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On the verification of existence of backward bifurcation for a mathematical model of cholera dynamics

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Abstract

A cholera transmission model, which incorporates preventive measures, is studied qualitatively. The stability results together with the center manifold theory are used to investigate the existence of backward bifurcation for the model. The epidemiological consequence of backward bifurcation is that the disease may still persist in the population even when the classical requirement of the reproductive number R_0 being less than one is satisfied.

Keywords: Model, Stability, Center manifold theory, Bifurcation
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1 Introduction

Cholera is a waterborne disease that causes severe diarrhea and vomiting which lead to dehydration of the body and can prove fatal unless treated quickly. Outbreaks result from contaminated food, poor sanitation and drinking dirty water [1]. Cholera is generally a disease of the poor affecting regions that lack a heightened sense of hygiene and access to safe drinking water [2]. Worldwide, there areestimated 3 – 5 million cholera cases and 100,000 – 130,000 deaths due to cholera every year [3]. Similar infection that is often transmitted through contaminated food or water can be found in [4].

The main concern of epidemiologists is to establish a condition under which a disease can be eradicated from a population. This condition is governed by the size of the basic reproduction number which is defined by [5] as the average number of secondary infections caused by an infectious individual during his or her entire period of infectiousness. However, contrary to the analysis of the basic reproduction number, there may not be a total elimination of a disease in a population even if the reproduction number R_0 is less than unity. The bedrock of this idea is bifurcation theory whose earliest work is attributable to the French mathematician Henri Poincare (1854 – 1912).

Bifurcation is the study of changes in the qualitative or topological structure of dynamical system. It occurs when a small smooth change toa parameter (bifurcation parameter) of a system of differential equation causes a sudden qualitative or topological change in the behavior of the system. In epidemiology, bifurcation is a phenomenon which shows how theequilibrium of an epidemic model divides into a branch at the bifurcation point (i.e. $R_0 = 1$) thereby, resulting into changes in the stability and qualitative behavior of the model. Bifurcation provides a way to understand changes in the stability properties of epidemic models as some parameters of the models vary.

Researchers have identified forward and backward bifurcations in the analysis of disease transmission models. A forward bifurcation occurs in a disease model if a stable disease-free equilibrium of the model losses its stability and becomes a stable endemic equilibrium as the reproduction numberR_0 of the model takes off through one. The existence of forward bifurcation does not have a major health implication as the basic requirement $R_0 < 1$ remains the necessary and sufficient condition for disease eradication. On the other hand, numerous studies [6], [7], [8] have shown that the classical requirement of the basic reproduction number R_0 being less than unity is just a necessary condition for community-wide eradication of a disease but not sufficient.

These studies have verified this fact by exploring the phenomenon of bistability, where multiple stable equilibria co-exist, in some epidemic models. These models, in general, undergo backward bifurcations which are sufficient for the existence of stable endemic equilibria when $R_0 < 1$. In other words, these studies have shown that a stable endemic equilibrium can co-exist with a stable disease-free equilibrium at the bifurcation point. Thus, unlike in many classical disease transmission models [2], [9], [10], reducing R_0 to values less than unity does not guarantee the community-wide eradication of a disease. This means that the occurrence of a backward bifurcation may have serious public health implications in the control or eradication of an epidemic since the condition $R_0 < 1$ is not sufficient for disease eradication.

Cholera, which is capable of causing periodic epidemic disease, has varying degree of sensitivity to a number of factors and responds differently to various levels of potency of interventions [11], [12]. The aim of this work is to verify the existence of backward bifurcation for a cholera model that incorporates vaccination and education as preventive measures using thecentre manifold theory.

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2 Model Formulation

The model is made up of human and pathogen populations. The human population is divided into three compartments which are S(t), I(t) and R(t) respectively while the pathogen population is denoted by B(t). Each of the compartments is a function of time meaning that the population of each compartment can fluctuate with time and all the parameters are non-negative. The model is thus given as

$$\frac{dS}{dt} = \pi - \mu S - (1-\theta)\lambda S - \nu S \tag{2.1}$$

$$\frac{dI}{dt} = (1-\theta)\lambda S - (\mu + \mu_c)I \tag{2.2}$$

$$\frac{dB}{dt} = (1-\theta)\varepsilon I - \delta B \tag{2.3}$$

$$\frac{dR}{dt} = vS - \mu R \tag{2.4}$$

Where

$$\lambda = \left[\frac{\beta_1 B}{B + \aleph} + \beta_2 I \right] \tag{2.5}$$

Eqn. (2.1) stands for the compartment S (t) which is the population of individuals who have not been infected with the disease at time t but are capable of being infected. The population of these people increases by π which is the recruitment rate into susceptibility. The population however decreases by the preventive measures θ and v,($0 < \theta < 1$) together with the natural death rate μ . θ is the education parameter that reduces the force of infection λ while v is the vaccination rate that confers permanent immunity to those individuals who have been successfully vaccinated. The force of infection λ is a function of rate of exposure to contaminated water β_1 andcontact rate between the susceptible and infectious individuals β_2 while $\aleph$ is the concentration of pathogen in the water reservoir that will make 50% of the susceptible population ill. $\frac{B}{B+\aleph}$ is the probability of getting cholera on consuming contaminated water.

Eqn. (2.2) stands for the compartment I (t) which is the population of individuals who have been infected with the disease at time t and are capable of spreading the disease to those in susceptible category. The population of infectious individuals increases by the force of infection λ but decreases by death rate due to cholera $\mu_c$and natural death rate μ respectively. Eqn. (2.3) stands for the compartment B(t) which is the population of the bacteria that are responsible for the transmission of cholera. The bacteria population increases by the parameter ε which is the rate at which infectious individuals contribute to the growth of the bacteria. ε is however reduces by the education parameter θ. The bacteria population also reduces by their natural death rate δ.

Eqn. (2.4) stands for the compartment R (t) which is the population of individuals who have been infected with the disease at one time or the other but are now permanently cure of the disease. The population increases by the permanent immunity acquired through vaccination while it decreases by the natural death rate μ. Eqn. (2.4) is not needed in the model analysis and shall be dropped. The reason is based on the assumption of permanent immunity which makes individuals who have been recovered from the disease to remain in compartment R (t) throughout the analysis unlessthere is natural death. Hence, the dynamics of compartment R (t) is given by the dynamics of compartments S (t) and I (t).

3 Disease Eradication

We begin the analysis by considering a scenario where there is no infection agents, that is, $\lambda = I = B = 0$ hence, the resulting equations (2.1) – (2.3) reduce to $\pi - (\mu + v)S = 0$ and the disease is eradicated at the point $E_0 = (S_0, I_0, B_0) = \left(\frac{\pi}{\mu + v}, 0, 0\right)$ (3.1)

3.1 Effective Reproductive Number

The average number of new infections generated, when an infectious individual gets into the population, is obtained following the next generation matrix approach as

$$R_e = \frac{\pi(1-\theta)[(1-\theta)\beta_1\varepsilon + \aleph\beta_2\delta]}{\aleph\delta(\mu+v)(\mu+\mu_c)} \quad (3.2)$$

4.0 Central Manifold Theorem

In what follows, the existence of backward bifurcation shall be investigated for the model system (2.1) – (2.3). To verify the possibility of backward bifurcation involving the disease-free equilibrium E_0 of the model, certain conditions on the parameters that guarantee the existence of backward bifurcation in the model shall be investigated. The parameter that coincides with the bifurcation point shall be considered and used to investigate the possible occurrence of backward bifurcation. To achieve this, the centre manifold theory due to [13], which has been applied in various epidemic models [14], [15], [16] shall be adopted.

The centre manifold theory outlines the role of the coefficients a and b of the normal form denoting the system on the centre manifold in determining the type of bifurcation existing at $\phi = 0$. The coefficients a and b show what happen to other parameters when there is a change in the bifurcation parameter. Generally, if $a < 0$ and $b > 0$, then such a bifurcation is forward but if $a > 0$ and $b > 0$, then the bifurcation at $\phi = 0$ is backward. To represent it, consider the following general system of ordinary differential equations with a parameter ϕ such that

$$\frac{dx}{dt} = (x, \phi),\ f : \Re^n \times \Re \to \Re^n \text{ and } f \in C^2\left(\Re^n \times \Re^n\right) \quad (4.1)$$

where $x = 0$ is an equilibrium point of the system (8) for all values of ϕ, i.e., $f(0, \phi) \equiv 0,$ for all $\phi = 0$.

4.1 Theorem 1 (Castillo-Chavez & Song, 2004)

Suppose

(a) $A = D_x f(0, 0) = \left[\frac{\partial f_i}{\partial x_j}(0, 0)\right]$ is the linearization matrix of the system (4.1) around the equilibrium 0 with ϕ at 0.

(b) Zero is a simple eigenvalue of A and all other eigenvalues of A have negative real parts;

(c) Matrix A has a right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue. If f_k is the kth component of f and

$$a = \sum_{k,i,j-1}^{n} v_k w_i w_k \frac{\partial^2 f_k}{\partial x_j \partial \phi}(0,0), \tag{4.2}$$

$$b = \sum_{k,i,j-1}^{n} v_k w_i w_k \frac{\partial^2 f_k}{\partial x_j \partial \phi}(0,0), \tag{4.3}$$

Then the local dynamics of the system (4.1) around the equilibrium point is totally obtained by the signs of a and b .

(1) $a > 0, b > 0$. When $\phi < 0$ with $|\emptyset| \ll 1, x = 0$ is locally asymptotically stable, and there exists a positive unstable equilibrium; when $|\emptyset| \ll 1, x = 0$ is unstable and there exists a negative and locally asymptotically stable equilibrium;

(2) $a > 0$, $b < 0$. When $\phi < 0$ with $|\emptyset| \ll 1, x = 0$ is unstable; when $|\emptyset| \ll 1, x = 0$ is locally asymptotically stable and there exists a positive unstable equilibrium;

(3) $a > 0$, $b < 0$ .When $\phi < 0$ with $|\emptyset| \ll 1, x = 0$ is unstable and there exists a locally asymptotically stable negative equilibrium; when $|\emptyset| \ll 1, x = 0$ is stable and a positive unstable equilibrium appears;

(4) $a < 0$, $b > 0$ .When ϕ changes from negative to positive, $x = 0$ changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium becomes positive and locally asymptotically stable. Particularly if $a < 0, b > 0$, then a backward bifurcation occurs at $\phi = 0$

4.2 Application ofthe Centre Manifold Theory

The Jacobian matrix of the model equations (2.1) – (2.3) evaluated at the disease-free equilibrium E_0 shall be considered to investigate the existence of backward bifurcation for the model. Tocome about this, a change of variable is made such that $S = x_1, I = x_2$ and $B = x_3$ and the model system (2.1) – (2.3) is transformed to vector form as

$$\frac{dX}{dt} = Q = (q_1, q_2, q_3) \tag{4.4}$$

where

$$x_1'(t) = q_1 = \pi - (\mu + v)x_1 - (1 - \theta)\lambda x_1 \tag{4.5}$$

$$x_2'(t) = q_2 = (1 - \theta)\lambda x_1 - (\mu + \mu_c)x_2 \tag{4.6}$$

$$x_3'(t) = q_3 = (1 - \theta)\varepsilon x_2 - \delta x_3 \tag{4.7}$$

Substituting the value of λ in eqn. (2.5) into eqns. (4.5) and (4.6), the Jacobian matrix of the resulting equations together with eqn. (4.7) at the disease-free equilibrium E_0 i.e. eqn. (2.6) is

$$J(E_0) = \begin{pmatrix} -a_3 & -b_1\aleph\beta_2 & -b_1\beta_1 \\ 0 & b_1\aleph\beta_2 - a_2 & b_1\beta_1 \\ 0 & a_1\varepsilon & -\delta \end{pmatrix} \quad (4.8)$$

where $a_1 = (1-\theta)$, $a_2 = (\mu + \mu_c)$, $a_3 = (\mu + v)$ and $b_1 = \frac{\pi a_1}{\aleph a_3}$

The characteristic equation of the above matrix is given by

$$-(a_3 + \lambda)[\lambda^2 + A_4\lambda + A_5] = 0 \quad (4.9)$$

where $A_4 = (a_2 + \delta - b_1\aleph\beta_2)$

$A_5 = -(\delta b_1\aleph\beta_2 + a_1 b_1\varepsilon\beta_1 - a_2\delta)$

The eigenvalues of the characteristic equation (4.9) are

$$\lambda_1 = -a_3 \text{ and } \lambda_{2,3} = \frac{-A_4 \pm \sqrt{A_4^2 + 4A_5}}{2} \quad (4.10)$$

From eqn. (4.10)

(1) the *DFE* , E_0 is locally asymptotically stable if $\lambda_2 < 0$ and $\lambda_3 < 0$;

(2) the *DFE* , E_0 is unstable if one of λ_2 or λ_3 is positive;

Going by R_e in eqn. (3.1), it follows that

(i) if $R_e < 1$ then $\lambda_2 < 0$ and $\lambda_3 < 0$ and E_0 is locally asymptotically stable;

(ii) if $R_e > 1$ then either $\lambda_2 > 0$ or $\lambda_3 > 0$ so that the DFE, E_0 is unstable.

If the effective reproduction number, R_e in eqn. (3.1) is introduced and $\beta_1 = \beta_1^*$ is chosen as the critical bifurcation parameter at the bifurcation point $R_e = 1$ then, from eqn. (3.1)

$$R_e = \frac{\pi a_1(\beta_1\varepsilon a_1 + \aleph\beta_2\delta)}{\aleph a_2 a_3\delta}$$

so, if $\beta_1 = \beta_1^*$ is expressed in terms of R_e at the bifurcation point i.e. $R_e = 1$, then

$$\beta_1 = \beta_1^* = \frac{\aleph a_2 a_3\delta - \pi\aleph\beta_2\delta a_1}{\pi\varepsilon a_1^2} \quad (4.11)$$

It follows that E_0 is locally asymptotically stable if $\beta_1 < \beta_1^*$, but loses its stability if $\beta_1 > \beta_1^*$. The linearised system of the model evaluated around the disease-free equilibrium E_0 with the bifurcation parameter β_1^* is given as

$$J(E_0, \beta_1^*) = \begin{pmatrix} -a_3 & -b_1\aleph\beta_2 & -b_1\beta_1^* \\ 0 & b_1\aleph\beta_2 - a_2 & b_1\beta_1^* \\ 0 & a_1\varepsilon & -\delta \end{pmatrix} \quad (4.12)$$

Eqn. (4.12) has at least one simple zero eigenvalue. Consequently, the right and left eigenvectors associated with the zero eigenvalue can be calculated. Suppose the right eigenvector associated with the zero eigenvalue is denoted by

$$w = [w_1, w_2, w_3]^T$$

then,

$$J_{right}(E_0, \beta_1^*) = \begin{pmatrix} -a_3 & -b_1\aleph\beta_2 & -b_1\beta_1^* \\ 0 & b_1\aleph\beta_2 - a_2 & b_1\beta_1^* \\ 0 & a_1\varepsilon & -\delta \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad (4.13)$$

and the following equations are obtained

$$\begin{aligned} -a_3 w_1 - b_1\aleph\beta_2 w_2 - b_1\beta_1^* w_3 &= 0 \\ (b_1\aleph\beta_2 - a_2)w_2 + b_1\beta_1^* w_3 &= 0 \\ a_1\varepsilon w_2 - \delta w_3 &= 0 \end{aligned}$$

from which, $w_1 = \dfrac{-b_1\left(\delta\aleph\beta_2 + a_1\varepsilon\beta_1^*\right)}{a_1 a_3\varepsilon} w_3$, $w_2 = \dfrac{\delta}{a_1\varepsilon} w_3$ and $w_3 = w_3 > 0$. (4.14)

Also, the left eigenvector denoted by $v = [v_1, v, v_3]^T$ which satisfies the condition $v.w = 1$, is evaluated as

$$J_{left}(E_0, \beta_1^*) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \begin{pmatrix} -a_3 & -b_1\aleph\beta_2 & -b_1\beta_1^* \\ 0 & b_1\aleph\beta_2 - a_2 & b_1\beta_1^* \\ 0 & a_1\varepsilon & -\delta \end{pmatrix} \quad (4.15)$$

which resulted into the following system of equations

$$\begin{aligned} -a_3 v_1 &= 0 \\ -b_1\aleph\beta_2 v_1 + (b_1\aleph\beta_2 - a_2)v_2 + a_1\varepsilon v_3 &= 0 \\ -b_1\beta_1^* v_1 + b_1\beta_1^* v_2 - \delta v_3 &= 0 \end{aligned}$$

and the left eigenvector is obtained as $v_1 = 0$, $v_2 = \dfrac{\delta}{b_1\beta_1^*} v_3$, and $v_3 = v_3 > 0$ (4.16)

4.3 Estimation of the Coefficient of a and b

The values of a and b can be computed by using the theorem outlined above i.e. eqns (4.2) and (4.3). The model system (4.5) – (4.7) shall be considered by substituting the value of λ in eqn. (2.5) for only non-zero components of the left eigenvector v i.e. $k = 2, 3$ for v_2 and v_3 only. The values of a and b are determined by the associated non-zero second order partial derivative at the disease-free equilibrium which are obtained as

$$\begin{aligned} a = {} & v_2 w_1^2 \frac{\partial^2 f_2}{\partial x_1^2}(E_0, \beta_1^*) + 2v_2 w_1 w_2 \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(E_0, \beta_1^*) \\ & + v_2 w_2^2 \frac{\partial^2 f_2}{\partial x_2^2}(E_0, \beta_1^*) + 2v_2 w_1 w_3 \frac{\partial^2 f_2}{\partial x_1 \partial x_3}(E_0, \beta_1^*) \\ & + 2 v_2 w_2 w_3 \frac{\partial^2 f_2}{\partial x_2 \partial x_3}(E_0, \beta_1^*) + v_2 w_3^2 \frac{\partial^2 f_2}{\partial x_3^2}(E_0, \beta_1^*) \end{aligned}$$

and,

$$b = v_2 w_1 \frac{\partial^2 f_2}{\partial x_1 \partial \beta_1}(E_0, \beta_1^*) + v_2 w_2 \frac{\partial^2 f_2}{\partial x_2 \partial \beta_1}(E_0, \beta_1^*) + v_2 w_3 \frac{\partial^2 f_2}{\partial x_3 \partial \beta_1}(E_0, \beta_1^*)$$

Hence, on simplification,

$$a = -2\left(\frac{\beta_{1\delta^2(\aleph\beta_{2\delta}+\, a_1\varepsilon\beta_1^*)}}{a_1a_3\beta_1^*\aleph\varepsilon^2}\; v_3w_3^2 + \frac{\beta_{2\delta^2(\aleph\beta_{2\delta}+\, a_1\varepsilon\beta_1^*)}}{a_1a_3\beta_1^*\varepsilon^2}\; v_3w_3^2\right) \tag{4.17}$$

and,

$$b = a_1\left(\frac{\pi\delta}{b_1\aleph(\mu+v)\beta_1^*}\; v_3\, w_3\right)$$

5 Conclusion

According to theorem 1, the model system (2.1) – (2.3) does not undergo backward bifurcation since $a < 0$ and $b > 0$. Therefore, the effective reproduction number R_e for the system is slightly above unity and there exists a unique endemic equilibrium for the model which is locally asymptotically stable at the bifurcation point i.e. R_e = 1.The result shows that there is a mild cholera outbreak in the population when there is a change in the parameter that represents contact rate between susceptible individuals and contaminated water (i.e. β_1) which is chosen as the bifurcation parameter. However, the outbreak can be easily brought under control as soon as the effective reproduction number reduced below unity.

The implication of the result is that even though no intervention, be it vaccination or education, offers 100% efficacy yet, backward bifurcation is sure to be prevented for cholera disease if these interventions are on ground at all time in a community and their effectiveness is high enough.

Conflict of Interest

The authors declare that there are no conflicts of interest regarding the publication of the paper.

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