IJMAO

INTERNATIONAL JOURNAL OF MATHEMATICAL ANALYSIS AND OPTIMIZATION: THEORY AND APPLICATIONS
VOL. 2018, PP. 276 -290

Partial Differential Equation Approach for Valuation of European Put Option Price using Two Stochastic Financial Models.

H. C. Chinwenyi[1*], E. S. Onah[2] And T. Aboiyar[3]

1. Raw Materials Research and Development Council (RMRDC), 17 Aguiyi Ironsi Street, Maitama District, FCT, Abuja, Nigeria.
2. National Mathematical Center (NMC), Abuja-Lokoja Road, Sheda, FCT, Abuja, Nigeria.
3. Department of Mathematics/Statistics/ Computer Science, University of Agriculture, Makurdi, Benue State, Nigeria.

*Corresponding author: chinwenyi@yahoo.com

Article Info
Received: 22 March 2018 Revised: 16 July 2018
Accepted: 20 July 2018 Available online: 26 July 2018

Abstract

An option is defined as a financial contract that provides the holder the right but not the obligation to buy or sell a specified quantity of an underlying asset in the future at a fixed price (called a strike price) at or before the expiration date of the option. This paper looked at two different models in finance which are the Constant Elasticity of Variance (CEV) model and the Black-Karasinski model. We obtained the various Partial Differential Equations (PDEs) option price valuation formulas for these two models using the riskless portfolio method through the elimination of the stochastic components from their respective Stochastic Differential Equations (SDEs). Also, numerical implementation of the derived Black-Scholes Parabolic PDEs options pricing equations for the SDEs models was carried out using the Explicit Finite Difference method which was implemented using MATLAB. Furthermore, results from the numerical examples using published data from the Nigeria Stock Exchange (NSE) show the various effects of increase in the underlying asset value (stock price) and increase in the risk free interest rate on the value of the European Put Option for these models. From the results, we see that an increase in the stock price and interest rate both yields a decrease in the value of the option price and hence guides the holder of the option with quality decision to not exercise his right on the option and may allow the option to expire.

Key Words: Option price valuation, Explicit Finite Difference method, Partial Differential Equations, Black-Scholes PDEs, European Put Option.

MSC2010: 60H15, 60H30

1 Introduction

Financial derivatives are financial instruments that are linked to a specific financial indicator or commodity and through which specific financial risks can be traded in financial markets in their own right. Transactions in financial derivatives should be treated as separate transactions rather than as an integral part of the value of underlying transactions to which they may be linked. The value of a financial derivative is derived from the price of an underlying item, such as an asset or index. Since the future reference price is not known with certainty, the value of the financial derivative at maturity can only be anticipated or estimated. Financial derivatives are used for a number of purposes including risk management, hedging, arbitrage between markets and speculations. Financial derivatives enable parties to trade specific financial risk such as interest rate risk, currency, equity and commodity price risk, and credit risk, etc. to other entities who are more willing, or better suited, to take or manage these risks, typically, but not always, without trading in a primary asset or commodity. The risk embodied in a financial derivative contract can be traded either by trading the contract itself, such as with options, or by creating a new contract which embodies risk characteristics that match, in a countervailing manner, those of the existing contract owned. We have four main types of financial derivatives namely: futures, forwards, swaps and options [1].

To value or price financial derivative products such as options is one of the most common problems in mathematical finance. In order to value an option using the Partial Differential Equations (PDEs) approach, a set of risky assets is used to form a riskless portfolio that mimics the payoff of the financial derivative. This corresponds to a portfolio whose stochastic component has been removed, so the Stochastic Differential Equation driving the portfolio dynamics becomes a PDE. We specify a boundary condition corresponding to the financial derivative. This PDE can sometimes be solved using the Feynman-Kac theorem. In other cases it must be solved numerically [2].

However, this paper will be centered on the use of advanced mathematical tools (PDE approach) in the valuation of options price which is a type of financial derivatives. The advantages of the PDE approach are that it is generally faster and that it gives the results for all initial prices (and even for all strikes or all maturities T in some cases).

2 Preliminaries

Definition 2.1 (Self-financing trading strategy). A trading strategy is an N –dimensional stochastic process $a_1(t), \cdots, a_N(t)$ that represents the allocations into the assets at time, t. The time– t value of the portfolio is $\Pi(t) = \sum_{i=1}^{N} a_i(t)S(t)$. A trading strategy is **self-financing** if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds. A **portfolio** is defined as an investment in several assets at the same time. This implies the strategy is self-financing if

$$d\Pi = d\left(\sum_{i=1}^{N} a_i(t)S(t)\right) = \sum_{i=1}^{N} a_i(t)dS(t),$$

in other words, if

$$\Pi(t) = \Pi(0) + \sum_{i=1}^{N} \int_0^t a_i(u)\, dS_t(u)$$

Where S_i is the values of the asset at different time intervals. In the case of two assets the portfolio value is $\Pi(t) = a_1(t)S_1(t) + a_2(t)S_2(t)$ and the strategy $(a_{1,}a_2)$ is self-financing if $d\Pi(t) = a_1(t)dS_1 + a_2(t)dS_2(t)$.

Definition 2.2 (Arbitrage opportunity). An **arbitrage opportunity** is a self-financing trading strategy for which $\Pi(t) \geq 0$ and $\Pr[\Pi(T) > 0] = 1$. This implies that the initial value of the portfolio (at time zero) is zero or negative, and the value of the portfolio at time T will be greater than zero with absolute certainty. This means we start with a portfolio with zero value, or with debt (negative value). At some future time we have positive wealth, and since the strategy is self-financing, no funds are required to produce this wealth.

We also require that the trading strategy be **predictable**. This implies that the allocations $a_i(t)$ are stochastic, but revealed at time t. This is reasonable, since the portfolio is rebalanced at time t so the allocations are known at that time, and at all instants prior. After time t the allocations are again stochastic, up until the next rebalancing time. This implies that when we integrate $d\Pi(t)$ we can write the individual terms as $\int_0^t a_i(u)dS_i(u)$ rather than as $\int_0^t d(a_i(u)S_i(u))$.

Definition 2.3 (Options and replication). An option is defined as a financial contract that gives the holder the right but not the obligation to buy or sell a specified quantity of the underlying asset in the future at a fixed price at or before the expiration time of the contract. The payoff V_T at time T of an option is a function of a risky asset. Arbitrage is ruled out if we identify a self-financing trading strategy that produces the same payoff as the option, so that $\Pi_{t=}V_t$ where $\Pi_t > 0$. The trading strategy is then a **replicating strategy** and the portfolio is a **replicating portfolio**. If a replicating strategy exists, then the option which is a financial contract that provides the holder the right but not the obligation to buy or sell a specified quantity of an underlying asset in the future at a fixed price (called a strike price) at or before the expiration date of the option is **attainable**, and if all options are **attainable**, the economy is **complete**.

In the absence of arbitrage, the trading strategy produces a unique value for the value V_T of the option; otherwise an arbitrage opportunity would exist. Not only that, at every time t the value of the option, V_t must be equal to the value of the replicating strategy, Π_t, so that $\Pi_t = V_t$. Otherwise an arbitrage opportunity exists [3].

Definition 2.4 (Self-financing portfolio). A portfolio allocation $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ with price (value) V_t given by

$$V_t = \xi_t S_t + \eta_t A_t, \qquad t \in \mathbb{R}_+$$

Where A_t or $B_t = \text{Riskless asset}$ is self-financing if and only if the relation

$$dV_t = \eta_t dA_t + \xi_t dS_t$$

holds, where ξ_t is the number of shares in S_t (could be any real number) and η_t is the amount in the bank. The self-financing condition is not entirely obvious at first, but it helps to think of one's personal decision-making in a financial market. Usually, one chooses a portfolio allocation in stock and bonds, and then allows a certain amount of change to occur in the market before adjusting their allocation. If one does not remove any cash for consumption and does not inject any cash for added investment, then the portfolio is self-financing [4].

3 Methods

3.1 Assumptions, Parameters and Variables

3.1.1 Assumptions

The assumptions made in this research are as follows:

i. All assets (stock) considered in this research work are non-dividend yielding stocks.

ii. There exists a risk-free investment that gives a guaranteed return on investment with no chance of default. Hence, the choice of numeraire (B_t) used in this study is the Bank account or government bond. Numeraire is any asset with a positive price and is an item or commodity acting as a measure of value or as a standard for currency exchange

iii. There exists no arbitrage in the market which implies a complete market economy.

3.1.2 Parameters and variables

The value of the option depends on the following parameters:

$S(t)$ or $X(t)$ or S_t = The underlying asset (Stock price) at time t

σ = The volatility of the underlying asset (Measure of the standard deviation of the returns of the asset)

k = The exercise or strike price

T = Maturity date or time of expiry in years

dt = Time step − size

r = Risk − free interest rate

μ = Drift factor (Measure of average rate of growth of the asset)

ds or dx = Change of stock price (Always positive).

We will also introduce some notations and their respective meanings:

u = This is a function of current value of the underlying asset $S(t)$ and time, i.e., $u = u(S, t)$.

$\xi_t = \xi(t)$ = Unit of stock at time t

η_t = Unit of options

A_t or B_t = Riskless asset

V_s or V_t = Value of portfolio strategy

$u(S, t)$ = Value of the option (Option price)

W_t or B_t = Brownian motion

Π = Riskless replicating portfolio value

π_t = Arbitrage free price

α = The elasticity constant

Lemma 3.1: Let $(\xi_t, \eta_t)_{t \in \mathbb{R}_+}$ be a portfolio strategy with value

$$V_t = \eta_t A_t + \xi_t S_t, \qquad t \in \mathbb{R}_+$$

The following statements are equivalent:

i) the portfolio strategy $(\xi_t, \eta_t)_{t \in \mathbb{R}_+}$ is self-financing,

ii) we have

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u \, dX_u, \qquad t \in \mathbb{R}_+ \tag{3.1}$$

Lemma 3.2: (Ito formula for Ito processes; [5]). For any Ito process $(X_t)_{t\epsilon\mathbb{R}_+}$ of the form $X_t = X_0 + \int_0^t \mu_t \, dt + \int_0^t \sigma_t \, dW_t$, $t \in \mathbb{R}_t$ and any $f\epsilon C^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ and $Z_t = f(t, X_t)$ we have

$$Z_t = f(0, X_0) + \int_0^t \mu_t \frac{\partial f}{\partial t}(t, X_t)dt + \int_0^t \sigma_t \frac{\partial f}{\partial t}(t, X_t)dB_t + \int_0^t \frac{\partial f}{\partial t}(t, X_t)dt + \frac{1}{2}\int_0^t |\sigma_t|^2 \frac{\partial^2 f}{\partial t^2}(t, X_t)dt$$

or in differential form

$$dZ_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

$$= \left(\frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\sigma_t^2\right)dt + \frac{\partial f}{\partial x}(t, X_t)\sigma_t dW_t \qquad (3.2)$$

Proposition 3.1: Let $(\xi_t, \eta_t)_{t\epsilon\mathbb{R}_+}$ be a portfolio strategy such that

i) $(\xi_t, \eta_t)_{t\epsilon\mathbb{R}_+}$ is self-financing

ii) The value $V_t = \eta_t A_t + \xi_t S_t$, $t \in \mathbb{R}_+$, takes the form

$$V_t = g(t, S_t)$$

For some $g\epsilon\, C^{1,2}\left((0,\infty) \times (0,\infty)\right)$

Then the function $g(t, x)$ satisfies the Black-Scholes PDE

$$rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx\frac{\partial g}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{dx^2}(t, x), \qquad x > 0, \quad t \in [0, T] \qquad (3.3)$$

and ξ_t is given by

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t), \qquad t \in \mathbb{R}_+$$

3.2 European put options

In the case of a European Put Option with strike price K the payoff function is given by $f(x) = (K - x)^+$ and the Black-Scholes PDE reads

$$\begin{cases} rg_p(t, x) = \frac{\partial g_p}{\partial t}(t, x) + rx\frac{\partial g_p}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g_p}{\partial x^2}(t, x) \\ g_p(T, x) = (K - x)^+ \end{cases} \qquad (3.4)$$

Options are financial tools that give the holder the right, but not the obligation, for a transaction of a certain asset at a given time for a given price [6]. The buyer pays a price for the right. At expiration, if the value of the underlying asset$(S) < \text{StrikePrice}\ (K)$; the buyer makes the difference $K - S$ but if the value of the underlying asset $(S) > \text{StrikePrice}\ (K)$, the buyer does not exercise the right.

3.3 Derivation of the Model Equations (option valuation equations) by PDEs Approach

A portfolio is a pair of adapted processes (in other words, processes that at date t only depend on the past of the Brownian motion up to time t [7].The basic idea of this approach is that we convert a set of Stochastic Differential Equations (SDE) that govern the price dynamics of the assets (stock, options, bonds, etc.) into a Partial Differential Equation (PDE) along with a boundary condition. The PDE is not stochastic and therefore easier to deal with. It is well known that a PDE has infinitely many solutions unless a set of initial values or boundary conditions are specified. In the case of options, this boundary condition is the **payoff** of the option. To price an option under this approach, we set up a replicating portfolio that is made up of our risky assets, but which itself is riskless. In some cases, the PDE will have a solution that is available in closed form. In general, however, the PDE will not have a closed- form solution and will need to be solved numerically, working from the boundary conditions.

We are going to be considering some existing models in financial mathematics and then go ahead to use the PDE approach to derive their option valuation equations. The models considered in this research work are the Constant Elasticity of Variance (CEV) model and the Black-Karasinski model.

3.3.1 The Constant Elasticity of Variance model (CEV)

We consider the pricing of European put option in the Constant Elasticity of Variance (CEV) model introduced by [8]. There are two assets; the bank account B is given by $B_t = e^{rt}$ and the stock X follow the SDE

$$dX_t = rX_t dt + \sigma X_t^{\alpha} dW_t, \qquad X_0 > 0 \tag{3.5}$$

with constants $\sigma > 0$ and $\alpha > 0$, where W_t is a standard Brownian motion, $t \in [0, \infty]$, [9]. Applying Ito's lemma (Lemma 3.2), the value $u = u(X_t, t)$ of the option written on X follows the SDE

$$du = \left(\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2}\right)dt + \frac{\partial u}{\partial x}\sigma x^{\alpha} dW \tag{3.6}$$

The idea is to transform the SDE for u into a PDE, and apply a boundary condition, which will allow a solution to the PDE. We set up a riskless replicating portfolio π made up of η units of the option and ξ units of the stock, and assume the portfolio is self-financing so that

$$d\Pi = \eta du + \xi dX$$

The portfolio thus follows the SDE

$$d\Pi = \eta\left[\left(\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2}\right)dt + \frac{\partial u}{\partial x}\sigma x^{\alpha} dW\right] + \xi[rxdt + \sigma x^{\alpha} dW]$$

$$= \eta\left(\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2}\right)dt + \eta\sigma x^{\alpha}\frac{\partial u}{\partial x}dW + \xi rxdt + \xi\sigma x^{\alpha} dW$$

$$d\prod = \left[\eta\frac{\partial u}{\partial t} + \eta rx\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} + \xi rx\right]dt + \left[\xi\sigma x^{\alpha} + \eta\sigma x^{\alpha}\frac{\partial u}{\partial x}\right]dW \tag{3.7}$$

Next, we make the following choices $\eta = 1$ and $\xi = -\frac{\partial u}{\partial x}$ which enables us to remove the stochastic component from $d\prod$ and renders the portfolio riskless. Since the portfolio is riskless, it earns the risk-free rate and the return on the portfolio in small time increment dt is

$$d\Pi = r\Pi dt$$

We substitute for ξ in equation (3.7) and equate with $r\prod dt$, which produces the PDE (Black-Scholes) for u.

$$d\Pi = \left[\eta\frac{\partial u}{\partial t} + \eta r x\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial t}\right]dt + \left[-\sigma x^\alpha\frac{\partial u}{\partial x} + \eta\sigma x^\alpha\frac{\partial u}{\partial x}\right]dW = r\Pi dt$$

$$= [r\eta u + r\xi x]dt = \left(r\eta u - rx\frac{\partial u}{\partial x}\right)dt$$

$$\Rightarrow \eta\frac{\partial u}{\partial t} + \eta r x\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} = r\eta u - rx\frac{\partial u}{\partial x}$$

$$\Rightarrow \eta\frac{\partial u}{\partial t} + \eta r x\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} - r\eta u = 0$$

but $\eta = 1$, therefore

$$\Rightarrow \frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2 x^{2\alpha}\frac{\partial^2 u}{\partial x^2} - ru = 0 \qquad (3.8)$$

Hence, we are able to set up a riskless portfolio with X and u and end up with a PDE because X and u are each driven by a single source of uncertainty, namely the Brownian motion W. This makes it possible to remove the stochastic component and end up with a PDE. We can then proceed to solve the PDE in equation (3.8) numerically given a boundary condition $u(T, X_T) = (K - x)^+$.

3.3.2 Black-Karasinski term structure model

Let us consider another example which is the pricing of a zero coupon bond in the term structure model of [10].They describe the short rate r by the SDE

$$d(\log r_u) = \varphi(u)(\log\mu(u) - \log r_u)du + \sigma(u)dW_u, \qquad r_0 > 0 \qquad (3.9)$$

with deterministic functions φ, μ, σ. The price at time t of a zero coupon bond with maturity T is then

$$E\left[\exp\left(-\int_t^T r_s\, ds\right)|\mathcal{F}_t\right] = u(t, r_t)$$

by the Markov property of r and we want to obtain the PDE option price valuation formula for the function u.

Next, we derive the PDE option price valuation formula of the SDE model in equation (3.9), we consider the risk-neutral measure $\mathbb{B}$ and the short rate r is described by the SDE

$$d(\log r_u) = \varphi(u)(\log\mu(u) - \log r_u)du + \sigma(u)dW_u, \qquad r_0 > 0$$

where W is $\mathbb{B}$ −Brownian motion. Applying Ito's lemma, the value $u = (X_t, t)$ of a derivative written on X follows the SDE

$$du = \left(\frac{\partial u}{\partial t} + \varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2}\right)dt + \left(\sigma(t)\frac{\partial u}{\partial x}\right)dW_t \qquad (3.10)$$

Again, the idea is to transform the SDE for u into a PDE, and apply a boundary condition, which allow a solution to the PDE. We set up a riskless replicating portfolio $\prod$ made up of η units of the option and ξ units of the stock, and assume that the portfolio is self-financing so that

$$d\Pi = \eta du + \xi dX$$

The portfolio thus follows the SDE

$$d\Pi = \eta du + \xi dX$$

$$d\Pi = \eta\left[\frac{\partial u}{\partial t} + \left(\varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2}\right)dt + \left(\sigma(t)\frac{\partial u}{\partial x}\right)dW_t\right] + \xi[\varphi(t)(\log\mu(t) - \log r_t)dt + \sigma(t)dW_t]$$

$$\left[\eta\frac{\partial u}{\partial t} + \eta\varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2(t)\frac{\partial^2 u}{\partial x^2} + \xi\varphi(t)(\log\mu(t) - \log r_t)\right]dt + \left[\eta\sigma(t)\frac{\partial u}{\partial x} + \xi\sigma(t)\right]dW_t \qquad (3.11)$$

The choices of $\eta = 1$ and $\xi = -\frac{\partial u}{\partial x}$ removes the stochastic component from $d\Pi$ and renders the portfolio riskless. Since the portfolio is riskless, it earns the risk-free and the return on the portfolio in small time increment dt is $d\Pi = r\Pi dt$. We substitute for ξ in equation (3.11) and equate with $r\Pi dt$, we have

$$\left[\eta\frac{\partial u}{\partial t} + \eta\varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\eta\sigma^2(t)\frac{\partial^2 u}{\partial x^2} + \xi\varphi(t)(\log\mu(t) - \log r_t)\right]dt + \underbrace{\left[\sigma(t)\frac{\partial u}{\partial x} - \sigma(t)\frac{\partial u}{\partial x}\right]}_{=0}dW_t$$

$$= r\prod dt = [r\eta u + r\xi x]dt$$

$$\Rightarrow \frac{\partial u}{\partial t} + \varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2} - \varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} = ru - rx\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2} - \varphi(t)(\log\mu(t) - \log r_t)\frac{\partial u}{\partial x} - ru + rx\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2} - ru = 0 \qquad (3.12)$$

Hence, we are able to set up a riskless portfolio with X and u and end up with a PDE because X and u are each driven by a single source of uncertainty, namely the Brownian motion W. This makes it possible to remove the stochastic component and end up with a PDE. We can then proceed to solve the PDE in (3.12) numerically given a boundary condition, $u(T, X_T) = 1,\ x \in (0, \infty)$.

4 Numerical Solutions

4.1 Derivation of the explicit finite difference scheme for derived PDEs

The finite difference scheme for the two different PDEs derived from the given stochastic models studied in this research paper is as follows:

$$\frac{\partial u}{\partial t}+rx\frac{\partial u}{\partial x}+\frac{1}{2}\sigma^2x^{2\alpha}\frac{\partial^2 u}{\partial x^2}-ru=0 \tag{4.1}$$

Which is the PDE for the CEV model.

The Explicit Finite Difference methods for European put options are used to solve the Black-Scholes Partial Differential Equations with the following initial and boundary conditions for the CEV model [11] given as

Initial condition: $u(x,t)=\max(k-x,0)^{+}$

Boundary conditions:

$$\begin{cases} u(0,t)=ke^{-rt} \\ u(x,t)\to 0, \qquad \text{for } x\to\infty \end{cases}$$

Hence, we have

$$\frac{\partial u}{\partial t}=\frac{u_{n,j+1}-u_{n,j}}{\Delta t}+0(\Delta t) \tag{4.2}$$

$$\frac{\partial u}{\partial x}=\frac{u_{n+1,j}-u_{n-1,j}}{2\Delta x}+0((\Delta x)^2) \tag{4.3}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{n+1,j}-2u_{n,j}+u_{n-1,j}}{(\Delta x)^2}+0(\Delta x) \tag{4.4}$$

Substituting (4.2), (4.3) and (4.4) into equation (4.1), we have

$$\frac{u_{n,j+1}-u_{n,j}}{\Delta t}+rx\left(\frac{u_{n+1,j}-u_{n-1,j}}{2\Delta x}\right)+\frac{1}{2}\sigma^2x^{2\alpha}\left(\frac{u_{n+1,j}-2u_{n,j}+u_{n-1,j}}{(\Delta x)^2}\right)-ru_{n,j}=0$$

$$\text{let } \Delta t=p \text{ and } \Delta x=h, \text{we have}$$

$$\frac{u_{n,j+1}-u_{n,j}}{p}+rx\left(\frac{u_{n+1,j}-u_{n-1,j}}{2h}\right)+\frac{1}{2}\sigma^2x^{2\alpha}\left(\frac{u_{n+1,j}-2u_{n,j}+u_{n-1,j}}{h^2}\right)-ru_{n,j}=0$$

Therefore,

$$\begin{aligned} u_{n,j+1} &= -rpx\left(\frac{u_{n+1,j}-u_{n-1,j}}{2h}\right)-\frac{1}{2}p\sigma^2x^{2\alpha}\left(\frac{u_{n+1,j}-2u_{n,j}+u_{n-1,j}}{h^2}\right) \\ &\quad + u_{n,j}(rp+1) \\ &= \left[rpx\left(\frac{u_{n-1,j}}{2h}\right)-\frac{1}{2}p\sigma^2x^{2\alpha}\left(\frac{u_{n-1,j}}{h^2}\right)\right]+\left[p\sigma^2x^{2\alpha}\left(\frac{2u_{n,j}}{h^2}+u_{n,j}(rp+1)\right)\right] \\ &\quad +\left[-rpx\left(\frac{u_{n+1,j}}{2h}\right)-\frac{1}{2}p\sigma^2x^{2\alpha}\left(\frac{u_{n+1,j}}{h^2}\right)\right] \\ &= \left[\frac{2h(hrpx-p\sigma^2x^{2\alpha})}{4h^3}\right]u_{n-1,j}+\left[\frac{p\sigma^2x^{2\alpha}+h^2rp+h^2}{h^2}\right]u_{n,j}-\left[\frac{2h(hrpx+p\sigma^2x^{2\alpha})}{4h^3}\right]u_{n+1,j} \end{aligned}$$

$$= \frac{1}{2h^2}(hrpx - p\sigma^2x^{2\alpha})u_{n-1,j} + \frac{1}{h^2}(p\sigma^2x^{2\alpha} + h^2rp + h^2)u_{n,j} - \frac{1}{2h^2}(hrpx + p\sigma^2x^{2\alpha})u_{n+1,j}$$

Substituting $p = \Delta t$ and $h = \Delta x$, we have

$$u_{n,j+1} = \frac{1}{2(\Delta x)^2}((\Delta x)r(\Delta t)x - (\Delta t)\sigma^2x^{2\alpha})u_{n-1,j} + \frac{1}{(\Delta x)^2}((\Delta t)\sigma^2x^{2\alpha} + (\Delta x)^2r(\Delta t) + (\Delta x)^2)u_{n,j} - \frac{1}{2(\Delta x)^2}\big((\Delta x)r(\Delta t)x + (\Delta t)\sigma^2x^{2\alpha}\big)u_{n+1,j} \quad (4.5)$$

$$u_{n,j+1} = au_{n-1,j} + bu_{n,j} + cu_{n+1,j} \quad (4.6)$$

where,

$$a = \frac{1}{2(\Delta x)^2}[((\Delta x)r(\Delta t)x - (\Delta t)\sigma^2x^{2\alpha})]$$

$$b = \frac{1}{(\Delta x)^2}[((\Delta t)\sigma^2x^{2\alpha} + (\Delta x)^2r(\Delta t) + (\Delta x)^2)]$$

$$c = -\frac{1}{2(\Delta x)^2}[(\Delta x)r(\Delta t)x + (\Delta t)\sigma^2x^{2\alpha}]$$

Next, we consider the PDE for the Black-Karasinski term structure model which is given below:

$$\frac{\partial u}{\partial t} + rx\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 u}{\partial x^2} - ru = 0 \quad (4.7)$$

The initial and boundary condition for a zero coupon bond for a short rate model such as the Black-Karasinski model is given as:

Initial condition: $u(0,t) = \max(K - x, 0)$

Boundary condition $u(x,t) = 1$, for $x \to \infty$

Thus we have that,

$$\frac{\partial u}{\partial t} = \frac{u_{n,j+1} - u_{n,j}}{\Delta t} + 0(\Delta t) \quad (4.8)$$

$$\frac{\partial u}{\partial x} = \frac{u_{n+1,j} - u_{n-1,j}}{2\Delta x} + 0((\Delta x)^2) \quad (4.9)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{(\Delta x)^2} + 0(\Delta x) \quad (4.10)$$

Substituting (4.8), (4.9) and (4.10) into equation (4.7), we have

$$\frac{u_{n,j+1} - u_{n,j}}{\Delta t} + rx\left(\frac{u_{n+1,j} - u_{n-1,j}}{2\Delta x}\right) + \frac{1}{2}\sigma^2(t)\left(\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{(\Delta x)^2}\right) - ru_{n,j} = 0$$

Let $\Delta t = p$ and $\Delta x = h$

$$\frac{u_{n,j+1} - u_{n,j}}{p} + rx\left(\frac{u_{n+1,j} - u_{n-1,j}}{2h}\right) + \frac{1}{2}\sigma^2(t)\left(\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{h^2}\right) - ru_{n,j} = 0$$

$$u_{n,j+1} = -rpx\left(\frac{u_{n+1,j} - u_{n-1,j}}{2h}\right) - \frac{1}{2}p\sigma^2(t)\left(\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{h^2}\right) + pru_{n,j} + u_{n,j}$$

Therefore,

$$u_{n,j+1} = -rpx\left(\frac{u_{n+1,j} - u_{n-1,j}}{2h}\right) - \frac{1}{2}p\sigma^2(t)\left(\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{h^2}\right) + u_{n,j}(rp+1)$$

$$u_{n,j+1} = -rpx\left(\frac{u_{n+1,j} - u_{n-1,j}}{2h}\right) - \frac{1}{2}p\sigma^2(t)\left(\frac{u_{n+1,j} - 2u_{n,j} + u_{n-1,j}}{h^2}\right) - u_{n,j}(rp+1)$$

$$= rpx\left(\frac{u_{n-1,j}}{2h}\right) - rpx\left(\frac{u_{n+1,j}}{2h}\right) - \frac{1}{2}p\sigma^2\left(\frac{u_{n+1,j}}{h^2}\right) + p\sigma^2\left(\frac{u_{n,j}}{h^2}\right) - \frac{1}{2}p\sigma^2\left(\frac{u_{n-1,j}}{h^2}\right) + u_{n,j}(rp+1)$$

$$= \left[rpx\left(\frac{u_{n-1,j}}{2h}\right) - \frac{1}{2}p\sigma^2\left(\frac{u_{n-1,j}}{h^2}\right)\right] + \left[p\sigma^2\left(\frac{u_{n,j}}{h^2}\right) + u_{n,j}(rp+1)\right] + \left[rpx\left(\frac{u_{n+1,j}}{2h}\right) - \frac{1}{2}p\sigma^2\left(\frac{u_{n+1,j}}{h^2}\right)\right]$$

$$= \left(\frac{rpx}{2h} - \frac{p\sigma^2}{2h^2}\right)u_{n-1,j} + \left(\frac{p\sigma^2}{h^2} + rp + 1\right)u_{n,j} - \left(\frac{rpx}{2h} + \frac{p\sigma^2}{2h^2}\right)u_{n+1,j}$$

$$= \frac{1}{2h^2}(hrpx - p\sigma^2)u_{n-1,j} + \frac{1}{h^2}(p\sigma^2 + h^2rp + h^2)u_{n,j} - \frac{1}{2h^2}(hrpx + p\sigma^2)u_{n+1,j}$$

Substituting back $p = \Delta t$ and $h = \Delta x$, we have

$$u_{n,j+1} = \frac{1}{2(\Delta x)^2}((\Delta x)r(\Delta t)x - (\Delta t)\sigma^2)u_{n-1,j} + \frac{1}{(\Delta x)^2}((\Delta t)\sigma^2 + (\Delta x)^2 r(\Delta t) + (\Delta x)^2)u_{n,j} - \frac{1}{2(\Delta x)^2}((\Delta x)r(\Delta t)x + (\Delta t)\sigma^2)u_{n+1,j} \quad (4.11)$$

$$u_{n,j+1} = au_{n-1,j} + bu_{n,j} + cu_{n+1,j} \quad (4.12)$$

where,

$$a = \frac{1}{2(\Delta x)^2}((\Delta x)r(\Delta t)x - (\Delta t)\sigma^2)$$

$$b = \frac{1}{(\Delta s)^2}((\Delta t)\sigma^2 + (\Delta x)^2 r(\Delta t) + (\Delta x)^2)$$

$$c = -\frac{1}{2(\Delta x)^2}((\Delta x)r(\Delta t)x + (\Delta t)\sigma^2)$$

4.2 Numerical Examples

Empirical data obtained from the Nigeria Stock Exchange (NSE) will be used to plot some graphs to investigate the effect of increase in the underlying asset (i.e. positive correlation) and the increase in the level of interest rate on the option value (price) for the different financial models examined in this paper. In order to carry out the above task, the parameters $K, x, \sigma, T, t, \alpha, \varphi, \lambda, r, \Delta x, \Delta t, xmax$ as defined in section (3.1) are assigned some data values as shown in Table, 1. Computer programs coded in MATLAB were used for solving the systems of the derived finite difference scheme in equations (4.6) and (4.12) respectively for the two models. The graphs for the various parameter values for the CEV and Black-Karasinski models in Table 1 are presented in figures 1 -4.

Cases / Parameters	1	2	3
xmax	150	150	150
K	50	50	50
Δx	1	1	1
x	4	6	8
φ	0.2	0.2	0.2
T	1	1	1
α	0.2	0.2	0.2
Δt	0.1	0.1	0.1
r	0.2	0.4	0.6
σ	0.5	0.5	0.5

Table 1: Parameter Values for Numerical Experiment for the CEV and Black-Karasinski model
Source: Data from the Nigerian Stock Exchange all Share Index (2015) except for **xmax**, **Δx** and **K**.

We shall discuss the result of our numerical experiments carried out by increasing the value of the underlying assets and risk free interest rate for the Constant Elasticity of Variance (CEV) and the Black-Karasinski models respectively at various parametric values. The parameter values for the experiments are shown in Table 1 above. The graphs, plotted using these values, are shown in Figures 1 – 4 below.

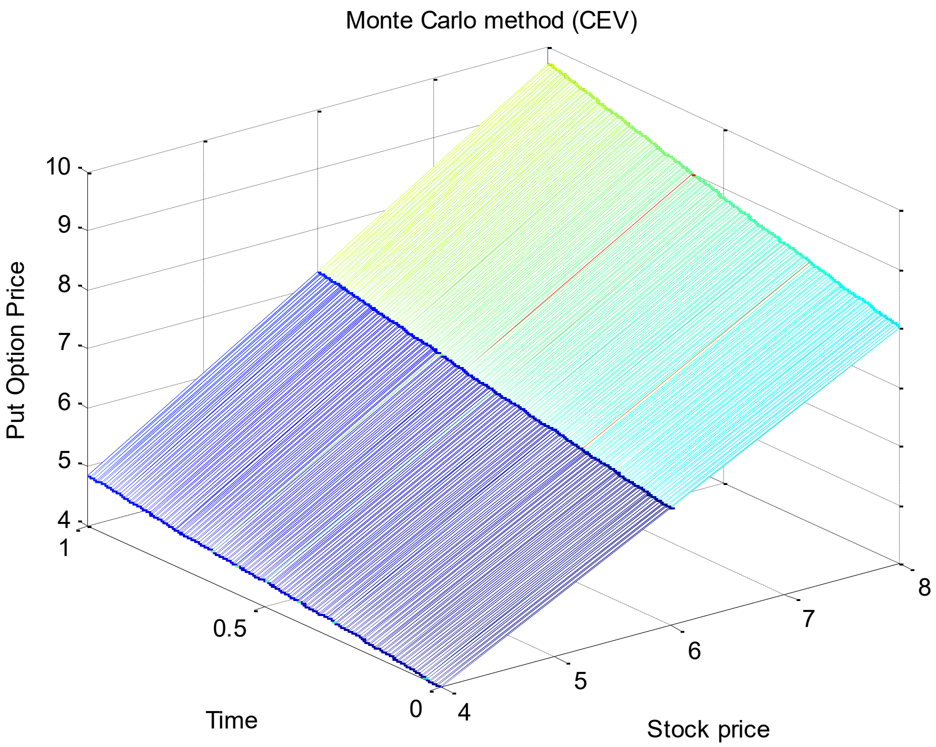

Fig. 1: Graph of Option Value, Expiration Time and Stock Price using the following Parameter values: $x = 4:8$; $xmax = 150$; $K = 50$; $\Delta x = 1$; $T = 1$; $\alpha = 0.2$; $\Delta t = 0.1$; $r = 0.2:0.6$ and $\sigma = 0.5$ for the CVE model.

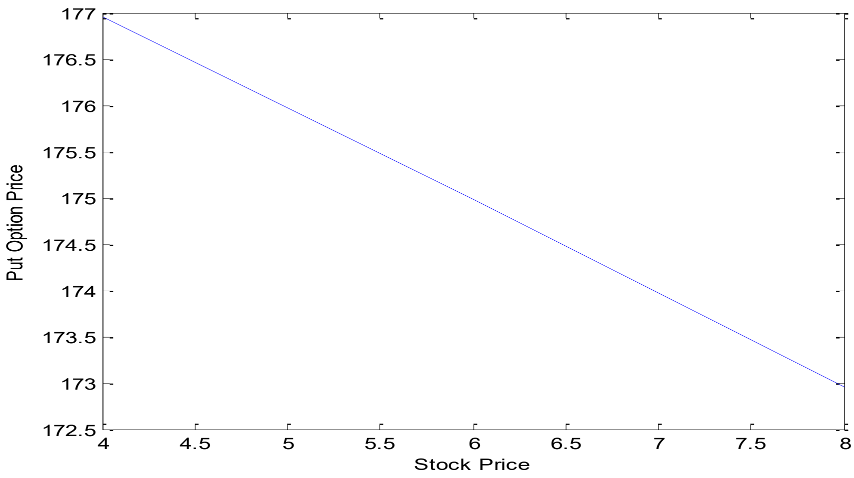

Fig. 2: Graph of Option Value vs. Expiration Time and Stock Price using the following Parameter values: $x = 4:8$; $xmax = 150$; $K = 50$; $\Delta x = 1$; $T = 1$; $\alpha = 0.2$; $\Delta t = 0.1$; $r = 0.2:0.6$ and $\sigma = 0.5$ for the CEV model.

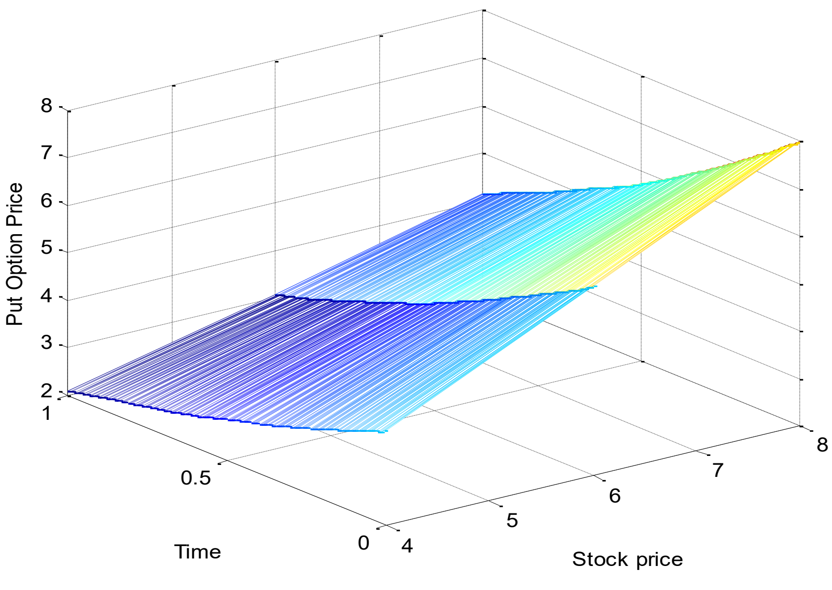

Fig. 3: Graph of Option Value, Expiration Time and Stock Price using the following Parameter values: $x = 4:8$; $xmax = 150$; $K = 50$; $\Delta x = 1$; $T = 1$; $\varphi = 0.2$; $t = 10$; $\Delta t = 0.1$; $r = 0.2:0.6$ and $\sigma = 0.5$ for the Black-Karasinski model.

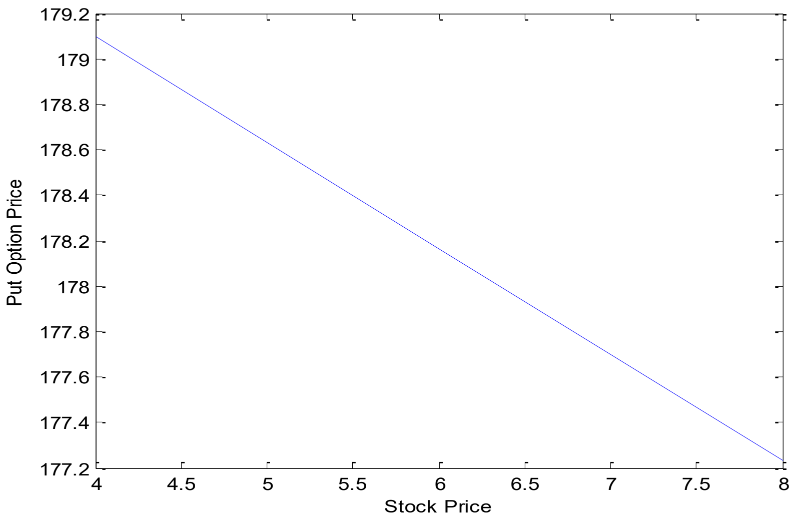

Fig. 4: Graph of Option Value, Expiration Time and Stock Price using the following Parameter values: $x = 4{:}8$; $xmax = 150$; $K = 50$; $\Delta x = 1$; $T = 1$; $\varphi = 0.2$; $t = 10$; $\Delta t = 0.1$; $r = 0.2{:}0.6$ and $\sigma = 0.5$ for the Black-Karasinski model.

5 Results and Discussions

Experiment one

Here, we investigate the situation where the underlying asset (stock price) value is $x = 4$ and the interest rate, $r = 0.2$ for the CEV model and the Black-Karasinski term structure model respectively. The result displayed in Fig. 2 and Fig. 4 showed that the value of the Put Option (Option Price) is 176.9 for the CEV model and 179.1for the Black Karasinski term structure model.

Experiment two

In this experiment, we investigate the situation where the underlying asset (stock price) value is increased to $x = 6$ and the interest rate also increased to $r = 0.4$ for the CEV model and the Black-Karasinski model respectively. The result displayed in fig. 2 and fig. 4 showed that the value of the Put Option (Option price) is 175.0 for the CEV model and 178.2 for the Black-Karasinski term structure model.

Experiment three

Here, we investigate the situation where the value of the underlying asset (stock price) is increased to the value of $x = 8$ and the interest rate is increased to $r = 0.6$ for the CEV model and the Black-Karasinski model respectively. The result displayed in fig. 2 and fig. 4 showed that the value of the Put Option (Option price) is 173.0 for the CEV model and 177.25 for the Black Karasinski term structure model.

6 Conclusion

In this research work, we have reviewed and studied two Stochastic Differential Equation models in finance which are the Constant Elasticity of Variance (CEV) model and the Black-Karasinski term structure model. The explicit finite difference method for PDEs was used in the numerical experiments. From this study, we observed that the PDE approach for the valuation of option price using the riskless portfolio method eliminates the stochastic component from the SDE, hence making it easier for obtaining the solution of the resulting PDE. Also, numerical experiments, using published data show that as the price of the underlying asset (stock price) and risk free interest rate increases, the value of the option decreases. Hence, the right to sell at a fixed price (Puts) will become less valuable and the buyer decides not to exercise the right on the options and may allow the option to expire.

Acknowledgements

My sincere gratitude goes to my Ph.D. supervisor Professor E. S. Onah, Executive Director/CEO of National Mathematical Centre, Kwali, Abuja and Professor T. Aboiyar of the Department of Mathematics/Statistics/Computer Science, University of Agriculture, Makurdi, Benue and also to Nigeria Stock Exchange (NSE) for providing the data used in this research work.

Competing financial interests

The author(s) declares no competing financial interests.

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