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The equivalence of some iteration schemes with their errors for uniformly continuous strongly successively pseudo-contractive operators

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Abstract

Some iteration schemes may converge faster for certain types of functions or structures in an arbitrary space. In this paper, we show that the convergence of modified Mann iteration, modified Mann iteration with errors, modified Ishikawa iteration, modified Ishikawa iteration with errors, modified Noor iteration, modified Noor iteration with errors, modified multistep iteration and modified multistep iteration with errors are equivalent for uniformly continuous strongly successively pseudo-contractive maps in an arbitratry real Banach space. The results generalize and extend the results of several authors, including Huang and Bu [1], Rhoades and Soltuz [2–4] and improve the results of Huang et al. [5].

Keywords: Modified Mann-Ishikawa Iterations (with errors), Modified Noor-Multistep Iteration(with errors), Uniformly Continuous Maps, Strongly Successively Pseudocontractive Maps.
MSC2010: 47H10, 54H25.

1 Introduction and Preliminaries

Fixed point theory plays a crucial role in the study of nonlinear analysis, particularly in the existence and uniqueness of solutions to various equations. Researchers related to contraction mappings, such as Banach's fixed point theorem, provide a fundamental framework for iterative methods in solving differential and integral equations. Extensions of contraction principles in metric, fuzzy, and other generalized spaces have further broadened their applications in diverse Mathematical and real-world problems (see [6–8]). The study of pseudocontractive maps and their coupled approximation methods for finding fixed points remains an active area of research till today. For example, see [9–11]. Yang [12] established the equivalence between Mann, Ishkawa and multistep iterations in a real Banach space. Rhoades and Soltuz [2] established the equivalence of the convergence between Mann and Ishikawa iterations for non-Lipschitzian operator. Rhoades and Soltuz [3] proved the equivalence of the convergences between the Mann and Ishikawa iterations for strongly successively pseudo-contractive maps without Lipschitzian assumption. In 2009, Olaleru and Odumosu [13] established the equivalence of the convergence of iterative procedures with errors for uniformly Lipschitzian strongly successively pseudo-contractive operators. This research improves on the results of Huang et al. [5] by proving the equivalence between the convergence of the known iterative scheme viz modified Mann, Ishikawa, Noor and multi-step iteration with errors for strongly successively pseudo-contractive mapping without Lipschitzian assumption in an arbitrary real Banach space. Our main results will also generalize and extend the results of several authors, including Huang and Bu [1], Rhoades and Soltuz [3] and improve the results of Huang et al. [5].

Let X be a real Banach space, and K a non-empty subset of X, T a self mapping of K and $F(D), D(T)$ and I are the set of fixed points, domain of T and identity operator respectively. Let J denote the normalized duality mapping from X to 2^{X^*} defined by;

$$J(X) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\} \text{ for all } x \in X$$

where X^* denotes the dual space of X and $\langle ., .\rangle$ denotes the generalized duality pairing.
Let X be a real Banach space and $D(T), R(T)$ represent the domain and range of T respectively, then, we have the following definitions.

Definition 1 [14, 15]: A mapping $T : X \to X$ is said to be non-expansive on X if,

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in D(T).$$

Definition 2 [4]: A map $T : K \to K$ is said to be Lipschitzian if there exists $L \geq 1$, such that,

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in D(T).$$

If $L = 1$ in Definition 2, then T is said to be non-expansive.

Definition 3 [4]: A map $T : K \to K$ is said to be uniformly Lipschitzian if there exists $L \geq 1$, such that,

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in D(T).$$

If $T^n = T$ in Definition 3, then T is said to be Lipschitzian. (Definition 2).

Definition 4 [16]: A map $T : K \to K$ is said to be pseudo-contractive if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that,

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 \text{ for all } x, y \in D(T).$$

Definition 5 [16]: A map $T : K \to K$ is said to be successively pseudo-contractive if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that,

$$\langle T^n x - T^n y, j(x - y)\rangle \leq \|x - y\|^2 \text{ for all } x, y \in D(T)$$

If $T^n = T$, a successively pseudocontractive map is a pseudocontractive map.

Definition 6 [16]: A map $T : K \to K$ is said to be strongly pseudocontractive if for each $x, y \in D(T)$ there exists $k \in (0, 1)$ and $j(x - y) \in J(x - y)$, such that,

$$\langle Tx - Ty, j(x - y)\rangle \leq (1 - k)\|x - y\|^2$$

This is equivalent to;

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\| \text{ where } r > 0.$$

A strongly pseudocontractive map is a pseudocontractive map but the converse is not true.

Definition 7 [16]: A map $T : K \to K$ is said to be strongly successively pseudocontractive if there exists $k \in (0,1)$ and $n \in \mathbb{N}$ such that;

$$\langle T^n x - T^n y, j(x-y)\rangle \leq (1-k)\|x-y\|,$$

Equivalently, T is strongly successively pseudocontractive if there exists $k \in (0,1)$ such that, for all $x, y \in K, n \in \mathbb{N}^+$ and $j(x-y) \in J(x-y)$

$$\|x-y\| \leq \|x-y+r[(I-T^n-kI)x-(I-T^n-kI)y]\| \tag{1.1}$$

for all $x, y \in K$ and $r > 0$.
The Mann iteration scheme was introduced in 1953 [17] to obtain a fixed point for many functions for which Banach principle fails. In 1974, Ishikawa [18], introduced another iteration scheme also referred to as two-step iteration scheme. Noor [19] introduced a three-step iterative scheme and used it to approximate solution of variational problems in Hilbert spaces. Noor, Rassias and Hung [20] extend the procedure to solving non-linear equations in Banach spaces. Golwinski and Le Tallec [21] used the scheme to approximate solutions of the elastovis-coplasticity problem, liquid crystal theory and eigen computation.
The modified Mann iteration with errors is defined as; (see [3])
$x_1 \in K$,

$$x_{n+1} = (1-b_n)x_n + b_n T^n x_n + c_n(s_n - x_n), n \geq 1 \tag{1.2}$$

where $\{s_n\}$ is a bounded sequence in K and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1)$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$.
Observe that (2) is equivalent to

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n s_n, n \geq 1$$

Remark 1.
1. If $c_n = 0$ for each n, then we have the modified Mann iterative scheme.
2. If T^n is replaced by T in (2), we obtain the modified Mann iterations with errors in the sense of Xu [22]. If in addition, $c_n = 1$, then (2) is called the Mann iteration with errors in the sense of Liu [23].
3. If T^n is replaced by T in (2), and $c_n = 0$, then (2) is called the Mann iteration.
In 2004, Rhoades and Soltuz [3] introduced the modified multistep iteration as follows,

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1}, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n, p \geq 2 \end{aligned} \tag{1.3}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ in $(0,1)$ satisfy certain conditions.
Remark 2
1. If T^n is replaced by T, the modified multistep iteration (3) is referred to as multistep iteration.
2. If $p = 3$, (3) becomes the modified Noor or three-step iteration procedure and if in addition, T^n is replaced by T, it is called Noor or three step iteration.
3. If $p = 2$, (3) becomes the modified Ishikawa iteration procedure and if in addition, T^n is replaced by T, it is called Ishikawa iteration.
The modified multistep iteration with errors introduced by Liu and Kang [24] is defined by,

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + w_n \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + w_n^1, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + w_n^{p-1} p \geq 2 \end{aligned} \tag{1.4}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ are in $[0,1)$ and the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ...p-1)$ are convergent sequences in K, all satisfying certain conditions.

Remark 3.

1. If T^n is replaced by T, the modified multistep iteration with errors (4) reduces to the Noor and Ishikawa iteration with errors respectively when $p = 3$ and;
2. If in addition, $w_n = w_n^i = 0, (i = 1, 2, ...)$ for all $n \in \mathbb{N}$, then (4) reduces to Noor and Ishikawa iterations (without errors) respectively.

The Ishikawa and Mann iteration with errors of (4) was introduced by Liu [24]. Numerous papers have been published that utilize this iteration procedure with error terms.. For example, see [5, 16, 23, 25].

However, it should be noted that the iteration process with errors in (4) is not satisfactory. The errors can occur in a random way. The condition then imposed on the error terms which say that they tend to zero as n tends to infinity are therefore unreasonable (see [26]). This informed the introduction of a better modified iterative processes with errors by Xu [22].

The Xu's modified multistep with errors is defined as follows:

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1 - b_n)u_n + b_n T^n v_n^1 + c_n(w_n - u_n) \\ v_n^i &= (1 - b_n^i)u_n + b_n^i T^n v_n^{i+1} + c_n^i(w_n^i - u_n), i = 1, ..., p-2 \\ v_n^{p-1} &= (1 - b_n^{p-1})u_n + b_n^{p-1} T^n u_n + c_n^{p-1}(w_n^{p-1} - u_n), p \geq 2 \end{aligned} \tag{1.5}$$

where the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ...p-1)$ are bounded sequences and $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ in $[0,1)$ satisfy certain conditions $n \in \mathbb{N}$.

Observe that the modified multistep iteration with errors (5) is equivalent to

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1 - b_n)u_n + b_n T^n v_n^1 + c_n w_n \\ v_n^i &= (1 - b_n^i)u_n + b_n^i T^n v_n^{i+1} + c_n^i w_n^i, i = 1, ..., p-2 \\ v_n^{p-1} &= (1 - b_n^{p-1})u_n + b_n^{p-1} T^n u_n + c_n^{p-1} w_n^{p-1}, \geq 2 \end{aligned} \tag{1.6}$$

where the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ...p-1)$ are bounded sequences in K, and $\{a_n\}$, $\{a_n^i\}$, $\{b_n\}$, $\{b_n^i\}(i = 1, ...p-1)$ in $[0,1)$ satisfying
$a_n + b_n + c_n = a_n^i + b_n^i + c_n^i = 1, i = 1, 2, ..., p-1$.

In this paper, we show that the modified Mann, Ishikawa, Noor and multistep iteration with errors (5) (using the more satisfactory definition by Xu [22]) and that these iterations without errors are all equivalent for strongly successively pseudo-contractive mapping without Lipschitzian assumption in an arbitrary real Banach space.

The result generalize and extend the results of several authors, including Huang and Bu [1], Rhoades and Soltuz [3] and improve the results of Huang et al. [5].

In the proof of our results, Lemma 1 [27] and Lemma 2 [28] below are useful.

Lemma 1 [27]: Let E be a real normed linear space. Then, the following inequalities holds

$$\|x - y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \text{ for all } j(x + y) \in J(x + y).$$

Lemma 2 [28]: Let $(\alpha_n)_n$ be a non-negative sequence which satisfies
the following inequality

$$\alpha_{n+1} \leq (1 - \lambda_n)\alpha_n + \delta_n$$

where $\lambda_n \in (0,1), \forall n \in \mathbb{N}$, $\sum_{n \to 1}^{\infty} \lambda_n = \infty$ and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

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2 Main results

Theorem 1: Let X be an arbitrary Banach space. Let T be uniformly continuous strongly successively pseudo-contractive operator such that

$$\langle T^n x_n - T^n y_n, j(x_n - y_n)\rangle \leq (1-k)\|x-y\|^2, \tag{2.1}$$

where $k \in (0,1)$ a self map of X. Suppose x^* is a fixed point of T and if $\{x_n\}$ be a sequence in X as defined in

$$\begin{aligned} x_1 &\in K \\ x_{n+1} &= (1-b_n)u_n + b_n T^n y_n^1 + c_n(w_n - x_n) \\ y_n^i &= (1-b_n^i)x_n + b_n^i T^n y_n^{i+1} + c_n^i(w_n^i - x_n), i = 1, ..., p-2 \\ y_n^{p-1} &= (1-b_n^{p-1})x_n + b_n^{p-1} T^n x_n + c_n^{p-1}(w_n^{p-1} - x_n), p \geq 2 \end{aligned} \tag{2.2}$$

where the sequences $\{w_n\}$ and $\{w_n^i\}$, are bounded sequences in X, with $\{b_n\}$, $\{c_n\}$, $\{b_n^i\}, \{b_n^i\} \in [0,1], i = 1, ..., p-1$.
satisfying the conditions:

1. $a_n + b_n + c_n = 1 = a_n^i + b_n^i + c_n^i, n \geq 0, i = 1, ..., p-1$, where $a_n = 1-(b_n + c_n)$ and $a_n^i = 1 - (b_n^i + c_n^i)$
2. $\lim_{n\to\infty} b_n = 0, c_n^{p-1} = b_n^{p-1} = o(b_n), i = 1, ..., p-1,$
3. $\sum_{n\to 1}^{\infty} c_n < \infty, \sum_{n\to 1}^{\infty} b_n = 0$

Then, there exists $t_0 > 0$ such that if $b_n + c_n < t_0$, then $\{x_n\}, n \in \mathbb{N}$ is bounded.

Proof: If $x_0 = T^n x_0, \forall n \geq 0, n \in \mathbb{N}$, then we are done. Suppose this is not the case, that is suppose there exists $n_0 \in \mathbb{N}$, the smallest positive integer such that $x_0 \neq T^n x_0$. Let $c_n + b_n < \frac{1}{2}$ for all $n \geq n_0$. Without loss of generality define $x_{n_0} = x_0$.
Then inequality (7) implies

$$\begin{aligned} \langle T^n x_0 - T^n x^*, j(x_0 - x^*)\rangle &\leq (1-k)\|x_0 - x^*\|^2 \\ \implies \|T^n x_0 - T^n x^*\|\|x_0 - x^*\| &\leq (1-k)\|x_0 - x^*\|^2 \\ k\|x_0 - x^*\| &\leq \|x_0 - x^*\| - \|T^n x_0 - T^n x^*\| \\ &\leq \|x_0 - x^* - T^n x_0 + T^n x^*\| \\ &\leq \|x_0 - T^n x_0\| \end{aligned}$$

Hence,

$$\|x_0 - x^*\| \leq \frac{1}{k}\|x_0 - T^n x_0\| = \lambda q,$$

where $\lambda = \frac{1}{k}$ and $q = \|x_0 - T^n x_0\|$.
Since T is uniformly continuous, there exists a δ such that
$\|T^n x - T^n y\| < \epsilon$ whenever $\|x - y\| < \delta$.

Set $\epsilon = \dfrac{1-(\delta+k)}{4} > 0$.

Claim $\|x_0 - x^*\| \leq 2\lambda q, \forall n \geq 0$
The claim holds for $n = 0$.
Assume that it holds for some n, that is, assume that

$$\|x_n - x^*\| \leq 2\lambda q$$

To prove that

$$\|x_{n+1} - x^*\| \leq 2\lambda q$$

Suppose this is not the case.
Then,

$$\|x_{n+1} - x^*\| > 2\lambda q$$

This implies

$$k\|x_{n+1} - x^*\| > 2\lambda qk \tag{2.3}$$

Now define $t_0 = \frac{1}{2}\min\{1, \frac{\delta}{2(2\lambda)(2M_A+2\lambda q)}, \frac{1}{1+2\lambda(N_A+2\lambda q)}\}$. Since $\{w_n\}$ and $\{w_n^i\}, n \geq 0, i = 1, ...p-1$, are bounded and x^* is a fixed point of T. There exists a positive constant n such that
$N_A = \sup_{n,i}\{\|w_n - x^*\|, \|w_n^i - x^*\|, i = 1, ...p-1\}$ and
$M_A = \sup_{n,p}\{\|x - T^n x\|, \|x - T^n y^{p-1} n : \|x - x_0\| \leq 7\lambda q + N_A\}$
then, $M_A < \infty$.
It follows from (8) and (9) that,

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n T^n y_n^1 + c_n(w_n - x_n) \\ y_n^{p-1} &= (1-b_n^{p-1})x_n + b_n^{p-1}T^n x_n + c_n^{p-1}(w_n^{p-1} - x_n), \end{aligned}$$

and

$$\begin{aligned} \|w_n - x_n\| &\leq \|w_n - x^*\| + \|x^* - x_n\| \\ &\leq N_A + 2\lambda q. \end{aligned}$$

Similarly,

$$\begin{aligned} \|w_n^{p-1} - x_n\| &\leq \|w_n^{p-1} - x^*\| + \|x^* - x_n\| \\ &\leq N_A + 2\lambda q. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq \|x_n - x^*\| + b_n\|T^n y_n^1 - x_n\| + c_n\|w_n - x_n\| \\ &\leq 2\lambda q + b_n M_A + c_n[N_A + 2\lambda q] \\ \|y_n^{p-1} - x^*\| &\leq \|x_n - x^*\| + b_n^{p-1}\|T^n y_n^1 - x_n\| + c_n^{p-1}\|w_n^{p-1} - x_n\| \\ &\leq 2\lambda q + b_n^{p-1}M_A + c_n^{p-1}[N_A + 2\lambda q] \\ &\leq 6\lambda q, p \geq 2. \end{aligned}$$

Also,

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq b_n\|T^n y_n^1 - x_n\| + c_n\|w_n - x_n\| \\ &\leq b_n M_A + c_n[N_A + 2\lambda q] \\ &\leq b_n[M_A + M_A + 2\lambda q], \\ &\leq t_0[2M_A + 2\lambda q] < \delta. \end{aligned}$$

So that

$$\|T^n x_{n+1} - T^n x_n\| < \tfrac{\epsilon}{2}.$$

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Also

$$\begin{aligned}\|y_n^{p-1}-x_n\| &\le b_n^{p-1}\|T^n x_n - x_n\| + c_n^{p-1}\|w_n^{p-1}-x_n\|\\ &\le b_n^{p-1}M_A + c_n^{p-1}[N_A+2\lambda q]\\ &\le t_0[M_A+M_A+2\lambda q] < \delta,\end{aligned}$$

so that

$$\|T^n y_n^{p-1} - T^n x_n\| < \tfrac{\epsilon}{2}.$$

Hence

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n^{p-1}\| &\le \|T^n x_{n+1} - T^n x_n\| + \|T^n x_n - T^n y_n^{p-1}\|\\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2}.\\ &= \epsilon.\end{aligned}$$

Using the above estimates yields

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(x_n - x^*) - b_n(x_n - T^n y_n^1) + c_n(w_n - x_n)\|^2\\ &\le \|x_n - x^*\|^2 - 2b_n\langle x_n - T^n y_n^1, j(x_{n+1}-x^*)\rangle\\ &+ 2c_n\langle w_n - x_n, j(x_{n+1}-x^*), \rangle\end{aligned}$$

which implies

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|x_n - x^*\|^2\\ &+ 2b_n\langle T^n y_n^1 - T^n x_{n+1} + T^n x_{n+1} + x^* - x^* - x_n, j(x_{n+1}-x^*)\rangle\\ &+ 2c_n\langle w_n - x_n, j(x_{n+1}-x^*)\rangle\\ &\le \|x_n - x^*\|^2 - 2b_n(1-k)\|x_{n+1}-x^*\|\\ &+ 2b_n\|x_{n+1}-x^*\|\|x_{n+1}-x_n\| + 2b_n\|T^n x_{n+1} - T^n y_n^1\|\\ &\|x_{n+1}-x^*\| + 2c_n\|w_n - x_n\|x_{n+1}-x^*\|\\ &\le \|x_n - x^*\|^2 - 4b_n(1-k)\lambda q + 12b_n\lambda q\delta + 12b_n\lambda q\epsilon\\ &+ 12c_n[2M_A+\lambda q]\lambda q\\ &= \|x_n - x^*\|^2 - 4b_n\lambda q + 12b_n\lambda qk + 12b_n\lambda q\delta + 12b_n\lambda q\epsilon\\ &+ 12c_n[2M_A+\lambda q]\lambda q\\ &= \|x_n - x^*\|^2 - 4b_n\lambda q + 3b_n\lambda q\\ &+ 12c_n[2M_A+\lambda q]\lambda q\\ &= \|x_n - x^*\|^2 - b_n\lambda q\\ &+ 12c_n[2MA+\lambda q]\lambda q\\ &= \|x_n - x^*\|^2 - b_n M_1 + c_n + g_n,\end{aligned}$$

where $M_1 = \lambda q$, and $g_n = 12\lambda q[2M_A + 2\lambda q]$.
Hence

$$\|x_{n+1}-x^*\|^2 \le \|x_n - u_n\|^2 - b_n M_1 c_n g_n.$$

This implies that

$$\begin{aligned}M_1\sum_{j=1}^{n} b_j &\le \sum_{j=i}^{n}(\|x_j - x^*\|^2 - \|x_{j+1}-x^*\|^2)\\ &+ h\sum_{j=1}^{n} c_j.\end{aligned}$$

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so that $\sum_{n=1}^{n} c_n < \infty$.

This implies that $\sum_{n=1}^{n} b_n < \infty$, a contradiction. Hence, $\{x_n\}$ is bounded.

Theorem 1 will be very useful in proving Theorem 2 below.

Theorem 2: Let X be a real Banach space with dual uniformity convex, K a non empty closed and convex subset of X and $T : K \to K$ be uniformly continuous and strongly successively pseudo-contractive operator such that

$$\|x-y\| \leq \|x-y-r[(I-T-kI)x-(I-T-kI)y]\|.$$

Suppose the modified Mann iteration with errors iteratively defined by the sequence $\{u_n\}$ and the modified multi-step iteration with errors iteratively defined by the sequence $\{x_n\}$ respectively are given by

$$u_{n+1} = (1-b_n)u_n + b_n T^n u_n + c_n(v_n - u_n), n \geq 1 \tag{2.4}$$

and

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n T^n y_n^1 + c_n(w_n - x_n)n \geq 1 \\ y_n^i &= (1-b_n^i)x_n + b_n^i T^n y_n^{i+1} + c_n^i(w_n^i - x_n), i = 1, ..., p-2 \\ y_n^{p-1} &= (1-b_n^{p-1})x_n + b_n^{p-1} T^n x_n + c_n^{p-1}(w_n^{p-1} - x_n), p \geq 2 \end{aligned} \tag{2.5}$$

where the sequences $\{w_n\}$ and $\{w_n^i\}$, are bounded sequences in X, with $\{b_n\}$, $\{c_n\}$, $\{b_n^i\}, \{b_n^i\} \in [0,1], i = 1, ..., p-1$.
satisfying the conditions;

1. $a_n + b_n + c_n = 1 = a_n^i + b_n^i + c_n^i, n \geq 0, i = 1, ..., p-1$, where $a_n = 1-(b_n + c_n)$ and $a_n^i = 1-(b_n^i + c_n^i)$
2. $\lim_{n\to\infty} b_n = 0, c_n^{p-1} = b_n^{p-1} = o(b_n), i = 1, ..., p-1$,
3. $\sum_{n=1}^{\infty} c_n < \infty, b_n < \frac{1-k}{2}, \sum_{n=1}^{\infty} b_n = \infty$.

Suppose x^* is a fixed point of T, then for any $u_1, x_1 \in K$, the following statements are equivalent:

(i) the modified Mann iteration (10) converges strongly to x^*;

(ii) the modified multi-step iteration with errors (11) converges strongly to x^*.

Proof: $(ii) \implies (i)$. If the modified multi-step iteration with errors (11) converges, then by setting $b_n^i = c_n^i = 0$ and $w_n = v_n, (i \in \mathbb{N})\forall n \in \mathbb{N}$ we obtain the convergence of modified Mann iteration with errors (10).
We shall prove that $(i) \implies (ii)$: Suppose $\{u_n\}$ converges to x^*. By assumption from Theorem 1,$\{x_n\}$ is bounded. Also for the fact that $\{u_n\}$ converges, it is also bounded. Thus by the uniform continuity of T, $\{T^n x_n\}$, $\{T^n u_n\}$, $\{T^n y_n^1\}$ and $\{T^n y_n^2\}$ are bounded.
Let $M = \max\{\sup\|u_n\|, \sup\|x_n\|, \sup\|T^n u_n\|, \sup\|v_n\|, \sup\|T^n y_n^1\|,$
$\sup\|T^n y_n^2\|, \sup\|w_n^1\|, \sup\|w_n\|, \forall n \in \mathbb{N}\}$.
Then, $M < \infty$.

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Using (10), (11) and Lemma 1, we have

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\|^2 &= \|(1-b_n)(x_n - u_n) + b_n(T^n y_n^1 - T^n u_n) - c_n(x_n - u_n + v_n - w_n)\|^2 \\
&\le (1-b_n)^2\|x_n - u_n\|^2 + 2b_n\langle T^n y_n^1 - T^n u_n, j(x_{n+1} - u_{n+1})\rangle \\
&- 2c_n\langle x_n - u_n + v_n - w_n, j(x_{n+1} - u_{n+1})\rangle \\
&= (1-b_n)^2\|x_n - u_n\|^2 + 2b_n\langle T^n y_n^1 - T^n u_n - T^n x_{n+1} \\
&+ T^n u_{n+1} + T^n x_{n+1} - T^n u_{n+1}, j(x_{n+1} - u_{n+1})\rangle \\
&- 2c_n\langle x_n - u_n + v_n - w_n, j(x_{n+1} - u_{n+1})\rangle \\
&\le (1-b_n)^2\|x_n - u_n\|^2 + 2b_n(1-k)\|x_{n+1} - u_{n+1}\|^2 \\
&+ 2b_n\|T^n y_n^1 - T^n u_n - T^n x_{n+1} + T^n u_{n+1}\|\|x_{n+1} - u_{n+1}\| \\
&- 2c_n\|x_n - u_n + v_n - w_n\|\|x_{n+1} - u_{n+1}\| \\
&= (1-b_n)^2\|x_n - u_n\|^2 + 2b_n(1-k)\|x_{n+1} - u_{n+1}\|^2 \\
&+ 2\rho_n\|x_{n+1} - u_{n+1}\|.
\end{aligned}$$

Hence

$$\begin{aligned}
(1 - 2b_n(1-k))\|x_{n+1} - u_{n+1}\|^2 \le (1-b_n)^2\|x_n - u_n\|^2 \\
+ 2b_n(1-k)\|x_{n+1} - u_{n+1}\|^2 \\
+ 2\rho_n\|x_{n+1} - u_{n+1}\| \qquad (2.6)
\end{aligned}$$

where $\rho_n = \|T^n y_n^1 - T^n u_n - T^n x_{n+1} + T^n u_{n+1}\| - c_n\|x_n - u_n + v_n - w_n\|$.
From (10) and Theorem (1), the following result is obtained

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le b_n\|T^n u_n - u_n\| + c_n\|v_n - u_n\| \\
&\le 2b_n M + 2c_n M \to 0 \text{ as } n \to \infty \\
&= 2M(b_n + c_n) \to 0 \text{ as } n \to \infty. \qquad (2.7)
\end{aligned}$$

It follows from (11) and Theorem 1 that

$$\begin{aligned}
\|x_{n+1} - y_n^1\| &\le b_n\|x_n - T^n y_n^1\| + b_n^1\|x_n - T^n y_n^2\| + c_n\|w_n - x_n\| \\
&+ c_n^1\|x_n - w_n^1\| \\
&\le 2Mb_n + 2Mc_n + 2M(b_n^1 + c_n^1) \to 0 \text{ as } n \to \infty \\
&= 2M((b_n + c_n) + (b_n^1 + c_n^1)) \to 0 \text{ as } n \to \infty. \qquad (2.8)
\end{aligned}$$

Since T is uniformly continuous,then (13) and (14) yield
$\|T^n y_n^1 - T^n u_n - T^n x_{n+1} + T^n u_{n+1}\| \le \|T^n y_n^1 - T^n x_{n+1}\| + \|T^n u_{n+1} + T^n u_n\| \to 0$ as $n \to \infty$.
Also in Theorem (1), the following is obtained

$$\begin{aligned}
c_n\|x_n - u_n + v_n - w_n\| &\le c_n M \\
&\le M(b_n + c_n) \to 0 \text{ as } n \to \infty \\
&\text{and so } \rho_n \to 0 \text{ as } n \to \infty. \qquad (2.9)
\end{aligned}$$

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Now, (10) and (11) gives

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\leq \|(1-b_n)(x_n-u_n)\| + b_n(T^n y_n^1 - T^n u_n) \\
&- c_n(x_n - u_n + v_n - w_n)\| \\
&\leq (1-b_n)\|x_n-u_n\| + b_n(T^n y_n^1 - T^n u_n) \\
&- c_n(x_n - u_n + v_n - w_n)\| \\
&\leq (1-b_n)(\|x_n - T^n x_n\| + \|T^n x_n - T^n u_n\| \\
&+ \|T^n u_n - u_n\|) + b_n\|T^n y_n^1 - T^n u_n\| \\
&- c_n\|x_n - u_n + v_n - w_n\| \\
&\leq \|x_n - T^n x_n\| + \|T^n x_n - T^n u_n\| \\
&+ \|T^n u_n - u_n\| + \|T^n y_n^1 - T^n u_n\| \\
&- c_n\|x_n - u_n + v_n - w_n\| \\
&\leq 12M. \qquad (2.10)
\end{aligned}$$

Substituting (16) in (12), gives

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\|^2 \leq (1-b_n)^2\|x_n-u_n\|^2 + 2b_n(1-k)\|x_{n+1}-u_{n+1}\|^2 \\
+ 24b_n\rho_n M.
\end{aligned}$$

Hence,

$$(1-2b_n(1-k))\|x_{n+1}-u_{n+1}\|^2 \leq (1-b_n)^2\|x_n-u_n\|^2 + 24b_n\rho_n M. \qquad (2.11)$$

Since by assumption $b_n \to 0$ as $n \to \infty$, then there is an $N > 0$ such that
$1 - 2b_n(1-k) > 0$ for all $n > N$. Thus from (16), we obtain

$$\|x_{n+1}-u_{n+1}\|^2 \leq \frac{(1-b_n)^2}{1-2b_n(1-k)}\|x_n-u_n\|^2 + \frac{24b_n\rho_n M}{1-2b_n(1-k)}. \qquad (2.12)$$

From the assumption that $b_n < \frac{1-k}{2}$, it is easy to see that

$$\begin{aligned}
(1-2b_n+b_n^2)(1-2b_n(1-k))^{-1} &< (1-2b_n+b_n^2)(1-b_n^2)^{-1} \\
&\leq 1 - 2b_n + 2b_n - 2b_n^3 + b_n^3 \\
&= 1 - b_n^3 \\
&\leq 1 - b_n. \qquad (2.13)
\end{aligned}$$

Thus, in view of (19), inequality (18) becomes

$$\|x_{n+1}-u_{n+1}\|^2 \leq (1-b_n)\|x_n-u_n\|^2 + b_n A_n$$

where $A_n = \dfrac{24\rho_n M}{1-2b_n(1-k)}$.
Setting $\sigma_n = b_n A_n, \lambda_n = b_n$ and $\beta_n = \|x_n - u_n\|^2$, then by Lemma 2, it implies that $\|x_n - u_n\|^2 \to \infty$ and thus $\|x_n - u_n\| \to 0$ as $n \to \infty$.
Since $u_n \to x^*$ as $n \to \infty$ by assumption, then the inequality

$$0 \leq \|x_n - x^*\| \leq \|u_n - x^*\| + \|x_n - u_n\|$$

imply that $x_n \to x^*$ as $n \to \infty$. This ends the proof.

Corollary 1. Let $X, K, T, \{b_n\}, \{b_n^i\}, \{c_n\}, \{c_n^i\}, \{w_n\}$ and $\{w_n^i\}, (n \in \mathbb{N})i = 1, ..., p-1 (p \geq 2)$ be as in Theorem 1, and x^* be the unique fixed point of T, then for any initial points $u_0, x_0 \in K$ the following statements are equivalent:
1. Modified Mann iteration with errors (10) converges strongly to x^*;

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2. Modified Ishikawa iteration with errors (if $p = 2$ in (11)), converges strongly to x^*;
3. Modified Noor iteration with errors (if $p = 3$ in (11)), converges strongly to x^*;
4. Modified multi-step iteration with errors (11), converges strongly to x^*;

Proof: If $p = 2, 3$ in Theorem 2, the result follows.

Corollary 2. Let $X, K, T, \{b_n\}, \{b_n^i\}, \{c_n\}, \{c_n^i\}, \{w_n\}$ and $\{w_n^i\}, (n \in \mathbb{N}) i = 1, ..., p-1 (p \geq 2)$ be as in Theorem 2, and x^* be the unique fixed point of T, then for any initial points $u_0, x_0 \in K$ the following statements are equivalent:
1, Modified Mann iteration (see [3]) converges strongly to x^*;
2. Modified Ishikawa iteration (if $p = 2$ in (3)), converges strongly to x^*;
3. Modified Noor iteration (if $p = 3$ in (3)), converges strongly to x^*.
4. Modified multi-step iteration (3), converges strongly to x^*;

Proof: If $s_n = w_n = w_n^i = 0$ for each $i = 1, ..., p-1$ in Theorem 2, the result follows.
In view of Corollary 1 and Corollary 2, we have the following theorem.

Theorem 3. Let X be a real Banach space with dual convex, K a non empty closed and convex subset of X and $T : K \to K$ a strongly successively pseudo-contractive operator. Suppose x^* is a fixed point of T. Suppose in addition $\{x_n\}$ is bounded, for any $u_0, x_0 \in K$ satisfying conditions of Theorem 2, the following statements are equivalent:
1. The modified Mann iteration converges strongly to x^*;
2. The modified Mann iteration with errors converges strongly to x^*;
3. The modified Ishikawa iteration converges strongly to x^*;
4. The modified Ishikawa iteration with errors converges strongly to x^*;
5. The modified Noor iteration converges strongly to x^*;
6. The modified Noor iteration with errors converges strongly to x^*;
7. The modified multi-step iteration converges strongly to x^*;
8. The modified multi-step iteration with errors converges strongly to x^*

Remark 4

(i) Assuming that the range of $T, R(T)$ is bounded, using the iteration methods (10) and (11), it then follows that the sequences $\{x_n\}, \{u_n\}, \{y_n^i\}, i \in \mathbb{N}$ are bounded, hence it is no more needful that $\{x_n\}$ is bounded. Thus the condition that $\{x_n\}$ is bounded is better than the boundedness of the range of T used by Rhoades and Soltuz [29], Rhoades and Soltuz [4], Huang et al. [5] and Hung and Bu [1].
(ii) In Huang et al. [5], it is assumed that X is a uniformly smooth Banach space while Theorem 1 is true for the general Banach spaces.

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