{aa} = \varepsilon_{bb}$, then $\varepsilon_{aa}\varepsilon_{aa} = \varepsilon_{aa}$, confirming idempotency.

Case 2: Proof of Stable Quasi-Idempotency

Let $\xi = \begin{pmatrix} A_i & A_j \\ A_i\xi & a_i \end{pmatrix}$, where $A_i \cap A_j = \emptyset$. Then:

$$\xi^2 = \begin{pmatrix} A_i & A_j \\ A_i\xi & a_i \end{pmatrix}\begin{pmatrix} A_i & A_j \\ A_i\xi & a_i \end{pmatrix} = \begin{pmatrix} A_i\xi & A_i\xi \\ A_i\xi & A_i\xi \end{pmatrix} \neq \xi.$$

However,

$$\xi^3 = \xi^2\xi = \begin{pmatrix} A_i\xi & A_i\xi \\ A_i\xi & A_i\xi \end{pmatrix}\begin{pmatrix} A_i & A_j \\ A_i\xi & a_i \end{pmatrix} = \begin{pmatrix} A_i\xi & A_i\xi \\ A_i\xi & A_i\xi \end{pmatrix} = \xi^2.$$

Thus, $\xi^n = \xi^2$ for all $n \geq 2$, confirming stable quasi-idempotency.

Case 3: Proof of Quasi-Idempotency

Let $\gamma = \begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}$. Then:

$$\gamma^2 = \begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}\begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix} = \begin{pmatrix} A_i\gamma & a_i \\ A_i\gamma & A_i\gamma \end{pmatrix}.$$

And for γ^4:

$$\gamma^4 = \begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}\begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix} = \begin{pmatrix} A_i\gamma & a_i \\ A_i\gamma & A_i\gamma \end{pmatrix}.$$

Thus, $\gamma \neq \gamma^2 = \gamma^4$, confirming quasi-idempotency. □

Theorem 3.12. The product of two idempotents ε_1^1 and ε_2^1 of defects 1 each, such that $\text{span}(\varepsilon_1^1) \cap \text{span}(\varepsilon_2^1) = \emptyset$, is an idempotent $\varepsilon_{1,2}^2$ of defect 2. Thus, the idempotent of defect 2, $\varepsilon_{1,2}^2$, is decomposable into a product of ε_1^1 and ε_2^1 (each of defect 1).

Proof. The result follows from the fact that

$$\begin{pmatrix} a_i \\ b_j \end{pmatrix}\begin{pmatrix} c_k \\ d_l \end{pmatrix} = \begin{pmatrix} a_i & c_k \\ b_j & d_l \end{pmatrix}$$

since

$$\begin{pmatrix} a_i \\ b_j \end{pmatrix}\begin{pmatrix} c_k \\ d_l \end{pmatrix} = \begin{pmatrix} a_i \\ b_j \end{pmatrix}$$

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or

$$\begin{pmatrix} a_i \\ b_j \end{pmatrix} \begin{pmatrix} c_k \\ d_l \end{pmatrix} = \begin{pmatrix} c_k \\ d_l \end{pmatrix}$$

behaving like $a \wedge b = a$, where $a \wedge b = \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b \leq a \end{cases}$.

Thus, the set of idempotents is a semilattice $L(\leq, \wedge, \vee)$. Also,

$$\begin{pmatrix} a_i & c_k \\ b_j & d_l \end{pmatrix} = \begin{pmatrix} a_i \\ b_j \end{pmatrix} \begin{pmatrix} c_k \\ d_l \end{pmatrix}$$

if $\text{span}(\varepsilon_1^1) \cap \text{span}(\varepsilon_2^1) = \emptyset$, where $\varepsilon_1^1 = \begin{pmatrix} a_i \\ b_j \end{pmatrix}$ and $\varepsilon_2^1 = \begin{pmatrix} c_k \\ d_l \end{pmatrix}$. □

Corollary 3.13. An idempotent of defect d, written as ε^d, is decomposable into d idempotents of defect 1, written as ε_i^1.

Proof. Let

$$\varepsilon^d = \begin{pmatrix} 1 & 2 & 3 & \cdots & d \\ 2d & 2d-1 & 2d-2 & \cdots & d+1 \end{pmatrix}.$$

To decompose ε^d into d idempotents of defect 1, we construct:

$$\varepsilon_i^1 = \begin{pmatrix} i \\ 2d-(i-1) \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, d.$$

Verification: 1. Idempotency of Each ε_i^1: Each ε_i^1 maps $i \to 2d-(i-1)$ and $2d-(i-1) \to 2d-(i-1)$, so:

$$\varepsilon_i^1 \circ \varepsilon_i^1 = \varepsilon_i^1.$$

2. Composition of ε^d: Combining all ε_i^1, we have:

$$\varepsilon^d = \varepsilon_1^1 \circ \varepsilon_2^1 \circ \cdots \circ \varepsilon_d^1.$$

Substituting the explicit maps ε_i^1, the combined transformation matches ε^d.

Example: Decomposition of ε^4. Let $d = 4$, so:

$$\varepsilon^4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix}.$$

Decomposing ε^4 into idempotents of defect 1:

$$\varepsilon_1^1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix}, \quad \varepsilon_2^1 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \quad \varepsilon_3^1 = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad \varepsilon_4^1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Now, combining these maps:

$$\varepsilon_1^1 \circ \varepsilon_2^1 \circ \varepsilon_3^1 \circ \varepsilon_4^1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{pmatrix} = \varepsilon^4.$$

Thus, the decomposition is constructive, and the corollary holds for all d. □

Theorem 3.14. A quasi-idempotent of defect $2d$ is decomposable into a product of a stable quasi-idempotent of defect $d-i$ and a quasi-idempotent of defect $d+i$.

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Proof. Given that

$$\gamma = \begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i & A_i & A_j \\ a_i & A_i\varepsilon & a_i \end{pmatrix}$$

by Lemma 3.7, then

$$\begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i & A_i & A_j \\ a_i & A_i\varepsilon & a_i \end{pmatrix} \neq \begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i & A_i & A_j \\ a_i & A_i\varepsilon & a_i \end{pmatrix}$$

$$= \begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i & A_i & A_j \\ a_i & A_i\varepsilon & a_i \end{pmatrix} \begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i & A_i & A_j \\ a_i & A_i\varepsilon & a_i \end{pmatrix}$$

and so $\gamma \neq \gamma^2 = \gamma^4$, making γ a quasi-idempotent of defect $2d$ decomposable into a product as

$$\begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} b_i \\ a_i \end{pmatrix} \begin{pmatrix} A_i & A_j \\ A_i\varepsilon & a_i \end{pmatrix}$$

and the result follows immediately. □

Corollary 3.15. A quasi-idempotent of defect $2d$ is decomposable into a stable quasi-idempotent of defect d and a stable quasi-idempotent of defect d.

Proof. Let

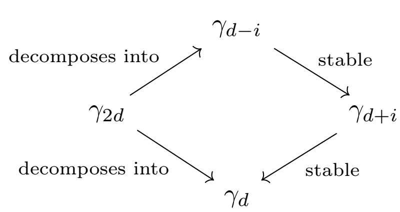
$$\gamma = \begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} A_i & A_j \\ A_i\varepsilon & a_i \end{pmatrix}$$

analogous to γ in Theorem 3.10. Then $\gamma \neq \gamma^2 = \gamma^4$, making γ a quasi-idempotent decomposable into

$$\begin{pmatrix} A_i & A_i\gamma & a_i \\ A_i\gamma & a_i & b_i \end{pmatrix} \begin{pmatrix} A_i & A_j \\ A_i\varepsilon & a_i \end{pmatrix}$$

and the result follows immediately. □

The digraph below illustrates the relationships among quasi-idempotents, their defects, and decompositions by representing each quasi-idempotent as a node labeled with its defect (d or $2d$), with directed edges showing how a quasi-idempotent of defect $2d$ decomposes into quasi-idempotents of defects $d - i$ and $d + i$; for instance, node γ, representing a quasi-idempotent of defect $2d$, has outgoing edges to nodes of defects $d-i$ and $d+i$, which can be recursively decomposed, providing an intuitive view of structural relationships and aligning with the algebraic decomposition to enhance the understanding of the proofs.



Corollary 3.16. The product of a stable quasi-idempotent and a quasi-idempotent is a quasi-idempotent.

Proof. This follows from Theorem 3.10 and Corollary 3.11. □

Corollary 3.17. The set of quasi-idempotents generates the set of idempotents.

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Proof. From Lemma 3.7, let

$$\gamma = \begin{pmatrix} A_i & A_i\gamma & B_i \\ A_i\gamma & a_i & b_i \end{pmatrix}$$

then $\gamma \neq \gamma^2 = \gamma^4$ and

$$\gamma = \begin{pmatrix} A_i & A_i\gamma & B_i \\ A_i\gamma & a_i & b_i \end{pmatrix} = \begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix} \begin{pmatrix} B_i \\ b_i \end{pmatrix}$$

where

$$\begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}$$

is a quasi-idempotent and

$$\begin{pmatrix} B_i \\ b_i \end{pmatrix}$$

is an idempotent. Since

$$\begin{pmatrix} B_i \\ b_i \end{pmatrix}$$

generates $T_n \setminus S_n$ (and $P_n \setminus S_n$ through Vagner), and

$$\begin{pmatrix} A_i & A_i\gamma \\ A_i\gamma & a_i \end{pmatrix}$$

generates S_n, since the products of transpositions are quasi-idempotents, then

$$\begin{pmatrix} A_i & A_i\gamma & B_i \\ A_i\gamma & a_i & b_i \end{pmatrix}$$

generates both T_n and P_n. □

4 Conclusion

This work establishes that every element of a finite partial transformation semigroup is digraphical, providing a structural perspective that bridges algebraic and combinatorial representations. The characterization of elements via digraphs enhances our understanding of the interplay between quasi-idempotents, idempotents, and their defects. By demonstrating that equivalence classes, idempotents, stable quasi-idempotents, and quasi-idempotents are inherently digraphical and decomposable, we provide a comprehensive framework for understanding these elements' interactions within P_n.

Significantly, the decomposition results, such as expressing idempotents of defect d into d idempotents of defect 1, and quasi-idempotents of defect $2d$ into components of defect $d-i$ and $d+i$, offer new pathways for studying the algebraic properties of transformation semigroups. These findings highlight the versatility of digraph-based approaches in unraveling complex algebraic structures and open new avenues for theoretical exploration and computational implementation.

This study deepens the connection between transformation semigroup theory and other areas of algebra, such as semilattice embedding, stability analysis, and decomposition theory. It strengthens the foundational understanding of partial transformation semigroups, particularly the interplay between quasi-idempotents and idempotents. Moreover, the identification of digraphical properties offers a visual and combinatorial toolset that could prove invaluable for algorithm design, computational verification, and practical applications in automata theory and formal language processing.

To build upon these findings, future research could explore computational algorithms for efficiently identifying and decomposing digraphical elements in larger classes of semigroups. Extending the results to other semigroup classes, such as infinite semigroups or transformation semigroups under different constraints, could broaden their applicability. Additionally, investigating the role of

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these decompositions in optimization problems, coding theory, or other combinatorial settings could provide insights into real-world applications.

By integrating these findings with existing theories and expanding their computational and practical scope, this work lays the groundwork for advancing both the theoretical and applied dimensions of transformation semigroup theory.

5 Competing Interests

The authors declare that they have no competing interests.

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