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Equivalence of the Convergences of some Modified Iterations with Errors for Uniformly Lipschitzian Asymtotically Pseudo-Contractive Maps

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Abstract

Certain iterative schemes demonstrate a faster convergence to a fixed point compared to others when used to solve various nonlinear differential equations. We show that the convergence of various iterative schemes, including the modified Mann iteration, modified Mann iteration with errors, modified Ishikawa iteration, modified Ishikawa iteration with errors, modified Noor iteration, modified Noor iteration with errors, modified multistep iteration and modified multi-step iteration with errors are all equivalent when applied to uniformly Lipschitzian asymptotically pseudo-contractive maps in an arbitrary real Banach space. Our results expand and generalize the earlier works of Rhoades and Soltuz [1], Olaleru and Odumosu [2] and Odumosu, Olaleru and Ayodele [3].

Keywords: Modified Iterations (with errors), Uniformly Lipschitzian Maps, Asymptotically Pseudocontractive Maps.
MSC2010: 47H10, 54H25.

1 Introduction and Preliminaries

Many researchers have explored whether the convergence of one iterative scheme to the fixed point of an operator is equivalent to the convergence of another iterative scheme to that same fixed point. The study of pseudocontractive maps and their approximation methods for finding fixed points remains an active area of research till today. For example, see [4–6]. The convergence of these iteration schemes in an arbitrary real Banach space has been considered by several authors, see [7–10]. Rhoades and Soltuz [11] proved the equivalence between the convergence of Ishikawa and Mann iterations for an assymtotically non expansive in the intermediate sense and strongly successively pseudo-contractive maps. In 2009, Olaleru and Odumosu [2] established the equivalence of the convergences of iterative procedures with errors for uniformly Lipschitzian strongly

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successively pseudo-contractive operators. Recently, Odumosu, Olaleru and Ayodele [3] established the convergences of modified iterative procedures with errors for uniformly continuous strongly successively pseudo-contractive operators. This research aims to prove the equivalence of convergence of iterative procedures with errors for uniformly Lipschitzian asymtotically pseudo-contractive operators. Our main results generalize and extend the results of several authors, including Rhoades and Soltuz [1], Olaleru and Odumosu [2] and Odumosu, Olaleru and Ayodele [3].
Let X be a real Banach space, and K a non-empty subset of X, T a self mapping of K and $F(D), D(T)$ and I are the set of fixed points, domain of T and identity operator respectively. Let J denote the normalized duality mapping from X to 2^{X^*} defined by;

$$J(X) = \{f \in X^* : \langle x, f\rangle = \|x\|^2, \|f\| = \|x\|\} \text{ for all } x \in X$$

where X^* denotes the dual space of X and $\langle ., .\rangle$ denotes the generalized duality pairing.
Let X be a real Banach space and $D(T), R(T)$ represent the domain and range of T respectively, and K, a closed convex subset of X then, we have the following definitions

Definition 1 [12, 13]: A mapping $T : X \to X$ is said to be non-expansive on X if,

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in D(T).$$

Definition 2 [14]: A map $T : K \to K$ is said to be Lipschitzian if there exists $L \geq 1$, such that,

$$\|Tx - Ty\| \leq L\|x - y\|, \text{ for all } x, y \in D(T).$$

If $L = 1$ in Definition 2, then T is said to be non-expansive.

Definition 3 [14]: A map $T : K \to K$ is said to be uniformly Lipschitzian if there exists $L \geq 1$, such that,

$$\|T^n x - T^n y\| \leq L\|x - y\|, \text{ for all } x, y \in D(T).$$

If $T^n = T$ in Definition 3, then T is said to be Lipschitzian.

Definition 4 [15]: A map $T : K \to K$ is said to be pseudo-contractive if for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$, such that,

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 \text{ for all } x, y \in D(T).$$

Definition 5 [1]: A mapping $T : K \to K$ is said to be asymtotically pseudo-contractive, if for each $x, y \in X$, there exist a sequence $k_n, k_n \in [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ and $j(x - y) \in J(x - y)$ such that,

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n\|x - y\|^2.$$

The Mann iteration scheme was introduced in 1953 [16] to find fixed points for various functions where the Banach contraction principle fails. In 1974, Ishikawa [17] developed another iterative approach known as the two-step iteration scheme. Following this, Noor [18] introduced a three-step iterative scheme and applied it to approximate solutions for variational problems in Hilbert spaces. The modified Mann iteration with errors is defined as; (see [11])
$x_1 \in K$,

$$x_{n+1} = (1 - b_n)x_n + b_n T^n x_n + c_n(s_n - x_n), n \geq 1 \tag{1.1}$$

where $\{s_n\}$ is a bounded sequence in K and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0, 1)$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$.
Observe that (1) is equivalent to

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n s_n, n \geq 1$$

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Remark 1.
1. If $c_n = 0$ for each n, then we have the modified Mann iterative scheme.
2. If T^n is replaced by T in (1), we obtain the modified Mann iterations with errors in the sense of Xu [19]. If in addition, $c_n = 1$, then (1) is called the Mann iteration with errors in the sense of Liu [20].
3. If T^n is replaced by T in (1), and $c_n = 0$, then (1) is called the Mann iteration.
In 2004, Rhoades and Soltuz [11] introduced the modified multistep iteration as follows,

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1}, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n, p \geq 2, \end{aligned} \tag{1.2}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ in $(0, 1)$ satisfy certain conditions.
Remark 2
1. If T^n is replaced by T, the modified multistep iteration (2) is referred to as multistep iteration.
2. If $p = 3$, (2) becomes the modified Noor or three-step iteration procedure, and if in addition, T^n is replaced by T, it is called Noor or three step iteration.
3. If $p = 2$, (2) becomes the modified Ishikawa iteration procedure and if in addition, T^n is replaced by T, it is called Ishikawa iteration.
The modified multistep iteration with errors introduced by Liu and Kang [21] is defined by,

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + w_n \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + w_n^1, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + w_n^{p-1}, p \geq 2 \end{aligned} \tag{1.3}$$

where the sequences $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ are in $[0, 1)$ and the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ...p-1)$ are convergent sequences in K, all satisfying certain conditions.
Remark 3.
1. If T^n is replaced by T, the modified multistep iteration with errors (3) reduces to the Noor and Ishikawa iteration with errors respectively when $p = 3$ and;
2. If in addition, $w_n = w_n^i = 0, (i = 1, 2, ...)$ for all $n \in \mathbb{N}$, then (3) reduces to Noor and Ishikawa iterations (without errors) respectively.
The Ishikawa and Mann iteration with errors of (3) was introduced by Liu [21]. Numerous papers have been published that utilize this iteration procedure with error terms.. For example, see [15, 20, 22, 23].
However, it should be noted that the iteration process with errors in (3) is not satisfactory. The errors can occur in a random way. The condition then imposed on the error terms which say that they tend to zero as n tends to infinity are therefore unreasonable (see [24]). This informed the introduction of a better modified iterative processes with errors by Xu [19].
The Xu's modified multistep with errors is defined as follows:

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + c_n(w_n - u_n) \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + c_n^i(w_n^i - u_n), i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + c_n^{p-1}(w_n^{p-1} - u_n), p \geq 2 \end{aligned} \tag{1.4}$$

where the sequences $\{w_n\}$, $\{w_n^i\}, (i = 1, ...p-1)$ are bounded sequences and $\{b_n\}$, $\{b_n^i\}, (i = 1, ...p-1)$ in $[0, 1)$ satisfy certain conditions $n \in \mathbb{N}$.

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Observe that the modified multistep iteration with errors (5) is equivalent to

$$\begin{aligned} u_1 &\in K \\ u_{n+1} &= (1-b_n)u_n + b_n T^n v_n^1 + c_n w_n \\ v_n^i &= (1-b_n^i)u_n + b_n^i T^n v_n^{i+1} + c_n^i w_n^i, i = 1, ..., p-2 \\ v_n^{p-1} &= (1-b_n^{p-1})u_n + b_n^{p-1} T^n u_n + c_n^{p-1} w_n^{p-1}, \geq 2 \end{aligned} \tag{1.5}$$

where the sequences $\{w_n\}$, $\{w_n^i\}$, $(i = 1, ...p-1)$ are bounded sequences in K, and $\{a_n\}$, $\{a_n^i\}$, $\{b_n\}$, $\{b_n^i\}(i = 1, ...p-1)$ in $[0, 1)$ satisfying
$a_n + b_n + c_n = a_n^i + b_n^i + c_n^i = 1, i = 1, 2, ..., p-1.$

In this paper, we show that the modified Mann, Ishikawa, Noor and multistep iteration with errors (2) (using the more satisfactory definition Xu [19]) and that these iterations without errors are all equivalent for asymptoticaly pseudo-contractive mapping with Lipschitzian assumption in an arbitrary real Banach space.
The result generalize and extend the results of several authors, including Rhoades and Soltuz [1], Olaleru and Odumosu [2] and Odumosu, Olaleru and Ayodele [3].
In the proof of our results, Lemma 1 [25] below is needed.

Lemma 1 [25]: Let $(\alpha_n)_n$ be a non-negative sequence which satisfies the following inequality

$$\alpha_{n+1} \leq (1-\lambda_n)\alpha_n + \delta_n,$$

where $\lambda_n \in (0, 1), \forall n \in \mathbb{N}, \sum_{n\to 1}^{\infty} \lambda_n = \infty$ and $\delta_n = o(\lambda_n)$. Then $\lim_{n\to\infty} \alpha_n = 0$.

2 Main results

Theorem 1: Let K be a closed convex subset of an arbitrary Banach space X. Let T be an asymptotically pseudo-contractive such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x-y\|^2, \tag{2.1}$$

and uniformly Lipschitzian with $L \geq 1$ self map of K. Suppose the modified Mann iteration with errors (1) iteratively defined by sequence $\{u_n\}$ and the modified multi-step iteration with errors (4) iteratively defined by sequence $\{x_n\}$ respectively given by

$$u_{n+1} = (1-b_n)u_n + b_n T^n u_n + c_n(v_n - u_n), n \geq 1$$

and

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n T^n y_n^1 + c_n(w_n - x_n) n \leq 1 \\ y_n^i &= (1-b_n^i)x_n + b_n^i T^n y_n^{i+1} + c_n^i(w_n^i - x_n), i = 1, ..., p-2 \\ y_n^{p-1} &= (1-b_n^{p-1})x_n + b_n^{p-1} T^n x_n + c_n^{p-1}(w_n^{p-1} - x_n), p \geq 2 \end{aligned} \tag{2.2}$$

satisfying the conditions:

$$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} b_n^1 \text{ and } \sum_{n=1}^{\infty} b_n = \infty.$$

Let x^* be a fixed point of T. If $u_1, x_1 \in K$, then the following statements are equivalent:

(i) modified Mann iteration with errors (1) converges to $x^* \in F(T)$

(ii) modified multi-step iteration (4) converges to $x^* \in F(T)$.

Proof: $(ii) \implies (i)$. If modified multistep iteration with errors converges to $x^* \in F(T)$, then setting $b_n^i = c_n^i = 0, \forall n \in \mathbb{N}$ in (4), we obtain the convergence of modified Mann iteration with errors (1).
Conversely, we shall prove $(i) \implies (ii)$

$$\begin{aligned}
x_n &= x_{n+1} + b_n x_n - b_n T^n y_n^1 - c_n(w_n - x_n)\\
&= (1 + (b_n^2))x_{n+1} + b_n(b_n k_n I - T^n)x_{n+1} - (1 + k_n)(b_n)^2 x_{n+1}\\
&+ b_n x_n + b_n(T^n x_{n+1} - T^n y_n^1) - c_n(w_n - x_n)\\
&= (1 + (b_n^2))x_{n+1} + b_n(b_n k_n I - T^n)x_{n+1}\\
&- (1 + k_n)(b_n)^2[x_n + b_n(T^n y_n^1 - x_n) + c_n(w_n - x_n)]\\
&+ b_n x_n + b_n(T^n x_{n+1} - T^n y_n^1) - c_n(w_n - x_n)\\
&= (1 + (b_n^2))x_{n+1} + b_n(b_n k_n I - T^n)x_{n+1} - (1 + k_n)(b_n)^2 x_n\\
&+ (1 + k_n)(b_n)^3(x_n - T^n y_n^1) + b_n x_n + b_n(T^n y_n^1) + b_n x_n + b_n(T^n x_{n+1} - T_n^1)\\
&+ [1 + (1 + k_n)(b_n)^2]c_n(x_n - w_n)\\
&= (1 + (b_n)^2)x_{n+1} + b_n(b_n k_n I - T^n)x_{n+1} + (1 - (1 + k_n)b_n)b_n x_n\\
&+ (1 + k_n)(b_n)^3(x_n - T^n y_n^1) + b_n(T^n x_{n+1} - T^n y_n^1)\\
&+ [1 + (1 + k_n)(b_n)^2]c_n(x_n - w_n).
\end{aligned} \tag{2.3}$$

Similarly,

$$\begin{aligned}
u_n &= (1 + (b_n)^2)u_{n+1} + b_n(b_n k_n I - T^n)u_{n+1} + [1 - (1 + k_n)b_n]b_n u_n\\
&+ (1 + k_n)(b_n)^3(u_n - T^n u_n) + b_n(T^n u_{n+1} - T^n u_n)\\
&+ [1 + (1 + k_n)(b_n)^2]c_n(u_n - v_n)
\end{aligned} \tag{2.4}$$

Subtract (9) from (8) gives

$$\begin{aligned}
x_n - u_n &= (1 + (b_n)^2)(x_{n+1} - u_{n+1}) + b_n[(b_n k_n I - T^n)x_{n+1}\\
&- (b_n k_n I - T^n)u_{n+1}] + [1 - (1 + k_n)b_n]b_n(x_n - u_n)\\
&+ (1 + k_n)(b_n)^3(x_n - u_n - T^n y_n^1 + T^n u_n)\\
&+ b_n(T^n x_{n+1} - T^n u_{n+1} - T^n y_n^1 + T^n u_n)\\
&+ [1 + (1 + k_n)(b_n)^2]c_n(x_n - w_n - u_n + v_n).
\end{aligned} \tag{2.5}$$

Note that

$$\begin{aligned}
(1 + (b_n)^2)(x_{n+1} - u_{n+1}) &+ b_n[(b_n k_n I - T^n)x_{n+1} - (b_n k_n I - T^n)u_{n+1}]\\
&= (1 + (b_n)^2)(x_{n+1} - u_{n+1})\\
&+ \frac{b_n}{1 + (b_n)^2}[b_n k_n I - T^n x_{n+1} - (b_n k_n I - T^n)u_{n+1}].
\end{aligned} \tag{2.6}$$

Taking the norm of the R.H.S of (11), and using the fact that T is asymtotically pseudo-contractive with $x = x_{n+1}, y = y_{n+1}$ yields

$$\begin{aligned}
(1 + (b_n)^2)\|x_{n+1} - u_{n+1}\| \leq (1 + (b_n)^2)\|x_{n+1} - u_{n+1} + \frac{b_n}{1 + (b_n)^2}\\
[(b_n k_n I - T^n)x_{n+1} - (b_n k_n I - T^n)u_{n+1}\|]
\end{aligned} \tag{2.7}$$

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Rewriting (10) gives

$$\begin{aligned}(1+(b_n)^2)(x_{n+1}-u_{n+1}) &+ \frac{b_n}{1+(b_n)^2}[b_n k_n I - T^n x_{n+1} - (b_n k_n I - T^n)u_{n+1}] \\ &= (x_n-u_n) - (1-(1+k_n)b_n)b_n(x_n-u_n) - (1+k_n)(b_n)^3(x_n-u_n \\ &- T^n y_n^1 + T^n u_n) - b_n(T^n x_{n+1} - T^n u_{n+1} - T^n y_n^1 - T^n u_n) \\ &- [1+(1+k_n)(b_n)^2]c_n(x_n - w_n - u_n + v_n). \end{aligned} \tag{2.8}$$

Taking the norm of both sides of (13) and using (12), we obtain

$$\begin{aligned}&(1+(b_n)^2)\|x_{n+1}-u_{n+1}\| \\ &\leq [1-(1-(1+k_n))b_n]\|x_n-u_n\| + (1+k_n)(b_n)^3\|x_n-u_n \\ &- T^n y_n^1 - T^n u_n\|\| + b_n\|T^n x_{n+1} - T^n u_{n+1} - T^n y_n^1 - T^n u_n\| \\ &+ c_n[1+(1+k_n)(b_n)^2]\|x_n - w_n - u_n + v_n\| \end{aligned}$$

Therefore

$$\begin{aligned}(1+(b_n)^2)\|x_{n+1}-u_{n+1}\| \leq\ & [1-(1-(1+k_n)b_n)b_n]\|x_n-u_n\| \\ &+ (1+k_n)(b_n)^3\|x_n - T^n y_n^1\| \\ &+ (1+k_n)(b_n)^3\|u_n - T^n u_n\| \\ &+ b_n\|T^n u_{n+1} T^n u_n\| + b_n\|T^n x_{n+1} - T^n y_n^1\| \\ &+ c_n[1+(1+k_n)(b_n)^2]\|x_n - w_n - u_n - v_n\|. \end{aligned} \tag{2.9}$$

Now evaluate $\|x_n - T^n y_n^1\|$.

$$\begin{aligned}\|x_n - T^n y_n^1\| &\leq \|x_n-u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n y_n^1\| \\ &\leq \|x_n-u_n\| + \|u_n - T^n u_n\| + L\|u_n - y_n^1\|. \end{aligned} \tag{2.10}$$

But from (1), the following result is obtained

$$\begin{aligned}\|u_n - y_n^1\| &= |u_n - (1-b_n^1)x_n - b_n^1 T^n y_n^2 - c_n^1(w_n^1 - x_n)| \\ &\leq \|u_n - x_n\| + b_n^1\|x_n - T^n y_n^2\| + \|c_n^1(w_n^1 - x_n)\| \\ &\leq \|x_n - u_n\| + b_n^1\|x_n-u_n\| + b_n^1\|u_n - T^n u_n\| \\ &+ b_n^1 L\|u_n - y_n^2\| + c_n^1\|w_n^1 - x_n\| \\ &= (1+b_n^1)\|x_n-u_n\| + b_n^1\|u_n - T^n u_n\| + b_n^1 L\|u_n - y_n^2\| \\ &+ c_n^1\|w_n^1 - x_n\| \\ &= (1+b_n^1)\|x_n-u_n\| + b_n^1\|u_n - T^n u_n\| \\ &+ b_n^1 L\|u_n - (1-(b_n)^2)x_n - (b_n)^2 T^n y_n^3 - c_n^2(w_n^2 - x_n)\| \\ &+ c_n^1\|w_n^1 - x_n\| \\ &\leq (1+b_n^1)\|x_n-u_n\| + b_n^1\|u_n - T^n u_n\| + b_n^1 L\|u_n - x_n\| \\ &+ b_n^1(b_n)^2 L\|x_n - T^n y_n^3\| + b_n^1 c_n^2 L\|w_n^2 - x_n\| + c_n^1\|w_n^1 - x_n\| \\ &\leq (1+b_n^1+b_n^1 L)\|x_n-u_n\| + b_n^1\|u_n - T^n u_n\| \\ &+ b_n^1 L\|x_n - T^n y_n^3\| + c_n^2 L\|w_n^2 - x_n\| + c_n^1\|w_n^1 - x_n\| \end{aligned}$$

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Therefore

$$\begin{aligned}
\|u_n - y_n^1\| &= (1 + 2b_n^1 L + b_n^1 L)\|x_n - u_n\| + (b_n^1 + b_n^1 L)\|u_n - T^n u_n\| \\
&+ b_n^1 L^2 \|u_n - y_n^3\| + c_n^2 L\|w_n^2 - x_n\| + c_n^1 \|w_n^1 - x_n\| \\
&\le (1 + 3b_n^1 L)\|x_n - u_n\| + 2b_n^1 L\|u_n - T^n u_n\| + c_n^2 L\|w_n^2 - x_n\| \\
&+ b_n^1 L^2 \|u_n - (1 - b_n^3)x_n - b_n^3 T^n y_n^4 - c_n^3(w_n^3 - x_n)\| + c_n^1\|w_n^1 - x_n\| \\
&\le (1 + 3b_n^1 L + b_n^1 L^2)\|x_n - u_n\| + 2b_n^1 L\|u_n - T^n u_n\| \\
&+ b_n^1 L^2 \|x_n - u_n + u_n - T^n u_n + T^n u_n - T^n y_n^4\| \\
&+ b_n^1 L^2 c_n^3 \|w_n^3 - x_n\| + c_n^2 L\|w_n^2 - x_n\| + c_n^1\|w_n^1 - x_n\|. \qquad (2.11)
\end{aligned}$$

This implies

$$\begin{aligned}
\|u_n - y_n^1\| &\le (1 + 3b_n^1 L + 2b_n^1 L^2)\|x_n - u_n\| + (2b_n^1 L + b_n^1 L^2)\|u_n - T^n u_n\| \\
&+ b_n^1 L^3 \|u_n - y_n^4\| + L^2 c_n^3 \|w_n^3 - x_n\| + c_n^2 L\|w_n^2 - x_n\| + c_n^1\|w_n^1 - x_n\| \\
&\le (1 + 5b_n^1 L^2)\|x_n - u_n\| + 3b_n^1 L^2 \|u_n - T^n u_n\| \\
&+ b_n^1 L^3 \|u_n - y_n^4\| + \sum_{i=1}^{3} L^{i-1}\|w_n^i - x_n\|.
\end{aligned}$$

Continuing in this way yields

$$\begin{aligned}
\|u_n - y_n^1\| &\le (1 + (2p-5)b_n^1 L^{p-3})\|x_n - u_n| + (p-2)b_n^1 L^{p-3}\|u_n - T^n u_n\| \\
&+ b_n^1 L^{p-2}\|u_n - y_n^{p-1}\| + \sum_{i=1}^{p-2} c^i L^{i-1}\|w_n^i - x_n\|, p \ge 3. \qquad (2.12)
\end{aligned}$$

In view of (1), the following result is obtained;

$$\begin{aligned}
\|u_n - y_n^{p-1}\| &= \|u_n - (1 - b_n^{p-1})x_n - b_n^{p-1} T^n x_n - c_n^{p-1}(w_n^{p-1} - x_n) \\
&\le \|u_n - x_n\| + b_n^{p-1}\|x_n - T^n x_n\| + c_n^{p-1}\|w_n^{p-1} - x_n\| \\
&\le \|u_n - x_n\| + b_n^{p-1}\|u_n - x_n\| + b_n^{p-1}\|u_n - T^n u_n\| \\
&+ b_n^{p-1} L\|u_n - x_n\| + c_n^{p-1}\|w_n^{p-1} - x_n\|
\end{aligned}$$

$$\begin{aligned}
&= (1 + b_n^{p-1} + b_n^{p-1} L)\|u_n - x_n\| + b_n^{p-1}\|u_n - T^n u_n\| \\
&+ c_n^{p-1}\|w_n^{p-1} - x_n\|. \qquad (2.13)
\end{aligned}$$

Inserting (18) into (17) gives

$$\begin{aligned}
\|u_n - y_n^1\| &\leq (1 + (2p-5)b_n^1 L^{p-3})\|x_n - u_n| + (p-2)b_n^1 L^{p-3}\|u_n - T^n u_n\| \\
&+ b_n^1 L^{p-2}\|[(1 + b_n^{p-1} + b_n^{p-1}L)\|x_n - u_n\| + b_n^{p-1}\|u_n - T^n u_n\|] \\
&+ b_n^1 L_n^{p-2} c_n^{p-1}\|w_n^{p-1} - x_n\| + \sum_{i=1}^{p-2} c_n^i L^{i-1}\|w_n^i - x_n\| \\
&\leq [(1 + (2p-5)b_n^1 L^{p-3}) + b_n^1 L^{p-2} + b_n^1 L^{p-1}]\|x_n - u_n\| \\
&+ (p-1)b_n^1 L^{p-2}\|u_n - T^n u_n\| + c_n^1 L^{p-2}\|w_n^i - x_n\| \\
&+ \sum_{i=1}^{p-2} c_n^i \|w_n^i - x_n\| \\
&\leq (1 + (p+1)2)b_n^1 L^{p-2}\|x_n - u_n\| \\
&+ (p-1)b_n^1 L^{p-2}\|u_n - T^n u_n\| + \sum_{i=1}^{p-1} c_n^i \|w_n^i - x_n\|. \qquad (2.14)
\end{aligned}$$

Putting (19) in (15) yields

$$\begin{aligned}
\|x_n - T^n y_n^1\| &\leq [1 + L(1 + (p+1)2b_n^1 L^{p-1})]\|x_n - u_n\| \\
&+ (1 + L^{p-1}(p-1)b_n^1)\|u_n - T^n u_n\| \\
&+ \sum_{i=1}^{p-1} c_n^i L^i \|w_n^i - x_n\|. \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}
\|x_n - T^n x_n\| &= \|x_n - u_n + u_n - T^n u_n + T^n u_n - T^n x_n\| \\
&\leq \|x_n - u_n\| + \|u_n - T^n u_n\| + \|T^n u_n - T^n x_n\| \\
&= (1 + L)\|x_n - u_n\| + \|u_n - T^n u_n\|. \qquad (2.16)
\end{aligned}$$

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Now it results from (4) that

$$\begin{aligned}
\|x_n - y_n^1\| &= \|x_n - (1-b_n^1)x_n - b_n^1 T^n y_n^2 - c_n^1\|(w_n^1 - x_n)\| \\
&\leq b_n^1\|x_n - T^n y_n^2\| + c_n^1\|(w_n^1 - x_n)\| \\
&\leq b_n^1\|x_n - T^n x_n\| + b_n^1 L\|x_n - y_n^2\| + c_n^1\|(w_n^1 - x_n)\| \\
&= b_n^1\|x_n - T^n x_n^2\| + b_n^1 L\|x_n - (1-(b_n)^2)x_n - (b_n)^2 T^n y_n^3 \\
&+ c_n^2(w_n^2 - x_n)\| + c_n^1\|(w_n^1 - x_n)\| \\
&\leq b_n^1\|x_n - T^n x_n\| + b_n^1\|x_n - T^n y_n^3\| + c_n^2(w_n^2 - x_n)\| \\
&+ c_n^1\|(w_n^1 - x_n)\| \\
&\leq b_n^1\|x_n - T^n x_n\| + b_n^1\|x_n - T^n x_n\| + b_n^1\|x_n - y_n^3\| \\
&+ c_n^1\|(w_n^1 - x_n)\| + c_n^2(w_n^2 - x_n)\| \\
&= (b_n^1 + b_n^1 L)\|x_n - T^n x_n\| + b_n^1 L^2\|x_n - (1-b_n^3)x_n \\
&- b_n^3 T^n y_n^4 - c_n^3(w_n^3 - x_n)\| - c_n^1\|(w_n^1 - x_n)\| + c_n^2(w_n^2 - x_n)\| \\
&\leq (b_n^1 + b_n^1 L)\|x_n - T^n x_n\| + b_n^1 L^2\|x_n - T^n y_n^4\| \\
&+ c_n^3 L^2\|(w_n^3 - x_n)\| c_n^2 L\|(w_n^2 - x_n)\| + c_n^1\|(w_n^1 - x_n)\| \\
&\leq 2b_n^1 L\|x_n - T^n x_n\| + b_n^1 L^2\|x_n - T^n x_n\| + b_n^1 L^3\|x_n - y_n^4\| \\
&+ c_n^3 L^2\|(w_n^3 - x_n)\| c_n^2 L\|(w_n^2 - x_n)\| + c_n^1\|(w_n^1 - x_n)\| \\
&\leq 3b_n^1 L^2\|x_n - T^n x_n\| + b_n^1 L^3\|x_n - y_n^4\| \\
&+ c_n^3 L^2\|(w_n^3 - x_n)\| c_n^2 L\|(w_n^2 - x_n)\| + c_n^1\|(w_n^1 - x_n)\| \\
&\leq (p-2)b_n^1 L^{p-3}\|x_n - T^n x_n\| + b_n^1 L^{p-2}\|x_n - y_n^{p-1}\| \\
&+ \sum_{i=1}^{p-2} c_n^i L^{i-1}\|w_n^i - x_n\| \\
&\leq (p-2)b_n^1 L^{p-3}\|x_n - T^n x_n\| + \sum_{i=1}^{p-2} c_n^i L^{i-1}\|w_n^i - x_n\| \\
&+ b_n^1 L^{p-2}\|x_n - (1-b_n^{p-1})x_n - b_n^{p-1} T^n x_n - c_n^{p-1}(w_n^{p-1} - x_n)\| \\
&\leq (p-1)b_n^1 L^{p-2}\|x_n - T^n x_n\| + \sum_{i=1}^{p-1} c_n^i L^{i-1}\|w_n^i - x_n\|. \qquad (2.17)
\end{aligned}$$

Inserting (21) in (22) yields

$$\|x_n - y_n^1\| \leq (p-1)b_n^1 L^{p-2}[(1+l)\|x_n - u_n\| + \|u_n - T^n u_n] + \sum_{i=1}^{p-1} c_n^i L^{i-1}\|w_n^i - x_n\| \qquad (2.18)$$

Now evaluate $\|T^n x_{n+1} - T^n y_n^1\|$.

$$\begin{aligned}
\|T^n x_{n+1} - T^n y_n^1\| &\leq L\|x_{n+1} - y_n^1\| \\
&= L\|(1-b_n)x_n + b_n T^n y_n^1 + c_n(w_n - x_n) - y_n^1\| \\
&\leq L\|x_n - y_n^1\| + Lb_n\|T^n y_n^1 - x_n\| + c_n L\|w_n - x_n\| \qquad (2.19)
\end{aligned}$$

Substituting (20) and (23) into (24) leads to

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n^1\| &\leq L(p-1)b_n^1 L^{p-2}[(1+L)\|x_n - u_n\| + \|u_n - T^n u_n\|] \\ &+ \sum_{i=1}^{p-1} c_n^i L^{i-1}\|w_n^i - x_n\| + Lb_n[1 + (L + (p+1))2b_n^1 L^p] \\ &\|x_n - u_n\| + (p-1)b_n^1 L^p\|u_n - T^n u_n\| \\ &+ Lb_n \sum_{i=1}^{p-1} c_n^i L^{i-1}\|w_n^i - x_n\| + c_n L\|w_n - x_n\| \\ &\leq [Lb_n(1 + L + 2(p+1)b_n^1 L^{p+1} + (p-1)b_n^1 L^{p-1}(1+L))] \\ &\|x_n - u_n\| + (p-1)b_n^1 L^{p-1}(1+L)\|u_n - T^n u_n\| \\ &+ 2\sum_{i=1}^{p-1} c_n^i L^{i-1}\|w_n^i - x_n\| + c_n L\|w_n - x_n\|. \end{aligned} \tag{2.20}$$

Putting (20) and (25) in (14) gives

$$\begin{aligned}(1 + (b_n)^2)\|x_{n+1} - u_{n+1}\| &\leq [1 - (1 - (1 + k - n)b_n)b_n]\|x_n - u_n\| \\ &+ (1+k_n)(b_n)^3[1 + L(1 + (p+1)2b_n^1)L^{p-1}]\|x_n - u_n\| \\ &+ (1+k_n)(b_n)^3[(1 + (p-1)b_n^1)L^{p-1}]\|u_n - T^n u_n\|\end{aligned}$$

$$\begin{aligned}&+ (1+k_n)(b_n)^3 \sum_{i=1}^{p-1} c_n^i L^i\|w_n^i - x_n\| \\ &+ (1+k_n)(b_n)^3\|u_n - T^n u_n\| + b_n\|T^n u_{n+1} - T^n u_n\| \\ &+ b_n[Lb_n(1 + L + 2(p+1)b_n^1)L^{p+1} \\ &+ (p-1)b_n^1 L^{p-1}(1+L)]\|u_n - u_n\| \\ &+ (p-1)b_n^1 L^{p-1}(1+L)]\|u_n - T^n u_n\| \\ &+ 2\sum_{i=1}^{p-1} c_n^i L^i\|w_n^i - x_n\| + c_n L\|w_n - x_n\| \\ &+ c_n[1 + (1+k_n)b_n^2]\|x_n - w_n - u_n - v_n\| \\ &\leq [1 - (1 - (1+k_n)b_n)b_n] \\ &+ (1+k_n)(b_n)^3[1 + L(1 + (p+1)2b_n^1)L^{p-1}] \\ &+ b_n[Lb_n(1 + L + 2(p+1)b_n^1)L^{p+1} \\ &+ (p-1)b_n^1 L^{p-1}(1+L)]\|x_n - u_n\| \\ &+ (1+k_n)(b_n)^3[1 + (p-1)b_n^1 L^{p-1} \\ &+ (p-1)b_n^1 L^{p-1}(1+L)]\|u_n - T^n u_n\| \\ &+ b_n\|T^n u_{n+1} - T^n u_n\| + b_n(1+k_n)(b_n)^3 \sum_{i=1}^{p-1} c_n^i L^i\|w_n^i - x_n\| \\ &+ c_n[1 + (1+k_n)b_n^2]\|x_n - w_n - u_n - v_n\|. \end{aligned} \tag{2.21}$$

Hence, (26) can be written as

$$\alpha_{n+1} \leq \gamma_n \alpha_n + \delta_n$$

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where

$$\alpha_n = \|x_n - u_n\|,$$
$$\gamma_n = [1-(1-(1+k_n)b_n)b_n] + (1+k_n)(b_n)^3[1+L(1+(p+1)2b_n^1)L^{p-1}] + b_n[Lb_n(1+L+2(p+1)b_n^1)L^{p+1} + (p-1)b_n^1 L^{p-1}(1+L)]\|x_n - u_n\|$$

$$\delta_n = (1+k_n)(b_n)^3[1+(p-1)b_n^1L^{p-1} + (p-1)b_n^1L^{p-1}(1+L)]\|u_n - T^nu_n\| + b_n\|T^nu_{n+1} - T^nu_n\| + b_n(1+k_n)(b_n)^3\sum_{i=1}^{p-1} c_n^i L^i\|w_n^i - x_n\| + c_n[1+(1+k_n)b_n^2]\|x_n - w_n - u_n - v_n\|. \tag{2.22}$$

Note that if the Mann iteration (1) converges by assumption, then

$$0 \leq \|u_{n+1} - u_n\| \leq \|u_{n+1} - x^*\| \to 0 \text{ as } n \to 0.$$

Which implies

$$\begin{aligned} 0 \leq \|u_{n+1} - T^nu_n\| &\leq \|T^nx^* - T^nu_n\| + \|u_{n+1} - x^*\| \\ &\leq L\|u_n - x^*\| + \|u_{n+1} - x^*\| \to 0 \text{ as } n \to 0. \end{aligned}$$

For the fact that $\lim_{n\to\infty} b_n = 0$, it follows that all n sufficiently large, we have

$$b_n \leq \frac{1}{5}\sup[\frac{1}{1+k_n}, \frac{1}{(1+k_n)^{\frac{1}{2}}(1+L+(p+1)2L^{p-1})}, \frac{1}{L+L^2+2(p+1)L^{p+2}}],$$
$$b_n^1 \leq \frac{1}{5}(\frac{1}{(p-1)(1+L)L^{p-1}}).$$

Note that $\frac{1}{2(p-1)L^p} \leq \frac{1}{1+L}$ since $L \geq 1$ thus,

$$\begin{aligned} \gamma_n &\leq [1-(1-(1+k_n)b_n)b_n] + (1+k_n)(b_n)^3[1+L+(p+1)2L^{p-1}] \\ &\quad + b_n^2[L+L^2+2(p+1)L^{p+2}] + b_nb_n^1(p-1)(1+L)L^{p-1} \\ &\quad [1-(1-(1+k_n)b_n)b_n] + (1+k_n)(b_n)^2[1+L(1+(p+1)2b_n^1)L^{p-1}] \\ &\quad + b_n^2[L+L^2+2(p+1)L^{p+2}] + b_nb_n^1(p-1)(1+L)L^{p-1} \\ &\leq 1 - \frac{4}{5}b_n + \frac{b_n}{25} + \frac{b_n}{5} + \frac{b_n}{5} \\ &= 1 - \frac{9}{25}b_n. \end{aligned}$$

Thus $\gamma_n \leq 1 - \frac{9}{25}b_n$ for all n sufficiently large, from which the relation the following relation holds

$$\alpha_{n+1} \leq (1-\lambda_n)\alpha_n + \delta_n$$

where $\lambda_n = \frac{9}{25}b_n < 1, \forall n$. Since Mann iteration converges by assumption, then $\lim_{n\to\infty} u_n = x^*$ or more precisely $\lim_{n\to\infty}\|u_n - x^*\| = 0$. It is easy to see that $\delta_n = o(\lambda_n)$. All the assumptions from Lemma 1 are now satisfied, so $\lim_{n\to\infty} \alpha_n = 0$. Hence the

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{2.23}$$

Since $\lim_{n\to\infty} u_n = x^*$ and (27) hold then,

$$\|x_n - x^*\| \leq \|x_n - u_n\| + \|u_n - x^*\| \to 0 \text{ as } n \to \infty$$

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which leads to

$$\lim_{n\to\infty} x_n = x^*.$$

This ends the proof.

From Theorem 1, the following results are obtained:

Corollary 1. Let $X, K, L, T, \{a_n\}\{b_n\}, \{c_n\}, \{a_n^i\}, \{b_n^i\}, \{c_n^i\}, \{w_n\}$ and $\{w_n^i\}, (n \in \mathbb{N}) i = 1, ..., p-1 (p \geq 2)$ be as in Theorem 1, and x^* be the unique fixed point of T, then for any initial points $u_1, x_1 \in K$ the following statements are equivalent:
1. Modified Mann iteration with errors (1) converges strongly to x^*;
2. Modified Ishikawa iteration with errors (if $p = 2$ in (4)), converges strongly to x^*;
3. Modified Noor iteration with errors (if $p = 3$ in (4)), converges strongly to x^*;
4. Modified multi-step iteration with errors (4), converges strongly to x^*;

Proof: If $p = 2, 3$ in Theorem (1), the result follows.

Corollary 2. Let $X, K, L, T, \{a_n\}\{b_n\}, \{c_n\}, \{a_n^i\}, \{b_n^i\}, \{c_n^i\}, \{w_n\}$ and $\{w_n^i\}, (n \in \mathbb{N}) i = 1, ..., p-1 (p \geq 2)$ be as in Theorem 1, and x^* be the unique fixed point of T, then for any initial points $u_1, x_1 \in K$ the following statements are equivalent:
2. Modified Ishikawa iteration (if $p = 2$ in (2)), converges strongly to x^*;
3. Modified Noor iteration (if $p = 3$ in (2)), converges strongly to x^*;
4. Modified multi-step iteration (3), converges strongly to x^*;

Proof: If $c_n = w_n = 0$ for each $i = 1, ..., p-1$ in Theorem 1, the result follows.
In view of Corollary 1 and Corollary 2, we have the following theorem.

Theorem 2. Let X be a real Banach space, K a non empty closed and convex subset of X and $T : K \to K$ an asymptotically pseudo-contractive and Lipschitzian with $L \geq 1$ self map of k.If $u_1, x_1 \in K$ and define $\{x_n\}$ and $\{u_n\}$ by (4) and (1) respectively with $\{v_n\}, \{w_n\}, \{w_n^i\}, i = 1, ..., p-1$ bounded sequences in K and $\{b_n\}, \{b_n^i\}, i = 1, ..., p-1 \forall n \in \mathbb{N}$ as sequence in $[0, 1)$ satisfying

$$\lim_{n\to\infty} b_n = 0 = \lim_{n\to\infty} b^{1_n}, i = 1, ..., p-1, n \geq 1,$$

and

$$\sum_{n\to\infty} b_n = \infty.$$

Then, the following statements are equivalent:
1. The modified Mann iteration converges strongly to x^*;
2. The modified Mann iteration with errors converges strongly to x^*;
3. The modified Ishikawa iteration converges strongly to x^*;
4. The modified Ishikawa iteration with errors converges strongly to x^*;
5. The modified Noor iteration converges strongly to x^*;
6. The modified Noor iteration with errors converges strongly to x^*;
7. The modified multi-step iteration converges strongly to x^*;
8. The modified multi-step iteration with errors converges strongly to x^*

Proof: In view of Theorem 1, Corollary 1 and Corollary 2, the result follows.

Remark 4:
(1.) Theorem 8 of Rhoades and Soltuz [1] is a special case of Theorem 1 in that error terms are not considered in Rhoades and Soltuz [1].
(2.) The same initial values considered in Theorem 8 of Rhoades and Soltuz [1] is dropped for any initial value of x_1 not necessarily equal to u_1.
Consequently, the theorems and corollaries obtained improve and generalize all the results in Rhoades and Soltuz [18], Olaleru and Odumosu [2] and Odumosu, Olaleru and Ayodele [3].

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