International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

Schwartz Space and Radial Distribution on the Euclidean Motion Group

U. E. Edeke [1*], U. N. Bassey [2]

1. Department of Mathematics, University of Calabar, Nigeria.
2. Department of Mathematics, University of Ibadan, Oyo State, Nigeria.
* Corresponding author: ueedeke@gmail.com*, drunbassey@gmail.com

Article Info
Received: 22 November 2024 Revised: 02 March 2025
Accepted: 30 March 2025 Available online: 10 April 2025

Abstract

Let $G = \mathbb{R}^2 \rtimes \mathbb{T}$ be the Euclidean motion group and let $K(\lambda, t) = I_0(\lambda)\delta(t)$ be a distribution on G, where $I_0(\lambda)$ is the Bessel function of order zero and $\delta(t)$ is the Dirac measure on $SO(2) \cong \mathbb{T}$, the circle group. In this work, it is proved, among other things, that the distribution $K(\lambda, t)$ is tempered, positive definite, bounded and radial. Further more, a description of temperature function on G ,realised as the positive definite solution of the Laplace-Beltrami operator on $SE(2)$, is presented.

Keywords: Schwartz space, Radial Distribution, Euclidean Motion Group.
MSC2010: 05B10.

1 Introduction and Preliminaries

Radial distributions on a locally compact group G is a probability distribution that depends on the radial distance of $g \in G$ from the identity e of G. It is widely applied in modeling uncertain motion in robotics and computer vision, estimating motion distributions for visual tracking and also analysing motion related signals.

In this research, a kind of radial distribution on $SE(2)$ obtained as the product of Bessel function of order zero I_0 and the dirac function on $SE(2)$ is studied. It is demonstrated that this distribution is tempered, positive definite and bounded. I_0, used in defining the radial distribution on $SE(2)$, is obtained by solving the Laplace-Beltrami operator $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$ radially using the method of separation of variable (see [1]). I_0 is further used to define a temperature function on $SE(2)$.

Preliminaries concerning the Euclidean motion group, its representation and invariant differential operators are presented in section two. Spaces of distributions on $SE(2)$ are presented in section three. It is also proved in this section that the Schwartz space of $SE(2)$ is a Frechect space and the convolution of functions is continuous in the Schwartz space of $SE(2)$. In section four, a radial distribution on $SE(2)$ is presented and is also shown to be tempered, bounded and positive definite. Lastly, a temperature function on $SE(2)$ is presented in section five.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

1.1 Euclidean Motion Group.

The Euclidean Motion group is a non-compact and non-commutative solvable Lie group realised as a semi direct product of the additive group $\mathbb{R}^n$ with the Orthogonal group $O(n)$. This means that if $S(n)$ denotes the group, then

$$S(n) = \mathbb{R}^n \rtimes O(n).$$

The special Euclidean motion group is the semi direct product of $\mathbb{R}^n$ with the special orthogonal group, $SO(n)$. That is,

$$SE(n) = \mathbb{R}^n \rtimes SO(n),$$

where $SO(n) = SL(n) \cap O(n)$.
SE(n) is also called group of transformation of the Euclidean plane. Henceforth, SE(n) is considered for this research when $n = 2$. Elements of $SE(2)$ are given by $g = (x, \alpha) \in SE(2)$, where $\alpha \in SO(2)$ and $x \in \mathbb{R}^2$. For any $g = (x_1, \alpha_1)$ and $h = (x_2, \alpha_2)$, the group law of $SE(2)$ is given as

$$gh = (x_1, \alpha_1)(x_2, \alpha_2) = (x_1 + \alpha_1 x_2, \alpha_1\alpha_2)$$

and the inverse g^{-1} is given as ([2])

$$g^{-1} = (-\alpha_1^T, x_1\alpha_1^T),\ where\ \alpha^T = \begin{pmatrix} cos\phi & -sin\phi \\ sin\phi & cos\phi \end{pmatrix}^T$$

Elements of $SE(2)$ may be identified as a 3×3 homogeneous transformation matrix of the form

$$H(g) = \begin{pmatrix} \alpha & x \\ 0^T & 1 \end{pmatrix},$$

where $0^T = (0, 0)$. $SE(2) = (\mathbb{R}^2 \rtimes SO(2)) \cong M \subseteq GL(3, \mathbb{R})$, where M is a subgroup of $GL(3, \mathbb{R})$. An element of $SE(2)$ may also be presented in rectangular coordinate as follows.

$$g(x_1, x_2, \phi) = \begin{pmatrix} cos\phi & -sin\phi & x_1 \\ sin\phi & cos\phi & x_2 \\ 0 & 0 & 1 \end{pmatrix},\ \phi \in [0, 2\pi],\ (x_1, x_2) \in \mathbb{R}^2.$$

Solvability of $SE(2)$ implies that there exist a sequence of closed subgroup $G_0 = G, G_1, ..., G_n, G_{n+1} = \{e\}$ such that G_{n+1} is a closed subgroup in G_k and G_k/G_{k+1} is abelian ([3–5]) . Since all abelian and solvable Lie groups are amenable, it means that the Euclidean motion group is also an amenable group. $SE(n)$ is a group of affine maps induced by orthogonal transformation. It is also called a group of rigid motions. The universal covering group of $SE(2)$ is the semi direct product group $\mathbb{R}^2 \rtimes \mathbb{R}$ whose multiplication is defined as

$$(x_1, \alpha_1)(x_2, \alpha_2) = (x_1 + e^{it}x_2, \alpha_1 + \alpha_2)$$

and its covering map is defined as

$$(x, \alpha) \mapsto (x, e^{it}).$$

For $x_1, x_2 \in \mathbb{R}^2$ and $\alpha \in SO(2)$, the invariant measure on $SE(n)$ is obtained as the product of Lebesque measure on $\mathbb{R}^2$ and the Haar measure on $SO(2)$ given as (see [6, 7])

$$d\mu[(x, \alpha)] = dx_1 dx_2 d\alpha.$$

Let $H = L^2(SE(2), \mu)$ be the Hilbert space of square integrable functions on $SE(n)$. For $u \in H$ and $x, x' \in \mathbb{R}^2$, the right and left regular representations T^R and T^L of SE(2) are defined respectively as

$$\begin{aligned}(T^R_{(x_2,\alpha_2)}u)[(x_1, \alpha_2)] &= u[(x_1, \alpha_1)(x_2, \alpha_2)] \\ &= u[(x_1 + x_2\alpha_1, \alpha_1 + \alpha_2)]\end{aligned}$$

and

$$
\begin{aligned}
(T^L_{(x_2,\alpha_2)}u)[(x_1,\alpha_1)] &= u[(x_2,\alpha_1)^{-1}(x_1,\alpha_1)] \\
&= u[(-x_{2-\alpha'}+x_{12\pi-\alpha'}, 2\pi-\alpha_2+\alpha_1)].
\end{aligned}
$$

Let $g(t_i)$ be the one-parameter subgroups of $SE(2)$ generated by X_i, $i=1,2,3..$ Then

$$
X_i u = \lim_{t\to 0}\left(\frac{T^L_{g(t_i)}-I}{t_i}u\right)
$$

where u is an element of the Garding domain [6]. Explicitly,

$$
X_1 = -\frac{\partial}{\partial x_1},
$$

$$
X_2 = -\frac{\partial}{\partial x_2},
$$

$$
X_3 = x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} - \frac{\partial}{\partial \alpha}.
$$

The generators of the left invariant Lie algebra of G are given as

$$
Y_1 = cos\alpha\frac{\partial}{\partial x_1} + sin\alpha\frac{\partial}{\partial x_2},
$$

$$
Y_2 = -sin\alpha\frac{\partial}{\partial x_1} + cos\alpha\frac{\partial}{\partial x_2},
$$

$$
Y_3 = \frac{\partial}{\partial \alpha},
$$

and they obey the following commutation relations $[Y_1,Y_2]=0$, $[Y_2,Y_3]=Y_1$ and $[Y_3,Y_1]=Y_2$, where $[A,B]$ is the standard Lie bracket defined as $[A,B]=AB-BA$.

The Laplace Beltrami operator defined on $SE(2)$ is given as [1]

$$
\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{1.1}
$$

A sub-laplacian on $SE(2)$ has the form $\mathfrak{L} = -\sum_j X_j$, where $X_j = \alpha_j X_3 + U_j$ for some $\alpha_j \in \mathbb{R}$ and $U_j \in Span\{X_1, X_2\}$. A radial solution (see [1]), by method of separation, of the Laplace-Beltrami equation below

$$
\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = 0 \tag{1.2}
$$

yields the following function

$$
\begin{aligned}
J_\lambda(mr) &= \Gamma(1)(\frac{\sqrt{\lambda r}}{2})^0 I_{\frac{2-2}{2}}(\sqrt{\lambda r}) && (1.3)\\
&= I_0(\sqrt{\lambda r}). \qquad \square && (1.4)
\end{aligned}
$$

I_0 is the Bessel function of order zero. It is the spherical function of $SE(2)$.

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

2 Spaces of Distributions on SE(2)

In this section, descriptions of spaces of distributions and their respective topologies are presented. Further more, Fourier transform of functions on $SE(2)$ is discussed

3.1 The space $C^\infty(G)$. Given a solvable Lie group G endowed with invariant measure $d\mu(g)$, and $\mathfrak{g}$ its Lie algebra. Lets denote by m the dimension of $\mathfrak{g}$. Fix $\{X_1, ..., X_m\}$ a basis of $\mathfrak{g}$. To each $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{N}^m$, we put $|\alpha| = \alpha_1 + ... + \alpha_n$ and associate a differential operator X^α, which is left invariant, on G acting on $f \in C^\infty(G)$, by

$$X^\alpha f(g) = \frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} ... \frac{\partial^{\alpha_m}}{\partial t_m^{\alpha_m}} f(g\ exp(t_1 X_1)...exp(t_m X_m))|_{t_1=...=t_m=0.}.$$

The space $C^\infty(G)$ may be given a topology defined by a system of seminorms specified as

$$|f|_{\alpha,m} = Sup_{|\alpha|\leq m}|X^\alpha f(g)|.$$

With this topology, C^∞(G) is metrizable, locally convex and complete, hence, it is a Frechet space. This Frechet space may be denoted as $\xi(G)$

3.2 The space $C_c^\infty(G)$**.** This space $C_c^\infty(G)$ is the space of complex-valued C^∞ function on G with compact support. For any $\epsilon > 0$, put

$$B_\epsilon = \left\{(\xi, \theta) \in G : ||\xi|| \leq \epsilon\right\}$$

and

$$\mathfrak{D}_\epsilon = \mathfrak{D}(B_\epsilon) = \left\{f \in C_c^\infty(G) : f(\xi, \theta) = 0, if\ ||\xi|| > \epsilon\right\}.$$

Then $\mathfrak{D}(B_\epsilon)$ is a Frechet space with respect to the family semi norms defined as

$$\left\{P_\alpha(f) = \|D^\alpha f\|_\infty : \alpha \in \mathbb{N}^3\right\}.$$

$\mathcal{D}(G) = \bigcup_{n=1}^\infty \mathfrak{D}(B_n)$ is topologised as the strict inductive limit of $\mathcal{D}(B_n)$. A linear functional on the topological vector space $\mathcal{D}$ that is continuous is known as a distribution on G denoted by $\mathcal{D}'(G)$.

Given a manifold M and a distribution T, T is said to vanish on a subset $V \subset M$, which is open, if $T = 0$. Let $\{U_\alpha\}_{\alpha\in\omega}$ represents the collection of all open sets on which T vanishes and let U stand for the union of $\{U_\alpha\}_{\alpha\in\omega}$. $\overline{M - U}$, regarded as the closure of the complement of M, is the support of T. We denote $\xi'(G)$ a distributions space with compact support.

3.3 The Schwartz space $\mathcal{S}(G)$**.** Consider the Euclidean motion group $SE(2)$ realised as $\mathbb{R} \rtimes \mathbb{T}$ where $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$. If we choose a system of coordinates([8]) (x, y, θ) on G with $x, y \in \mathbb{R}$ and $\theta \in \mathbb{T}$, then a complex - valued C^∞ function f on $G = SE(2)$ is called rapidly decreasing if for any $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^3$ we have

$$p_{N,\alpha}(f) = Sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} \mid (1 + ||\xi||^2)^N (D^\alpha f)(\xi, \theta) \mid < +\infty, \tag{2.1}$$

where

$$D^\alpha = \left(\frac{\partial}{\partial x}\right)^{\alpha_1} \left(\frac{\partial}{\partial y}\right)^{\alpha_2} \left(\frac{\partial}{\partial \theta}\right)^{\alpha_3},$$

$(\alpha = (\alpha_1, \alpha_2, \alpha_3); \xi = (x, y))$. The space of all rapidly decreasing functions on G is denoted by $\mathcal{S} = \mathcal{S}(G)$. Then $\mathcal{S}$ is a Frechet space in the topology given by the family of semi-norms $\{P_{N,\alpha} : N \in \mathbb{N}, \alpha \in \mathbb{N}^3\}$. This result is stated formally with proof in the next proposition,

3.4 Proposition. The Schwartz space $\mathcal{S}(SE(2))$ is a Frechet space.

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

Proof. Let us denote the system of seminorm defined in (5) by $||.||^{\infty}_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{N}^m$. $||.||^{\infty}_{n,\alpha}$ is countable and separable on $\mathcal{S}(SE(2))$. This is because $||\varsigma||^{\infty}_{0,0} = ||\varsigma||^{\infty}_{L^1(G)} = 0 \Rightarrow \varsigma = 0$. This separability condition defines a locally convex topology on $\mathcal{S}(SE(2))$. Next is to prove that $\mathcal{S}(SE(2))$ is a Frechet space. In order to do this, we need to show that it is complete. To this end, let $\{\varsigma\}_{n\in\mathbb{N}} \subset \mathcal{S}(SE(2))$ be a sequence that is Cauchy in nature for the semi norms $||.||^{\infty}_{n,\alpha}$. Let X^{α} be as defined in 2.1, $X^{\alpha}\varsigma_n$ converges to a bounded function $\varsigma_{n,\alpha}$ uniformly for every $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$. Next is for us to prove that

$$\varsigma_{n,\alpha} = X^{\alpha}\varsigma_{0,0} \; n \in \mathbb{N}, \;\; \alpha \in \mathbb{N}^m. \tag{2.2}$$

The prove of (6) is to establish that $\varsigma_{0,0} \in \mathcal{S}(SE(2))$ and $\varsigma_n \to \varsigma_{0,0}$ in $\mathcal{S}(SE(2))$ and by implication, it will mean that $\mathcal{S}(SE(2))$ is complete. Therefore, let us prove that (6) is true. For $n = 0$ and α of length one, say $\alpha = \alpha_i$ with all coordinates equal to zero but the i^{th} equal to one, we have for all $t \in \mathbb{N}$

$$\varsigma_n(gexp(tX_i) = \varsigma_n(g) + \int_0^t X_i\varsigma_n(\eta X_i))d\eta, \tag{2.3}$$

when $n \to \infty$ (7) becomes

$$\varsigma_{0,0}(gexp(tX_i)) = \varsigma_{0,0}(g) + \int_0^t \varsigma_{0,\alpha_i}(gexp(\eta X_i))d\eta. \tag{2.4}$$

If we differentiate (8) with respect to t at 0, it shows that $\varsigma_{0,0}$ is continuously differentiable in the direction X_i with

$$X_i\varsigma_{0,0}(g) = \varsigma_{0,\alpha_i}g.$$

If this argument is repeated, it shows that $\varsigma_{0,0} \in C^{\infty}(G)$ with $X^{\alpha}\varsigma_{0,0} = \varsigma_{0,\alpha}$, $\forall\alpha \in \mathbb{N}^m$. This means that for all $n \in \mathbb{N}$ and $\alpha \in \mathbb{N}^m$, $X^{\alpha}\varsigma_n$ converges pointwise to $X^{\alpha}\varsigma_{0,0}$. By hypothesis, $X^{\alpha}\varsigma_n$ converges to $\varsigma_{n,\alpha}$, therefore $\varsigma_{n,\alpha} = X^{\alpha}\varsigma_{0,0}$. Since $\varsigma_{n,\alpha} \in L^{\infty}(G)$, this shows that $\varsigma_{0,0} \in \mathcal{S}(SE(2))$ and that ς_n converges to $\varsigma_{0,0}$ in $\mathcal{S}(SE(2))$. Hence, $\mathcal{S}(SE(2))$ is complete and therefore Frechet. □

The space $\mathcal{S}'(G)$ of (continuous) linear functionals on $S(G)$ is referred to as the space of tempered distributions on $G = SE(2)$. This space can be topologised by strong dual topology, which is defined as the topology of uniform convergence on the bounded subsets of $\mathcal{S}(G)$ generated by the seminorms $p_{\varphi}(u) = |u(\varphi)|$, where $u : \mathcal{S}(G) \to \mathbb{R}$ and $\varphi \in \mathcal{S}(G)$.

Let $f_1, f_2 \in \mathcal{S}(G)$ or $L^2(G)$. The convolution of f_1 and f_2 is defined as

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(h)f_2(h^{-1}g)d\mu_G(h)\\ &= \int_G f_1(gh)f_2(h^{-1})d\mu_G(h).\end{aligned}$$

The convolution operation obeys the associativity property

$$(f_1 * f_2) * f_3 = f_1 * (f_2 * f_3),$$

whenever all the integrals are absolutely convergent (cf: [2,8,9]).

The next result shows continuity of convolution of functions in $\mathcal{S}(SE(2))$. It is presented below as proposition 3.5 with proof.

3.5 Proposition. Convolution of functions is continuous from $\mathcal{S}(SE(2)) \times \mathcal{S}(SE(2))$ to $\mathcal{S}(SE(2))$

Proof. Let us recall that the convolution of two functions on $SE(2)$, provided the integral converges, is defined as

$$\begin{aligned}(f_1 * f_2)(g) &= \int_G f_1(h)f_2(h^{-1}g)d\mu_G(h)\\ &= \int_G f_1(gh)f_2(h^{-1})d\mu_G(h).\end{aligned}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

Since the differential operators X^α are left invariant, they act on the convolution as follows

$$X^\alpha(f_1 * f_2) = f_1 * X^\alpha f_2,$$

therefore,

$$|X^\alpha(f_1 * f_2)(g)| = |\int_G X^\alpha f_1(h) f_2(h^{-1}g)dh|$$

We note that $SE(2)$ is unimodular (see [10], p.326), this means

$$\int_G f(hg)dg = \int_G f(gh)dg = \int_G f(g^{-1})dg = \int_G f(g)dg.$$

So, by putting $g = hg$, we get

$$\begin{aligned} |X^\alpha(f_1 * f_2)(g)| &= |\int_G X^\alpha f_1(h) f_2(h^{-1}(hg)dh| \\ &= |\int_G X^\alpha f_1(h) f_2(g)dh| \\ &\leq \int_G |X^\alpha f_1(h) f_2(g)dh|. \end{aligned}$$

We know that

$$X^\alpha(f_1 * f_2)(g) = f_1 * X^\alpha f_2(g).$$

But

$$f_1 * X^\alpha f_2(h) = \int_G f_1(h) X^\alpha(h^{-1}g)dh.$$

Therefore

$$\begin{aligned} |X^\alpha(f_1 * f_2)(g)| &\leq \int_G |f_1(h)||X^\alpha f_2(g)|dh \\ &\leq \int_G |f_1(h)|dh \frac{C}{|(1 + ||\xi||^2)^m|}, \end{aligned}$$

because $|X^\alpha f_2(g)| \leq \frac{C_{\alpha,N}}{|(1+||\xi||^2)^m|}$. Since $||\xi||$ is a positive real constant, we may put $Q_N = |(1 + ||\xi||^2)^N|$, so that

$$\begin{aligned} |X^\alpha(f_1 * f_2)(g)| &\leq \int_G |f_1(h)||X^\alpha f_2(g)|dh \\ &\leq \int_G |f_1(h)|dh \frac{C}{|(1 + ||\xi||^2)^N|} \\ &\leq \frac{C}{Q_N} \int_G |f_1(h)|dh \\ &= \frac{C}{Q_N} \|f_1\|_{L^1(G)} \end{aligned}$$

$$\begin{aligned} |(1 + ||\xi||^2)^N X^\alpha(f_1 * f_2)(\xi,\theta)| &= |(1 + ||\xi||^2)^N||X^\alpha(f_1 * f_2)(\xi,\theta)| \\ &\leq C||f||_{L^1(G)} < +\infty. \end{aligned}$$

On taking supremum, we have

$$P_{\alpha,N}(f_1 * f_2) = sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} |(1 + ||\xi||^2)^N X^\alpha(f_1 * f_2)(\xi,\theta)| < +\infty.$$

4. Radial Distribution on SE(2)

Let G be an arbitrary locally compact group and K its subgroup that is compact, the pair (G, K) is known as a Gelfand pair if $L^1(K\backslash G/K)$ is abelian under convolution. Also, let $C_c(K\backslash G/K)$ stand for the space of continuous functions with support on G that are compact. Any $f \in C_c(K\backslash G/K)$ that satisfies $f(k_1 g k_2) = f(g)\ \forall k_1, k_2 \in K$ is called a spherical function and $C_c(K\backslash G/K)$ equipped with convolution as a binary operation forms a Banach algebra that is commutative (see [10, 14]).

4.1 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called radial if $\exists\ \phi$ defined on $[0, \infty)$ in such a way that $f(x) = \phi(|x|),\ \forall\ x \in \mathbb{R}^n$. For a transformation ρ on $\mathbb{R}^n$, ρ is called orthogonal if $\exists$ linear operator on $\mathbb{R}^n$ such that $\langle \rho x, \rho y\rangle = \langle x, y\rangle\ \forall\ x, y \in \mathbb{R}^n$.

A Schwartz function φ is called radial if for all $A \in O(n)$ (that is to say, for all rotations on ^{n}) the following equation holds

$$\varphi = \varphi o A.$$

A collection of all radial Schwartz functions is denoted as $\mathcal{S}_{rad}()$ and $\mathcal{S}'()$ the space of tempered distributions on ^{n}. A distribution $u \in \mathcal{S}'()$ is called radial if for all $A \in O(n)$, we have

$$u = uoA.$$

This means that for all Schwartz functions φ on ^{n}, we have

$$\langle u, \varphi\rangle = \langle u, \varphi o A\rangle$$

and $\mathcal{S}'_{rad}()$ is the space of all radial tempered distributions on ^{n}.

4.2 Definition. A positive definite function

$$f : G \to \mathbb{C}$$

satisfies the following inequality

$$\sum_{i,j=1^m} \alpha_i \overline{\alpha_k} f(g_i^{-1} g_k) \geq 0 \tag{2.5}$$

for all subsets $\{g_1, ..., g_m\} \in G$ and all sequences $\{\alpha_1, ..., \alpha_m\} \in \mathbb{C}$. The integral analogue of the inequality (5) is given by

$$\int_G \int_G f(g_i^{-1} g_k)\varphi(g_i)\varphi(g_k) dg_i dg_k \geq 0 \tag{2.6}$$

where φ ranges over $L^1(G)$ or $C_c(G)$. If f is a continuous functions, (9) and (10) are equivalent.

A spherical function that also satisfies (5) is referred to as positive definite spherical function. Let $\widehat{(G, K)}$ stand for the set of spherical functions on G and let $\widehat{(G, K)}_+$ denotes the subset of $\widehat{(G, K)}$ that is positive definite. The set $\widehat{(G, K)}_+$ is isomorphic with $\mathbb{R}^+$. A measure π on $\widehat{(G, K)}$ such that for $f \in L^1(G, K)$ the plancherel theorem holds, that is

$$\int_{(K\backslash G/K)} |f(a)|^2 d(kak) = \int_{\widehat{(G,K)}} |\widehat{f}(\varphi)|^2 d\pi(\varphi).$$

π is referred to as plancherel measure and its support is the full set $\widehat{(G, K)}$.

4.3 Definition [11]. A positive definite distribution T on a Lie group G is a distribution that satisfies $T(\widetilde{\phi} * \phi) \geq 0\ \forall \phi \in \mathcal{D}(G)$. If in addition to the above condition, $\phi \in C_c(K\backslash G/K)$, such a distribution is known as a K-bi-invariant distribution on G.

Let us look at the following regular distributions on $SE(n)$

1. Let f be a continuous function on $\mathbb{R}^2$, μ a Radon measure on the compact subgroup of G. The linear functional $f \otimes \mu$ on $\mathfrak{D}(G)$ defined by

$$\varphi \mapsto \langle \varphi, f \otimes \mu\rangle = \int_{\mathbb{R}^2} \int_{SO(2)} f(a)\varphi(g) da dA$$

is a distribution on $G = SE(2)$, $\varphi \in \mathfrak{D}(G)$.

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

2. The character function χ_a is a linear functional on $\mathfrak{D}(G)$ or distribution on G defined by

$$\chi_a : f \mapsto \sum_{n=-\infty}^{\infty} \int_G f(g)(U_g^a \chi_n, \chi_n) dg = \sum_{n=-\infty}^{\infty} (U_g^a \chi_n, \chi_n) = TrU_f^a,$$

where

$$U_f^a = \int_G f(g) U_g^a dg.$$

It is our interest in this section to show that the type of distribution mentioned in 2 above is a radial, positive definite, tempered and bounded. We proceed as follows.

There is a relationship between the spherical function I_0 of SE(2) and χ_a on $\mathcal{D}(G)$. This may be found in [8] as theorem 3.3. It is stated here as theorem 4.4, without proof.

4.4 Theorem For any fixed $\sigma > 0$, the linear functional

$$\chi_a f \mapsto TrU_f^\sigma$$

is a distribution on $G = SE(2)$. In fact, χ_a is equal to $I_0(t||\xi||) \otimes \delta(t)$ where I_0 is the Bessel function of order 0 and δ is the Dirac measure of order zero on $T \cong K$ and TrU_f^σ stands for the trace of the representation U_f^σ. That is to say, our distribution under consideration is $K(\lambda, t) = I_0(\lambda) \otimes \delta(t)$, where $\lambda = t||\xi||$. $\square$

$\mathcal{S}'(SE(2))$, as earlier defined, is the space of tempered distributions on $SE(2)$. There are three kinds of topology that can be given to $\mathcal{S}'(SE(2))$, namely, strong dual topology, weak topology and the weak $*$ topology. A tempered distribution $u \in \mathcal{S}'(SE(2))$ is a continuous linear functional on $\mathcal{S}(SE(2))$.

Let $T \in \mathcal{S}'(SE(2))$ and let f be an arbitrary C^∞ function on G. There is a condition for $fT \in \mathcal{S}'(SE(2))$. This leads us to the following definition of a Lie group with polynomial growth.

4.5 Definition [12], p.7 Let G be a Lie group and let μ be the left Haar measure of G. G is said to have polynomial growth if $\exists$ a compact symmetric neighborhood U of $e \in G$ that generates G and such that the sequence $(\mu(U^n)_{n\in\mathbb{N}}$ has polynomial growth as $n \to \infty$. A function $f \in C^\infty(G)$ is said to have a polynomial growth if G has a polynomial growth. A Gelfand pair (G, K) has polynomial growth if G has polynomial growth ([12], p.7). The Gelfand pair $(SE(2), SO(2) \cong \mathbb{T})$ is a pair with polynomial growth, it follows from Def. 4.5 that $SE(2)$ is a a Lie group with polynomial growth.

Given $T \in \mathcal{S}'(G)$ and $f \in C^\infty(G)$, the condition for $fT \in \mathcal{S}'(G)$ is that f must be a function with polynomial growth. I_0 is the spherical function on $SE(2)$. It is bounded, positive definite and has polynomial growth.(see [10]). Also, $\delta(t) \in C^\infty(G)$. Following this development, we have that $I_0(\lambda)\delta(t) \in \mathcal{S}'(G)$, where $\lambda = \sigma||\xi||$. Further more, I_0 being the spherical function of $SE(2)$ is also a radial function. This is because elementary spherical functions are also radial functions. Since $I_o(\lambda)$ is radial, bounded and positive definite (see [10],Prop. 2.4) and $K(\lambda, t) = I_0(\lambda)\delta(t)$ is compactly supported in $\mathcal{S}'()$ at the identity, it therefore means that it also belongs to the space of tempered radial distributions $\mathcal{S}'_{rad}(G)$ on $G = SE(2)$.

5. Temperature function on the motion group $SE(2)$

In this section, we extend results obtained in [13] for Heisenberg group to the Euclidean motion group.

5.1 Definition. Let $\triangle_{SE(2)}$ be the Laplace-Beltrami operator on $SE(n)$. A temperature function on $SE(2)$ is a C^{∞} function $F \in \mathcal{S}(G)$ that satisfies

$$\left(\left(\frac{\partial}{\partial \xi} - \triangle_G\right)F\right)(\xi,\theta) = 0, (\xi,\theta) \in G.$$

Let $f \in L^{\infty}(SE(2))$, then f is positive definite on G if

$$\int_G f(\xi,\theta)(\varphi * \varphi^*)(\xi,\theta)d\xi d\theta \geq 0, \varphi \in \mathcal{S}(SE(2), \varphi^*(\xi,\theta) = \varphi((\xi,\theta)^{-1}).$$

A function f that is continuous on $SE(n)$ is called $U(n)$- invariant if

$$f(\alpha\xi,\theta) = f(\xi,\theta), (\xi,\theta) \in SE(2), \forall\alpha \in U(n),$$

where $U(n)$ stands for a collection of square unitary matrices of order n. Let $f \in \mathcal{S}'(SE(2))$, then for all $\alpha \in U(n)$, f_{α} is defined as

$$f_{\alpha}(\varphi) = f(\varphi_{\alpha^{-1}}),$$

where $\varphi \in \mathcal{S}(SE(2))$ and $\varphi_{\alpha^{-1}}(\xi,\theta) = \varphi(\alpha^{-1}\xi,\theta)$. A function $\varphi \in \mathcal{S}(SE(2)$ is $U(n)$-invariant if

$$\varphi_{\alpha} = \varphi,$$

a distribution $f \in \mathcal{S}'(SE(2))$ is $U(n)$-invariant if and only if $f = f^{\natural}$ where

$$f^{\natural}(\varphi) = f(\varphi^{\natural}),$$

and

$$\varphi^{\natural}(\xi,\theta) = \int_{U(n)} \varphi(\alpha\xi,\theta)d\alpha,$$

$d\alpha$ is the normalized measure on $U(n)$ such that

$$\int_{U(n)} d\alpha = 1.$$

It therefore means that $\varphi \in \mathcal{S}(SE(2))$ is $U(n)$-invariant if and only if $\varphi^{\natural} = \varphi$. Let $f \in \mathcal{S}'(SE(2))$ and $\varphi \in \mathcal{S}(SE(2)$, then the convolution $f * \varphi$ is defined as

$$(f * \varphi)(\xi,\theta) = \int df(\xi',\theta')\varphi((\xi',\theta')^{-1}(\xi,\theta)),\ (\xi,\theta) \in SE(2),\ (\xi',\theta').$$

and

$$(\varphi * f)(\xi,\theta) = \int df(\xi',\theta')\varphi((\xi,\theta)(\xi',\theta')^{-1}),\ (\xi,\theta) \in SE(2),\ (\xi',\theta').$$

The linear map

$$\mathcal{S}'(SE(2)) \times \mathcal{S}(SE(2)) \ni (f,\varphi) \mapsto f * \varphi \in \mathcal{S}'(SE(2))$$

is separately continuous and the convolution $f * \varphi$ is a smooth function on $SE(2)$ (see prop. 2.5 above). The following lemma is needed in the proof of theorem 5.3, which is the main result of this section. Following [13], we have the following lemma.

5.2 Lemma. The Bessel function $\{J_0(\lambda) : \lambda > 0\}$ associated with the Laplace-Beltrami Operator of $SE(2)$ satisfies the following conditions.
(i) $J_0(\lambda) \in \mathcal{S}(G)$, $\lambda > 0$.
(ii) For every $\psi \in \mathcal{S}(G)$, $J_0(\lambda) * \psi \to \psi$ in $\mathcal{S}(G)$ as $\lambda \to 0$.

The following result is the main result of this research.

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
11(1), 2025, Pages 96 - 106
https://doi.org/10.5281/zenodo.15176025

5.3 Theorem. Let $f \in \mathcal{S}'(SE(2))$. The function F defined on G by

$$F(\xi,\theta) = (f * J_0(\lambda))(\xi,\theta)$$

satisfies the following conditions.
(i) $((\frac{\partial}{\partial\lambda} - \triangle_G)F)(\xi,\theta) = 0$, $(\xi,\theta) \in SE(2)$.
(ii) There exist positive constants C and N such that

$$|F(\xi,\theta)| \leq C \sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} |(1+||\xi||^2)^{-N}||f||_{L^1(SE(2))},$$

where $||f||_{L^1(SE(2))}$ is the L^1- norm of $SE(2)$.

Proof.(i) Let $f \in \mathcal{S}'(SE(2))$. Then by (i) of lemma 4.2, the function defined by $F(\xi,\theta) = (f * J_0(\lambda))(\xi,\theta)$ is a smooth function on $SE(2)$. Also, since the linear map $\mathcal{S}'(SE(2)) \times \mathcal{S}(SE(2)) \ni (f,\varphi) \mapsto f * \varphi \in \mathcal{S}'(SE(2))$ is separately continuous, it therefore means that $f * \varphi$ is smooth. Let $U(\mathfrak{g})$ be the the universal enveloping algebra of $SE(2)$. Elements of this algebra are left invariant differential operators on $SE(2)$. The Laplace-Beltrami operator on $SE(2)$ is a member of this algebra, therefore $\triangle_G$ is a left vector field on $SE(2)$. Therefore, it stands to reason that

$$\left(\left(\frac{\partial}{\partial r} - \triangle_G\right)F\right)(\xi,\theta) = 0.$$

(ii) For $f \in \mathcal{S}'(SE(2))$, there exists C and N such that

$$|f(\varphi)| \leq C \sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} |(1+||\xi||^2)^N (D^\alpha f)(\xi,\theta)|.$$

By (2.1),

$$|F(\xi,\theta)| = |(f * J_0(\lambda))(\xi,\theta)| \leq \sup_{\theta\in\mathbb{T},\xi\in\mathbb{R}^2} |(1+||\xi||^2)^N (D^\alpha f * J_0(\lambda))(\xi,\theta)|.$$

But

$$\begin{aligned} |D^\alpha(f * J_0(\lambda))(\xi,\theta)| &= |f * D^\alpha J_0(\lambda)(\xi,\theta)| \\ &\leq \int_{SE(2)} |f(\eta)||D^\alpha(\eta^{-1}\zeta)|d\eta \\ &\leq \int_{SE(2)} |f(\eta)||D^\alpha \zeta)|d\eta \quad \textit{since } SE(2) \textit{ is unimodular} \\ &\leq \int_{SE(2)} |f(\eta)|d\eta \frac{C}{|(1+||\xi||^2)^N|} \\ &= C|(1+||\xi||^2)^{-N}| \int_{SE(2)} |f(\eta)|d\eta \\ &= |(1+||\xi||^2)^{-N}||f||_{L^1(SE(2))} \end{aligned}$$

$\eta = (\xi,\theta)$ and $\zeta = (\xi',\theta')$.

References

[1] U. Edeke and U. Bassey. *The Radial Part Of An Invariant Differential Operator On The Euclidean Motion Groups.* Journal of the Nigerian Mathematical Society. 43(3). 237-251 (2024)

[2] G. S. Chirikjian, A. B. Kyatkin.*Engineering Applications of Non Commutative Harmonic Analysis.* CRC Press, New York.2001.

[3] Yarman, C. E., Yazici, B.; *A Wiener Filtering Approach over the Euclidean Motion group for Radon transform Inversion.* Proceeding of SPIE - The international Society for Optical Engineering. May 2003.

[4] U. E. Edeke, J. A. Odey, E.E. Essien, A.O. Otiko, D.O. Egete, J.N. Ezeorah, E.E. Bassey and B.I. Ele *Positive Definite Kernels and Radial Distributions on the Euclidean Motion Group,* Int. J. Anal. Appl., 23 (2025), 55.

[5] U. E. Edeke, U. N. Bassey. *Some Properties of Convolution and Spherical Analysis on the Euclidean Motion Group.* (2025). Journal of The Nigerian Society of Physical Sciences, 7(2), 2355. https://doi.org/10.46481/jnsps.2025.2355

[6] A. O. Barut, R. Raczka. *Theory of group representations and Applications.* Polish Scientific Publishers, Warszawa.

[7] U. N. Bassey, U. E. Edeke.*Convolution operators and equation on the Euclidean motion group.* JNSPS, Vol.6, issue 4, sept., 2024

[8] M. Sugiura. *Unitary Repreentation and Harmonic Analysis: An Introduction.*North- Holland, New York.1990.

[9] N. Ja. Vilenkin, A. U. Klimyk. *Representation of Lie Groups and Special functions Vol 1.*Kluwer Academic Publishers. 1991.

[10] G. V. Dijk. *Introduction to Harmonic Analysis and Generalized Gelfand Pairs.*Walter De Gruyter, Berlin. 2009. doi.org/ 978-3110220193

[11] M. E. Egwe. *A Radial Distribution on the Heisenberg Group.* Journal of Pure Mathematical Science, Vol. 3, 2014. no.3, 105-112, http://dx.org/10.12988/pms.2014.436.

[12] F. Astengo, B. Blasio and F. Ricci. *On The Schwartz Correspondence for Gelfand Pairs of Polynomial growth.* European Mathematical Scociety, Vol 32, No.1, 2021 pp 79-96. http://doi.org/ 10.4171/RLM/927

[13] K. Jinman, M. W. Wong. *Positive definite teemperature functions on the Heisenberg group.* Applicable Analysis: An international Journal. Vol.85, No. 8, 2006.

[14] F. R. Fabio. *The Topology of the Spectrum for Gelfand pairs on Lie group.* Bollettino dell'Unione Matematica Italiana 10-B.3 (2007): 569-579. http://eudml.org/doc/290444.

[15] Eke, K. S., Olaoluwa, H. O. and Olaleru, J. O. (2024). Common Fixed Point Results for Asymptotic Quasi-Contraction Mappings in Quasi-Metric Spaces. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(1), 115 - 127.

[16] Omidire, O. J. (2024). Common Fixed Point Theorems of Some Certain Generalized Contractive Conditions in Convex Metric Space Settings. International Journal of Mathematical Sciences and Optimization: Theory and Applications, 10(3), 1 - 9. https://doi.org/10.6084/zenodo.13152753.