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Direct Integrator for Fifth order Boundary Value Problems in Ordinary Differential Equations

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Abstract

Fifth-order Boundary Value Problems (BVPs) in Ordinary Differential Equations are solved using Block Unification Methods (BUM). Continuous Linear Multistep Methods (CLMMs) are developed via the collocation and interpolation approach. These methods constitutes the main and additional methods which are unified in block fashion for numerically solving general fifth order linear and nonlinear BVPs in ODEs. The method is analyzed for its order, local trucation error, consistency and its convergence. Sufficient numerical evidences were given to show the effectiveness of the new scheme in terms of accuracy and flexibility. Some numerical examples are used to show the robustness in handling such problems. The method performed favourably well viz-a-viz other existing methods in literature.

Keywords: Boundary Value Problems, Block Methods, Convergence, Linear multistep.
MSC2010: 65L05, 65L06, 65L10, 65L12.

1 Introduction

In this work, the numerical solution for the fifth-order Boundary Value Problems (BVPs) of the type:

$$y^{(v)} = f(x, y, y', \dots, y^{(iv)}), \quad a < x < b \tag{1.1}$$

subject to the boundary conditions:

$$y(a) = \alpha_0, \quad y'(a) = \alpha_1, \quad y''(a) = \alpha_2, \quad y(b) = \beta_0, \quad y'(b) = \beta_1 \tag{1.2}$$

are considered, where $\alpha_i, \quad \beta_j, \quad i = 0, 1, 2; \quad j = 0, 1$ are finite real arbitrary constants, $x \in (a, b)$, $f \in C[a, b]$, is a continuous function defined on the interval (a, b).

Higher order boundary value problems occur, for example, in the modeling of induction motors with two rotor circuits. The behavior of this kind of circuits is governed by a fifth-order differential equation. (see [1]). Other applications of this type of boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences.

Several numerical methods including spectral Galerkin and collocation, decomposition, nth-order B-spline have been developed for solving fifth-order boundary value problems. For instance, Wazwaz [2], developed decomposition method, for numerical solution of fifth-order boundary value problems. Noor and Mohyud in [3, 4, 5, 6], used variation of Parameters Method for Solving Fifth-Order Boundary Value Problems and variational iteration technique for solving higher order boundary value problems. Ghazala [7], develop searching the least value method for the solution of fifth order boundary value problem. Other authors who have worked on solution of fifth order BVPs include, Gamel [8], he analyzed Sinc-Galerkin method for the solution of fifth-order boundary value problems with two-point boundary conditions, Siddiqi *et al.* [9, 10] developed nonpolynomial spline method, polynomial sextic spline method, for the numerical solution of the fifth-order linear special case boundary value problems, just to mention few.

In this work, we derive a Block Unification Method (BUM). This method comprise of continuous linear multistep methods (CLMMs), unified in a block fashion and are derived using collocation approach, for the direct solution of higher order BVPs in Ordinary Differential Equations (ODEs). The Significance of this method is based on the ease of derivation and implementation.

The paper is organized as follows. In section two, the CLMMs which are used to formulate the BUM are derived and the analysis of the method is discussed. In section three, the computational aspects of the method is given. Section four is the numerical examples to illustrate the accuracy of the method. Finally, section five discuses the conclusion of the paper.

2 Derivation of The Method

In this section, 5-step continuous difference formulae for the solution of fifth-order boundary value problems with appropriate conditions, on the interval $[x_n, x_{n+5}]$ is developed. It is initially assumed that the solution on the interval is locally approximated by polynomial of the form;

$$y(x) = \sum_{j=0}^{11} a_j x^j \tag{2.1}$$

whose fifth derivative yield

$$y^{(v)}(x) = \sum_{j=5}^{11} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5} \tag{2.2}$$

Where a_j are unknown coefficients to be determined. The values of these coefficients are derived by imposing collocation conditions to $u(x)$ and $u^{(v)}(x)$ at the points $x_n, x_{n+1}, \ldots, x_{n+5}$, where

$$\{u(x+ih) = y_{n+i} \ \text{ and } \ u^{(v)}(x+ih) = f_{n+i}\}$$

and y_{n+i} and f_{n+i} are approximations for the solution and the derivative at the given points, $y_{n+i} \simeq y(x_n+ih)$, $f_{n+i} \simeq y^{(v)}(x_n+ih) = f(x_n+ih, y(x_n+ih), \ldots, y^{(iv)}(x_n+ih))$. The collocation equations are given as;

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$$
\begin{aligned}
&a_0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + \cdots + a_{11} x_n^{11} - y_n = 0\\
&a_0 + a_1 x_{n+1} + a_2 x_{n+1}^2 + a_3 x_{n+1}^3 + \cdots + a_{11} x_{n+1}^{11} - y_{n+1} = 0\\
&a_0 + a_1 x_{n+2} + a_2 x_{n+2}^2 + a_3 x_{n+2}^3 + \cdots + a_{11} x_{n+2}^{11} - y_{n+2} = 0\\
&a_0 + a_1 x_{n+3} + a_2 x_{n+3}^2 + a_3 x_{n+3}^3 + \cdots + a_{11} x_{n+3}^{11} - y_{n+3} = 0\\
&a_0 + a_1 x_{n+4} + a_2 x_{n+4}^2 + a_3 x_{n+4}^3 + \cdots + a_{11} x_{n+4}^{11} - y_{n+4} = 0\\
&120a_5 + 720a_6 x_n + 2520a_7 x_n^2 + 6720a_8 x_n^3 + \cdots + 55440a_{11} x_n^6 - f_n = 0\\
&120a_5 + 720a_6 x_{n+1} + 2520a_7 x_{n+1}^2 + 6720a_8 x_{n+1}^3 + \cdots + 55440a_{11} x_{n+1}^6 - f_{n+1} = 0\\
&120a_5 + 720a_6 x_{n+2} + 2520a_7 x_{n+2}^2 + 6720a_8 x_{n+2}^3 + \cdots + 55440a_{11} x_{n+2}^6 - f_{n+2} = 0\\
&120a_5 + 720a_6 x_{n+3} + 2520a_7 x_{n+3}^2 + 6720a_8 x_{n+3}^3 + \cdots + 55440a_{11} x_{n+3}^6 - f_{n+3} = 0\\
&120a_5 + 720a_6 x_{n+4} + 2520a_7 x_{n+4}^2 + 6720a_8 x_{n+4}^3 + \cdots + 55440a_{11} x_{n+4}^6 - f_{n+4} = 0\\
&120a_5 + 720a_6 x_{n+5} + 2520a_7 x_{n+5}^2 + 6720a_8 x_{n+5}^3 + \cdots + 55440a_{11} x_{n+5}^6 - f_{n+5} = 0
\end{aligned}
$$

which translates into the matrix equation $A\underline{x} = \underline{b}$, where;

$$
A = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \cdots & x_n^{11}\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & \cdots & x_{n+1}^{11}\\
\vdots & & & & & & & & \\
1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & \cdots & x_{n+4}^{11}\\
0 & 0 & 0 & 0 & 0 & 120 & 720x_n & \cdots & 55440x_n^6\\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+1} & \cdots & 55440x_{n+1}^6\\
\vdots & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 120 & 720x_{n+5} & \cdots & 55440x_{n+5}^6
\end{bmatrix};
$$

$$
\underline{x} = (a_0, a_1, a_2, \ldots, a_{10}, a_{11})^T; \quad \underline{b} = (y_n, \ldots, , y_{n+4}, f_n, \ldots, f_{n+5})^T
$$

This is solved using the Gaussian elimination method, enhanced by CAS like Mathematica, to obtain the unknown coefficients a_j. these coefficients are substituted into (2.1) to get the continuous form;

$$
Y(x) = \sum_{i=0}^{4} \alpha_i(x) y_{n+i} + h^5 \sum_{i=0}^{5} \beta_i(x) f_{n+i} \tag{2.3}
$$

The form (2.3) has the following derivatives:

$$
\begin{cases}
Y'(x) & = \frac{1}{h}\left(\sum_{i=0}^4 \alpha_i'(x) y_{n+i} + h^5 \sum_{i=0}^5 \beta_i'(x) f_{n+i}\right)\\
\vdots & \\
Y^{(iv)}(x) & = \frac{1}{h^4}\left(\sum_{i=0}^4 \alpha_i^{(iv)}(x) y_{n+i} + h^5 \sum_{i=0}^5 \beta_i^{(iv)}(x) f_{n+i}\right)
\end{cases} \tag{2.4}
$$

where $y_{n+i} = y(x_n + ih)$, $f_{n+i} = y^{(v)}(x_n + ih)$; $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, \ldots, y_{n+i}^{(iv)})$ $i = 0, \ldots, 5$.

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The continuous coefficients $\alpha_i(x)$, $\beta_i(x)$, expressed as functions of t obtained are given below;

$$\begin{aligned}
&\alpha_0 = \tfrac{1}{24}(6t + 11t^2 + 6t^3 + t^4), \qquad \alpha_1 = \tfrac{1}{6}(-8t - 14t^2 - 7t^3 - t^4), \qquad \alpha_2 = \tfrac{1}{4}(12t + 19t^2 + 8t^3 + t^4)\\
&\alpha_3 = \tfrac{1}{6}(-24t - 26t^2 - 9t^3 - t^4), \qquad \alpha_4 = \tfrac{1}{24}(24 + 50t + 35t^2 + 10t^3 + t^4)\\
&\beta_0 = \tfrac{1}{7257600}(720t + 948t^2 - 820t^3 - 1405t^4 + 504t^6 + 120t^7 - 45t^8 - 20t^9 - 2t^{10})\\
&\beta_1 = \tfrac{1}{1451520}(14256t + 26460t^2 + 16104t^3 + 4387t^4 - 672t^6 - 144t^7 + 63t^8 + 24t^9 + 2t^{10})\\
&\beta_2 = \tfrac{1}{725760}(81648t + 153084t^2 + 84532t^3 + 12329t^4 + 1008t^6 + 168t^7 - 99t^8 - 28t^9 - 2t^{10})\\
&\beta_3 = \tfrac{1}{725760}(57168t + 133764t^2 + 109168t^3 + 34417t^4 - 2016t^6 - 48t^7 + 153t^8 + 32t^9 + 2t^{10})\\
&\beta_4 = \tfrac{1}{1451520}(-2160t + 4068t^2 + 20700t^3 + 24215t^4 + 12096t^5 + 2184t^6 - 360t^7 - 225t^8 - 36t^9 - 2t^{10})\\
&\beta_5 = \tfrac{1}{7257600}(2160t + 1932t^2 - 3400t^3 - 4265t^4 + 2016t^6 + 1200t^7 + 315t^8 + 40t^9 + 2t^{10})
\end{aligned}$$

where $t = \frac{x - x_{n+4}}{h}$. Evaluating (2.3) and (2.4) at the point $x = x_{n+5}$, the following 5-step discrete methods are obtained as the main methods.

$$\begin{aligned}
y_{n+5} &= y_n - 5y_{n+1} + 10y_{n+2} - 10y_{n+3} + 5y_{n+4} + h^5\left(\tfrac{1}{24}f_{n+1} + \tfrac{11}{24} + f_{n+2} + \tfrac{11}{24} + f_{n+3} + \tfrac{1}{24} + f_{n+4}\right)\\
hy'_{n+5} &= \frac{25y_n}{12} - \frac{61y_{n+1}}{6} + \frac{39y_{n+2}}{2} - \frac{107y_{n+3}}{6} + \frac{77y_{n+4}}{12} - h^5\left(\frac{f_n}{3360} - \frac{149f_{n+1}}{1680} - \frac{9679f_{n+2}}{10080} - \frac{10819f_{n+3}}{10080} - \frac{29f_{n+4}}{180} - \frac{f_{n+5}}{3360}\right)\\
h^2y''_{n+5} &= \frac{35y_n}{12} - \frac{41y_{n+1}}{3} + \frac{49y_{n+2}}{2} - \frac{59y_{n+3}}{3} + \frac{71y_{n+4}}{12} - h^5\left(\frac{23f_n}{43200} - \frac{7559f_{n+1}}{60480} - \frac{983f_{n+2}}{720} - \frac{1672f_{n+3}}{945} - \frac{29161f_{n+4}}{60480} - \frac{887f_{n+5}}{100800}\right)\\
h^3y'''_{n+5} &= \frac{5y_n}{2} - 11y_{n+1} + 18y_{n+2} - 13y_{n+3} + \frac{7y_{n+4}}{2} + h^5\left(\frac{17f_n}{6048} + \frac{5239f_{n+1}}{60480} + \frac{6323f_{n+2}}{5040} + \frac{54083f_{n+3}}{30240} + \frac{4531f_{n+4}}{4320} + \frac{277f_{n+5}}{4032}\right)\\
h^4y^{(iv)}_{n+5} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5\left(\frac{853f_n}{60480} - \frac{2879f_{n+1}}{60480} + \frac{22451f_{n+2}}{30240} + \frac{17659f_{n+3}}{30240} + \frac{84149f_{n+4}}{60480} + \frac{19097f_{n+5}}{60480}\right)
\end{aligned} \tag{2.5}$$

The additional methods are derived by evaluating (2.4) at $x = x_{n+i}$ $i = 0, 1, 2, \ldots, 4$ respectively;

$$\begin{aligned}
hy'_n &= -\frac{25y_n}{12} + 4y_{n+1} - 3y_{n+2} + \frac{4y_{n+3}}{3} - \frac{y_{n+4}}{4} + h^5\left(\frac{f_n}{3360} + \frac{107f_{n+1}}{1440} + \frac{199f_{n+2}}{1680} + \frac{3f_{n+3}}{560} + \frac{19f_{n+4}}{10080} - \frac{f_{n+5}}{3360}\right)\\
hy'_{n+1} &= -\frac{y_n}{4} - \frac{5y_{n+1}}{6} + \frac{3y_{n+2}}{2} - \frac{y_{n+3}}{2} + \frac{y_{n+4}}{12} + h^5\left(\frac{f_n}{3360} - \frac{f_{n+1}}{84} - \frac{361f_{n+2}}{10080} - \frac{f_{n+3}}{480} - \frac{f_{n+4}}{1680} + \frac{f_{n+5}}{10080}\right)\\
hy'_{n+2} &= \frac{y_n}{12} - \frac{2y_{n+1}}{3} + \frac{2y_{n+3}}{3} - \frac{y_{n+4}}{12} + h^5\left(\frac{-f_n}{10080} + \frac{11f_{n+1}}{2520} + \frac{25f_{n+2}}{1008} + \frac{11f_{n+3}}{2520} - \frac{f_{n+4}}{10080}\right)\\
hy'_{n+3} &= -\frac{y_n}{12} + \frac{y_{n+1}}{2} - \frac{3y_{n+2}}{2} + \frac{5y_{n+3}}{6} + \frac{y_{n+4}}{4} - h^5\left(\frac{f_{n+1}}{280} - \frac{341f_{n+2}}{10080} - \frac{3f_{n+3}}{224} + \frac{f_{n+4}}{1120} - \frac{f_{n+5}}{1008}\right)\\
hy'_{n+4} &= \frac{y_n}{4} - \frac{4y_{n+1}}{3} + 3y_{n+2} - 4y_{n+3} + \frac{25y_{n+4}}{12} + h^5\left(\frac{f_n}{10080} + \frac{11f_{n+1}}{1120} + \frac{9f_{n+2}}{80} + \frac{397f_{n+3}}{5040} - \frac{f_{n+4}}{672} + \frac{f_{n+5}}{3360}\right)\\
h^2y''_n &= \frac{35y_n}{12} - \frac{26y_{n+1}}{3} + \frac{19}{2}y_{n+2} - \frac{14y_{n+3}}{3} + \frac{11y_{n+4}}{12} + h^5\left(\frac{-887f_n}{100800} - \frac{21811f_{n+1}}{60480} - \frac{13079f_{n+2}}{30240} - \frac{41f_{n+3}}{1440} - \frac{209f_{4+n}}{60480} + \frac{23f_{n+5}}{43200}\right)\\
h^2y''_{n+1} &= \frac{11y_n}{12} - \frac{5y_{n+1}}{3} + \frac{y_{n+2}}{2} + \frac{y_{n+3}}{3} - \frac{y_{n+4}}{12} - h^5\left(\frac{23h^3f_n}{43200} - \frac{73f_{n+1}}{2240} - \frac{779f_{n+2}}{15120} + \frac{13f_{n+3}}{7560} - \frac{f_{n+4}}{576} + \frac{79f_{n+5}}{302400}\right)\\
h^2y''_{n+2} &= -\frac{y_n}{12} + \frac{4y_{n+1}}{3} - \frac{5y_{n+2}}{2} + \frac{4y_{n+3}}{3} - \frac{y_{n+4}}{12} + h^5\left(\frac{79f_n}{302400} - \frac{337f_{n+1}}{60480} - \frac{17f_{n+2}}{10080} + \frac{7f_{n+3}}{864} - \frac{11f_{n+4}}{8640} + \frac{17f_{n+5}}{100800}\right)\\
h^2y''_{n+3} &= -\frac{y_n}{12} + \frac{y_{n+1}}{3} + \frac{y_{n+2}}{2} - \frac{5y_{n+3}}{3} + \frac{11y_{n+4}}{12} - h^5\left(\frac{17f_n}{100800} + \frac{19f_{n+1}}{8640} + \frac{5f_{n+2}}{108} + \frac{23f_{n+3}}{630} - \frac{127f_{n+4}}{60480} + \frac{79f_{n+5}}{302400}\right)\\
h^2y''_{n+4} &= \frac{11y_n}{12} - \frac{14u_{n+1}}{3} + \frac{19y_{n+2}}{2} - \frac{26y_{n+3}}{3} + \frac{35y_{n+4}}{12} + h^5\left(\frac{79f_n}{302400} + \frac{7f_{n+1}}{192} + \frac{12757f_{n+2}}{30240} + \frac{11147f_{n+3}}{30240} + \frac{113f_{n+4}}{20160} + \frac{23f_{n+5}}{43200}\right)\\
h^3y'''_n &= -\frac{5y_n}{2} + 9y_{n+1} - 12y_{n+2} + 7y_{n+3} - \frac{3y_{n+4}}{2} + h^5\left(\frac{277f_n}{4032} + \frac{4081f_{n+1}}{4320} + \frac{19433f_{n+2}}{30240} + \frac{137f_{n+3}}{1260} - \frac{1061f_{n+4}}{60480} + \frac{17f_{n+5}}{6048}\right)\\
h^3y'''_{n+1} &= -\frac{3y_n}{2} + 5y_{n+1} - 6y_{n+2} + 3y_{n+3} - \frac{y_{n+4}}{2} - h^5\left(\frac{17f_n}{6048} - \frac{31f_{n+1}}{1344} - \frac{3251f_{n+2}}{15120} - \frac{49f_{n+3}}{4320} - \frac{41f_{n+4}}{10080} + \frac{41f_{n+5}}{60480}\right)\\
h^3y'''_{n+2} &= -\frac{y_n}{2} + y_{n+1} - y_{n+3} + \frac{y_{n+4}}{2} + h^5\left(\frac{41f_n}{60480} - \frac{419f_{n+1}}{15120} - \frac{395f_{n+2}}{2016} - \frac{419f_{n+3}}{15120} + \frac{41f_{n+4}}{60480}\right)\\
h^3y'''_{n+3} &= \frac{y_n}{2} - 3y_{n+1} + 6y_{n+2} - 5y_{n+3} + \frac{3y_{n+4}}{2} + h^5\left(\frac{1301f_{n+1}}{60480} + \frac{1523f_{n+2}}{7560} + \frac{67f_{n+3}}{2016} - \frac{13f_{n+4}}{1890} + \frac{41f_{n+5}}{60480}\right)\\
h^3y'''_{n+4} &= \frac{3y_n}{2} - 7y_{n+1} + 12y_{n+2} - 9y_{n+3} + \frac{5y_{n+4}}{2} - h^5\left(\frac{41f_n}{60480} - \frac{671f_{n+1}}{10080} - \frac{3019f_{n+2}}{4320} - \frac{6823f_{n+3}}{7560} - \frac{115f_{n+4}}{1344} + \frac{17f_{n+5}}{6048}\right)\\
h^4y^{(iv)}_n &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} - h^5\left(\frac{19097f_n}{60480} + \frac{81629f_{n+1}}{60480} + \frac{3799f_{n+2}}{30240} + \frac{8591f_{n+3}}{30240} - \frac{5399f_{n+4}}{60480} + \frac{853f_{n+5}}{60480}\right)\\
h^4y^{(iv)}_{n+1} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + u_{n+4} + h^5\left(\frac{853f_n}{60480} - \frac{4339f_{n+1}}{12096} - \frac{20557f_{n+2}}{30240} + \frac{1531f_{n+3}}{30240} - \frac{1867f_{n+4}}{60480} + \frac{281f_{n+5}}{60480}\right)\\
h^4y^{(iv)}_{n+2} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} - h^5\left(\frac{281f_n}{60480} - \frac{5059f_{n+1}}{60480} - \frac{181f_{n+2}}{6048} + \frac{3887f_{n+3}}{30240} - \frac{1367f_{n+4}}{60480} + \frac{181f_{n+5}}{60480}\right)\\
h^4y^{(iv)}_{n+3} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} + h^5\left(\frac{181f_n}{60480} + \frac{1153f_{n+1}}{60480} + \frac{17747f_{n+2}}{30240} + \frac{2591f_{n+3}}{6048} - \frac{2539f_{n+4}}{60480} + \frac{281f_{n+5}}{60480}\right)\\
h^4y^{(iv)}_{n+4} &= y_n - 4y_{n+1} + 6y_{n+2} - 4y_{n+3} + y_{n+4} - h^5\left(\frac{281f_n}{60480} - \frac{4387f_{n+1}}{60480} - \frac{12329f_{n+2}}{30240} - \frac{34417f_{n+3}}{30240} - \frac{4843f_{n+4}}{12096} + \frac{853f_{n+5}}{60480}\right)
\end{aligned} \tag{2.6}$$

Remark 2.1. *All the formulas in (2.5) and (2.6) considered together form the Block Unification Method (BUM), that should be applied sequentially on block of intervals of the form $[x_n, x_{n+5}]$. Here $n = 0, 5, \ldots, N-5$, where N, is the number of subintervals and must be a multiple of 5 in order to reach the final point $x_N = b$.*

2.1 Order and Local Truncation Error (LTE)

2.1.1 Local Truncation Error

The method (2.3) and its derivatives (2.4), are continuous formulae associated with LMM of the form:

$$y(x) = \sum_{i=0}^{k} \alpha_i y_{n+i} + h^5 \sum_{i=0}^{k} \beta_i f_{n+i} \tag{2.7}$$

Thus, the linear differential operator $\mathcal{L}[y(x);\ h]$ is defined as

$$\mathcal{L}[y(x);\ h] = \sum_{i=0}^{k} \Big(\alpha_i y(x+ih) - h^5 \beta_i y^{(5)}(x+ih) \Big) \tag{2.8}$$

Expanding (2.7) in Taylor series, we obtain

$$\mathcal{L}[y(x);\ h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^q y^{(p)}(x) + O(h^{(p+1)}) \tag{2.9}$$

The LMM (2.7) is of order p if $C_0 = C_1 = C_2 = \cdots = C_{p+4} = 0, \mathit{and} \quad C_{p+5} \neq 0$ in which

$$\mathcal{L}[y(x);\ h] = C_{p+5} h^{p+5} y^{(p+5)}(x) + O(h^{(p+6)}) \tag{2.10}$$

In this case, C_{p+5} is the error constant, [11].

Definition 2.2. *A linear difference operator $\mathcal{L}[y(x);\ h]$ is said to be consistent of order p if*

$$\mathcal{L}[y(x);\ h] = O(h^{(p+5)})$$

In the light of this, expanding each of the methods in (2.5) and (2.6), the Local Truncation Error (LTE), C_{p+5} as defined in (2.8), are as shown below, with each having order $p = 6$;

$$C_{p+5} = \tau_{n+5} = \tfrac{41}{221760}$$
$$C_{p+5} = (\tau'_n, \tau'_{n+1}, \tau'_{n+2}, \tau'_{n+3}, \tau'_{n+4}, \tau'_{n+5})^T = \left(-\tfrac{79}{362880}, -\tfrac{19}{266112}, \tfrac{1}{71280}, \tfrac{37}{1330560}, -\tfrac{5}{44352}, -\tfrac{91}{570240}\right)^T$$
$$C_{p+5} = (\tau''_n, \tau''_{n+1}, \tau''_{n+2}, \tau''_{n+3}, \tau''_{n+4}, \tau''_{n+5})^T = \left(-\tfrac{2321}{907200}, \tfrac{37}{226800}, -\tfrac{17}{201600}, \tfrac{89}{907200}, -\tfrac{571}{1814400}, -\tfrac{1}{3780}\right)^T$$
$$C_{p+5} = (\tau'''_n, \tau'''_{n+1}, \tau'''_{n+2}, \tau'''_{n+3}, \tau'''_{n+4}, \tau'''_{n+5})^T = \left(\tfrac{13}{1152}, \tfrac{227}{453600}, -\tfrac{43}{453600}, -\tfrac{23}{129600}, \tfrac{229}{907200}, -\tfrac{337}{113400}\right)^T$$
$$C_{p+5} = (\tau^{(iv)}_n, \tau^{(iv)}_{n+1}, \tau^{(iv)}_{n+2}, \tau^{(iv)}_{n+3}, \tau^{(iv)}_{n+4}, \tau^{(iv)}_{n+5})^T = \left(-\tfrac{1}{6048}, -\tfrac{361}{120960}, \tfrac{181}{120960}, -\tfrac{67}{40320}, \tfrac{341}{120960}, -\tfrac{277}{24192}\right)^T$$

2.2 Convergence Analysis

In order to show that the BUM converge, the main methods and additional methods can be compactly written in matrix form by introducing the following notations. Let A be a $5N \times 5N$ matrix defined by

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}$$

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Where A_{ij} are $N \times N$ matrices given as

$$A_{11} = \begin{bmatrix} -4 & 3 & \frac{-4}{3} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{26}{3} & \frac{-19}{2} & \frac{14}{3} & \frac{-11}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -9 & 12 & 7 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 5 & -10 & 10 & -5 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 3 & \frac{-4}{3} & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{26}{3} & \frac{-19}{2} & \frac{14}{3} & \frac{-11}{12} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -9 & 12 & 7 & \frac{3}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -10 & 10 & -5 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & 3 & \frac{-4}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{26}{3} & \frac{-19}{2} & \frac{14}{3} & \frac{-11}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -9 & 12 & 7 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & -6 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 5 & -10 & 10 & -5 & 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} \frac{5}{6} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{2}{3} & 0 & \frac{-2}{3} & \frac{-1}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{2} & \frac{3}{2} & \frac{-5}{6} & \frac{-1}{4} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{4}{3} & -3 & 4 & \frac{25}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{61}{6} & \frac{-39}{2} & \frac{107}{6} & \frac{77}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{5}{6} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{12} & \frac{2}{3} & 0 & \frac{-2}{3} & \frac{-1}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & \frac{-1}{2} & \frac{3}{2} & \frac{-5}{6} & \frac{-1}{4} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{4} & \frac{4}{3} & -3 & 4 & \frac{25}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & \frac{61}{6} & \frac{-39}{2} & \frac{107}{6} & \frac{77}{12} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4} & \frac{5}{6} & \frac{3}{2} & \frac{1}{2} & \frac{-1}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{12} & \frac{2}{3} & 0 & \frac{-2}{3} & \frac{-1}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{12} & \frac{-1}{2} & \frac{3}{2} & \frac{-5}{6} & \frac{-1}{4} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4} & \frac{4}{3} & -3 & 4 & \frac{25}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & \frac{61}{6} & \frac{-39}{2} & \frac{107}{6} & \frac{77}{12} \end{bmatrix}$$

$$A_{31} = \begin{bmatrix} \frac{5}{3} & \frac{-1}{2} & \frac{-1}{3} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-4}{3} & \frac{5}{2} & \frac{-4}{3} & \frac{1}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{2} & \frac{-5}{3} & \frac{-11}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{14}{3} & \frac{-19}{2} & \frac{26}{3} & \frac{-35}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{41}{3} & \frac{-49}{2} & \frac{59}{3} & \frac{71}{12} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-11}{12} & \frac{5}{3} & \frac{-1}{2} & \frac{-1}{3} & \frac{1}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & \frac{-4}{3} & \frac{5}{2} & \frac{-4}{3} & \frac{1}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{12} & \frac{-1}{3} & \frac{-1}{2} & \frac{-5}{3} & \frac{-11}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-11}{12} & \frac{14}{3} & \frac{-19}{2} & \frac{26}{3} & \frac{-35}{12} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-35}{12} & \frac{41}{3} & \frac{-49}{2} & \frac{59}{3} & \frac{71}{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-11}{12} & \frac{5}{3} & \frac{-1}{2} & \frac{-1}{3} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{12} & \frac{-4}{3} & \frac{5}{2} & \frac{-4}{3} & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{12} & \frac{-1}{3} & \frac{-1}{2} & \frac{-5}{3} & \frac{-11}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-11}{12} & \frac{14}{3} & \frac{-19}{2} & \frac{26}{3} & \frac{-35}{12} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-35}{12} & \frac{41}{3} & \frac{-49}{2} & \frac{59}{3} & \frac{71}{12} \end{bmatrix}$$

$$A_{41} = \begin{bmatrix} -5 & 6 & -3 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 3 & -6 & -5 & \frac{-3}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 7 & -12 & 9 & \frac{-5}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 11 & -18 & 13 & \frac{-7}{2} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{2} & -5 & 6 & -3 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & 0 & 1 & \frac{-1}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{2} & 3 & -6 & -5 & \frac{-3}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-3}{2} & 7 & -12 & 9 & \frac{-5}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \frac{-5}{2} & 11 & -18 & 13 & \frac{-7}{2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{3}{2} & -5 & 6 & -3 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -1 & 0 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2} & 3 & -6 & -5 & \frac{-3}{2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-3}{2} & 7 & -12 & 9 & \frac{-5}{2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-5}{2} & 11 & -18 & 13 & \frac{-7}{2} \end{bmatrix}$$

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$$A_{51} = \begin{bmatrix} 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 4 & -6 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -6 & 4 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 4 & -6 & 4 & -1 \end{bmatrix}$$

$A_{22} = A_{33} = A_{44} = A_{55} = I$, where I is an $N \times N$ identity matrix, and A_{ij} are $N \times N$ zero matrices for $i = 1, 2, 3, 4, 5$; $j = 2, 3, 4, 5$; $i \neq j$.

Similarly, Let B be a $5N \times 5N$ matrix defined by

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} \end{bmatrix}$$

where B_{ij} are $N \times N$ matrices. B_{11}, B_{21}, B_{31}, B_{41} and B_{51} can be derived similar as the above A_{ij}, while B_{ij} is an $N \times N$ zero matrices for $i = 1, 2, 3, 4, 5$; $j = 2, 3, 4, 5$.

The formulae in (2.8) and (2.9) together can be written in block form as;

$$AY - h^5 BF(Y) + C + L(h) = 0 \tag{2.11}$$

where

$$Y = \big(y(x_1), \ldots, y(x_N), hy'(x_1), \ldots, hy'(x_N), h^2y''(x_1), \ldots, h^2y''(x_N),$$
$$h^3y'''(x_1), h^3y'''(x_N), h^4y^{(iv)}(x_1), \ldots, h^4y^{(iv)}(x_N)\big)^T$$

$$F(Y) = (f_1, \ldots, f_N, f'_1, \ldots, f'_N, f''_1, \ldots, f''_N, f'''_1, \ldots, f'''_N, f_1^{(iv)}, \ldots, f_N^{(iv)},)^T$$

$$C = \Big(-hy'_0, -\tfrac{25y_0}{12}, h^5\tfrac{f_0}{3360}, -h^2y''_0, \tfrac{35y_0}{12}, h^5\tfrac{-887f_0}{100800}, -h^3y'''_0, -\tfrac{5y_0}{2}, h^5\tfrac{277f_0}{4032}, -h^4y_0^{(iv)}, y_0, h^5\tfrac{-19097f_0}{60480},$$
$$y_0, 0, \ldots, 0, -\tfrac{y_0}{4}, h^5\tfrac{f_0}{3360}, \tfrac{y_0}{12}, -\tfrac{y_0}{12}, \tfrac{y_0}{4}, h^5\tfrac{f_0}{10080}, \tfrac{25y_0}{12}, h^5\tfrac{-f_0}{3360}, 0, \ldots, 0, \tfrac{11y_0}{12}, h^5\tfrac{79f_0}{302400}, -\tfrac{y_0}{12},$$
$$-h^5\tfrac{17f_0}{100800}, h^5\tfrac{79f_0}{302400}, \tfrac{35y_0}{12}, h^5\tfrac{-23f_0}{43200}, 0, \ldots, 0, -\tfrac{3y_0}{2}, -h^5\tfrac{17f_0}{6048}, -\tfrac{y_0}{2}, h^5\tfrac{41f_0}{60480}, \tfrac{y_0}{2}, \tfrac{3y_0}{2}, -h^5\tfrac{41f_0}{60480},$$
$$\tfrac{5y_0}{2}, h^5\tfrac{17f_0}{6048}, 0, \ldots, 0, y_0, h^5\tfrac{853f_0}{60480}, y_0, h^5\tfrac{-281f_0}{60480}, y_0, h^5\tfrac{181f_0}{60480}, y_0, h^5\tfrac{-281f_0}{60480}, y_0, h^5\tfrac{853f_0}{60480}, 0, \ldots, 0\Big)^T$$

$$L(h) = (\tau_1 \ldots, \tau_N, h\tau'_1 \ldots, h\tau'_N, h^2\tau''_1 \ldots, h^2\tau''_N, h^3\tau'''_1 \ldots, h^3\tau'''_N, h^4\tau_1^{(iv)} \ldots, h^4\tau_N^{(iv)})^T$$

where $L(h)$ is the local truncation error. We also define E as

$$E = \overline{Y} - Y = (e_1, \ldots, e_N, he'_1, \ldots, he'_N, h^2e''_1, \ldots, h^2e''_N, h^3e'''_1, \ldots, h^3e'''_N, h^4e_1^{(iv)}, \ldots, h^4e_N^{(iv)})^T$$

where

$$\overline{Y} = (y_1, \ldots, y_N, hy'_1, \ldots, hy'_N, h^2y''_1, \ldots, h^2y''_N, h^3y'''_1, \ldots h^3y'''_N, h^4y_1^{(iv)}, \ldots h^4y_N^{(iv)})^T$$

We thus state the following theorem

Theorem 2.3. *[12, 13]*
Let Y, $\overline{Y}$, and E be as defined above. Let $\overline{Y}$ be an approximation of the solution vector Y for the system formed by combining the methods (2.5) and (2.6), and $e_i = |y(x_i - y_i|$, $he'_i = |hy'(x_i - hy'_i|$, $h^2e''_i = |h^2y''(x_i) - h^2y''_i|$, $h^3e'''_i = |h^3y'''(x_i) - h^3y'''_i|$ and $h^4e_i^{(iv)} = |h^4y^{(iv)}(x_i) - h^4y_i^{(iv)}|$ be as defined above for $i = 1, \ldots, N$ where the exact solution $Y(x) \in C^n[a, b]$. Define $||E||_\infty = ||Y - \overline{Y}||_\infty$, then the BUM is a sixth-order convergent method. That is $||E||_\infty = O(h^6)$.

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Proof. Following [14], the exact form of the system (2.5) and (2.6) is defined as in (2.11), while the approximate form is defined by

$$A\overline{Y} - h^5 BF(\overline{Y}) + C = 0 \tag{2.12}$$

subtracting (2.11) from (2.12), one obtains

$$A(\overline{Y} - Y) - h^5 B(F(\overline{Y} - F(Y)) = L(h) \tag{2.13}$$

using the fact that $E = \overline{Y} - Y$, we can write (2.13) as

$$AE - h^5 B(F(\overline{Y}) - F(Y)) = L(h) \tag{2.14}$$

Using Jacobian matrix J_F, we can approximate

$$F(\overline{Y}) = F(Y) + J_F(Y)(\overline{Y} - Y) + O||\overline{Y} - Y|| \tag{2.15}$$

where J_F as the approximate value $\frac{F(\overline{Y}) - F(Y)}{\overline{Y} - Y} = J_F(Y)$. Without lost of generality, we can use first derivative term so that

$$F(\overline{Y}) - F(Y) = J_F(Y)(\overline{Y} - Y) = J_F(Y)E = J_F E \tag{2.16}$$

where

$$J_F = \begin{pmatrix} J_{11} & \cdots & J_{15} \\ \vdots & & \vdots \\ J_{51} & \cdots & J_{55} \end{pmatrix}$$

whose entries J_{ij} are $N \times N$ matrices written as $J_{11} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N} \end{pmatrix}$;

$$J_{1,j+1} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N}{\partial y_N^{(j)}} \end{pmatrix} \text{ for } j = 1, 2, 3, 4$$

$$J_{i,j+1} = h^i \begin{pmatrix} \frac{\partial f_1^{(i)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_1^{(i)}}{\partial y_N^{(j)}} \\ \vdots & & \vdots \\ \frac{\partial f_N^{(i)}}{\partial y_1^{(j)}} & \cdots & \frac{\partial f_N^{(i)}}{\partial y_N^{(j)}} \end{pmatrix}; \text{ for } j = 0, \ldots, 4. \text{ and for } i = 2, \ldots, 5$$

Substituting (2.16) in (2.14), we get

$$AE - h^5 BJ_F E = L(h) \tag{2.17}$$

$$(A - h^5 B(J_F))E = L(h)$$

Now, consider the matrix

$$P = A - h^5 BJ_F \tag{2.18}$$

we claim that P is invertible for sufficiently small h. First, we claim that A is invertible. To see this, since $A_{ii} = I$, for $i = 2, 3, 4, 5$ and $A_{ij} = 0$ for $i = 1, 2, 3, 4, 5$; $j = 2, 3, 4, 5$; $i \neq j$, we thus write

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A as

$$A = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 \\ A_{31} & 0 & A_{33} & 0 & 0 \\ A_{41} & 0 & 0 & A_{44} & 0 \\ A_{51} & 0 & 0 & 0 & A_{55} \end{bmatrix} \tag{2.19}$$

Observe that A_{11} contains non zero diagonal matrices which are identical, so its determinant exists, hence its non singular. This is true for A_{21}, A_{31}, A_{41} and A_{51}. Note also that A_{ii}, $i = 2, 3, 4, 5$ are Identity matrices. Therefore A is square lower triangular matrix with all diagonal elements nonzero.

It is known that a square lower triangular matrix is invertible if and only if all elements in its main diagonal are nonzero. Then determinant of A is non-zero and therefore A^{-1} exists. Now, (2.18) can be written as

$$|P| = |A - h^5 BJ_F| = |A||I - C| \tag{2.20}$$

where $C = h^5 BJ_F A^{-1}$, then $|\lambda I - C|$ is a characteristic polynomial of C, so that

$$|\lambda I - C| = (\lambda - \lambda_1) \cdots (\lambda - \lambda_{5N})$$

where λ_i are eigenvalues of the matrix C. When $\lambda = 1$, we have

$$|I - C| = (1 - \lambda_1) \cdots (1 - \lambda_{5N})$$

for $|I - C| \neq 0$, then each $\lambda_i \neq 0$. If $\hat{\lambda_i}$ is an eigenvalue of C, so is $h^5\lambda_i$, thus we need $h^5\lambda_i \neq 1$. So we choose h such that $h^5 \notin \left\{ \frac{1}{\hat{\lambda_i}} | \hat{\lambda_i} \text{ are nonzero eigenvector of } BJ_F A^{-1} \right\}$ For such h, $|I - C| \neq 0$, so that

$$|P| = |A||I - h^5 BJ_F| \neq 0 \tag{2.21}$$

hence P is invertible. Then

$$\begin{aligned} PE &= L(h) \\ E &= P^{-1}L(h) \\ ||E|| &= ||P^{-1}L(h)|| \\ &\leq ||P^{-1}||||L(h)|| \\ &\leq O(h^{-5})O(h^{11}) \\ &= O(h^6) \end{aligned}$$

This shows that the method is 6th order convergent. □

3 Implementation

The method (2.5) and (2.6) can be expressed in block form as

$$A_0 V_\mu = A_1 V_{\mu-1} + h^5 B_1 F_{\mu-1} + h^5 B_0 F_\mu, \quad \mu = 1, \ldots, \Gamma, \quad n = 0, 5, \ldots, N - 5 \tag{3.1}$$

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where

$$\begin{aligned}
V_{\mu} &= (y_{n+1}, \dots y_{n+k}, hy'_{n+1}, \dots, hy'_{n+5}, h^2y''_{n+1}, \dots, h^2y''_{n+5}, h^3y'''_{n+1}, \dots, h^3y'''_{n+5}, h^4y^{(iv)}_{n+1}, \dots, h^4y^{(iv)}_{n+5})^T \\
V_{\mu-1} &= (y_n, \dots, y_{n-4}, hy'_n, \dots, hy'_{n-4}, h^2y''_n, \dots, h^2y''_{n-4}, h^3y'''_n, \dots, h^3y'''_{n-4}, h^4y^{(iv)}_n, \dots, h^4y^{(iv)}_{n-4})^T \\
F_{\mu-1} &= (f_n, \dots, f_{n-4}, hf'_n, \dots, hf'_{n-4}, h^2f''_n, \dots, h^2f''_{n-4}, h^3f'''_n, \dots, h^3f'''_{n-4}, h^4f^{(iv)}_n, \dots, h^4f^{(iv)}_{n-4})^T \\
F_{\mu} &= (f_{n+1}, \dots, f_{n+5}, hf'_{n+1}, \dots, hf'_{n+5}, h^2f''_{n+1}, \dots, h^2f''_{n+5}, h^3f'''_{n+1}, \dots, h^3f'''_{n+5}, h^4f^{(iv)}_{n+1}, \dots, h^4f^{(iv)}_{n+5})^T
\end{aligned}$$

The positive integer $\Gamma = \frac{N}{k}$ is the number of blocks, $k = 5$ is the step number.
The block unification method has been implemented using the system *Mathematica*, enhanced by the feature ***NSolve[]*** for linear problems while nonlinear problems were solved by Newton's method enhanced by the feature ***FindRoot[]***, as summarized in the algorithm below. We begin by noting that the solution of the problem (1.1) is sought in the subinterval $\pi_N = \{a = x_0 < x_1 < \dots < x_N = b\}$, where $h = \frac{b-a}{N}$ is a constant step-size of the partition of π_N, N is a positive integer and n the grid index.

We emphasize the block unification of (2.5) and (2.6) lead to a single matrix of finite difference equations, which is solved to provide all the solutions of (1.1) on the entire interval $[a, b]$.
Step 1: Use the block unification of (3.1) for $\mu = 1$, $n = 0$ to obtain V_1 on the rectangle $[y_0, y_5] \times [a, b]$, similarly, for $\mu = 2$, $n = 5$ so that V_2 is obtained on the rectangle $[y_5, y_{10}] \times [a, b]$, and on the rectangle $[y_{10}, y_{15}] \times [a, b], \dots, [y_{N-5}, y_N] \times [a, b]$ for $\mu = 3, \dots \Gamma$, $n = 10, 15, \dots, N - 5$, we thus obtain $\mathrm{V}_3, \dots, \mathrm{V}_\Gamma$.
Step 2: Solve the unified block given by the system $\mathrm{V}_1 \bigcup \mathrm{V}_2 \bigcup \dots \bigcup \mathrm{V}_\Gamma$, obtained in step 1.
Step 3:The solution of (1.1) is approximated by the solutions in step 2 as $y = [y(x_1), y(x_2), \dots, y(x_n)]^T$, $n = 1, 2, \dots, N$.

Algorithm.

Data: a, b (integration interval), N (number of steps)
Result: *sol*, discrete approximate solution of BUM (2.5)-(2.6)

```
1  For n = 0, 5, ..., N − 5, For μ = 1, 2, ..., Γ;
2  Generate V_1, V_2, ..., V_Γ.
3  Set System V_1 ⋃ V_2 ⋃ ... ⋃ V_Γ
4  Solve NSolve[System, data] if linear, otherwise go to 5
5  Solve FindRoot[System, data],
6  Let sol = y(x_n). for n = 1, 2, ..., N
7  if n = N then
8  |  go to 11
9  else
10 |  go to 4;
11 end
12 End
```

4 Numerical Examples

In this section, the implementation of this method is demonstrated using examples in literature. The absolute errors are computed thus, $Absolute\ error = |approximate\ solution - exact\ solution|$. Absolute errors are compared with those in other methods in literature. To show the efficiency of the method derived in this work, in terms of accuracy, maximum absolute errors are compared where applicable, with others in literature.

Problem 1. Consider the nonlinear fifth order boundary value problem ([7], [15]),.

$$y^{(v)}(x) + y^{(iv)}(x) + e^{-2x}y^2 = 2e^x + 1, \quad 0 \le x \le 1$$
$$y(0) = 1, \quad y(1) = e, \quad y'(0) = 1, \quad y'(1) = e, \quad y''(0) = 1$$

The exact solution is given by $y(x) = e^x$.

Table 1: Error for Problem 1, $h = 0.1$

x	y_{approx}	y_{ex}	$Error$
0.1	1.10517091807568	1.10517091807565	$3.82E-14$
0.2	1.22140275816042	1.22140275816017	$2.50E-13$
0.3	1.34985880757669	1.34985880757600	$6.88E-13$
0.4	1.49182469764255	1.49182469764127	$1.28E-12$
0.5	1.64872127070197	1.64872127070013	$1.84E-12$
0.6	1.82211880039261	1.82211880039051	$2.10E-12$
0.7	2.01375270747234	2.01375270747048	$1.87E-12$
0.8	2.22554092849368	2.22554092849247	$1.21E-12$
0.9	2.45960311115737	2.45960311115695	$4.15E-13$
1.0	2.71828182845905	2.71828182845905	0.00

Table 2: Maximum Error for Problem 1, $h = 0.1$

Methods	Maximum Absolute error ($h = 0.1$)
BUM	$2.098E-12$
Method in [7]	$1.0178E-07$
Method 1 in [15]	$1.121E-05$
Method 2 in [15]	$3.049E-05$

BUM=Block Unification Method
RKSM=Reproducing Kernel Space Method [7]
EPCICM=Embedded Perturbed Chebyshev Integral Collocation Method [15]
EPBICM=Embedded Perturbed Bernstein Integral Collocation Method [15].

Table 1 presents the approximate solution y_{app}, exact solution y_{ex} and the error Er obtained as $Er = |y_{app} - y_{ex}|$, at the points $x = 0.1, 0.2, ..., 1.0$, for Example 1. Table 2 presents the maximum error obtained using the BUM, method in Ghazala and Hamood (2013) and methods in Taiwo *et al.* (2015). The BUM performs better and is more accurate in terms of the errors obtained when compared with the other methods.

Problem 2. Consider the nonlinear fifth order boundary value problem

$$y^{(v)} - e^{-x}y^2 = 0, \quad 0 \le x \le 1,$$
$$y(0) = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e.$$

with theoretical solution $y(x) = e^x$.

Table 3: Error for Problem 2, $h = 0.1$

x	y_{Exact}	y_{Appro}	BUM	VIM	HPM	VOP	ADM
0.0	1.000000000000000	1.000000000000000	0.0000	0.00000	0.00000	0.00000	0.00000
0.1	1.105170918075683	1.105170918075648	$3.6E-14$	$1.0E-09$	$1.0E-09$	$1.3E-12$	$1.0E-09$
0.2	1.221402758160407	1.221402758160170	$2.4E-13$	$2.0E-09$	$2.0E-09$	$1.0E-11$	$2.0E-09$
0.3	1.349858807576668	1.349858807576003	$6.6E-13$	$1.0E-08$	$1.0E-08$	$3.2E-11$	$1.0E-08$
0.4	1.491824697642532	1.491824697641270	$1.3E-12$	$2.0E-08$	$2.0E-08$	$7.0E-11$	$2.0E-08$
0.5	1.648721270701983	1.648721270700132	$1.9E-12$	$3.1E-08$	$3.1E-08$	$1.2E-10$	$3.1E-08$
0.6	1.822118800392674	1.822118800390509	$2.2E-12$	$3.7E-08$	$3.7E-08$	$1.9E-10$	$3.7E-08$
0.7	2.013752707472445	2.013752707470477	$2.0E-12$	$4.1E-08$	$4.1E-08$	$2.8E-10$	$4.1E-08$
0.8	2.225540928493765	2.225540928492468	$1.3E-12$	$3.1E-08$	$3.1E-08$	$3.7E-10$	$3.1E-08$
0.9	2.459603111157402	2.459603111156950	$4.5E-13$	$1.4E-08$	$1.4E-08$	$4.7E-10$	$1.4E-08$
1.0	2.718281828459045	2.718281828459045	0.00000	0.00000	0.00000	$5.6E-10$	0.00000

Problem 3. Consider the following fifth-order boundary value problem [7].

$$\begin{aligned} &y^{(v)} + 24e^{-5y} = \tfrac{48}{(1+x)^5} \\ &y(0) = 0, \;\; y(1) = \ln 2, \\ &y'(0) = 1, \;\; y'(1) = 0.5, \\ &y''(0) = -1 \end{aligned}$$

Its exact solution is $y(x) = \ln(1 + x)$

Table 4: Error for Problem 3, $h = 0.1$

x	y_{app}	y_{ex}	Er
0.1	0.095310179947	0.095310179804	$1.428E-10$
0.2	0.182321557689	0.182321556793	$8.955E-10$
0.3	0.262364266679	0.262364264467	$2.212E-9$
0.4	0.336472240166	0.336472236621	$3.545E-9$
0.5	0.405465112502	0.405465108108	$4.394E-9$
0.6	0.470003633696	0.470003629245	$4.450E-9$
0.7	0.530628254717	0.530628251062	$3.655E-9$
0.8	0.587786667142	0.587786664902	$2.240E-9$
0.9	0.641853886912	0.641853886172	$7.402E-10$
1.0	0.693147180559	0.693147180559	0.00

Table 5: Error for Problem 3, $h = 0.1$

Method	Maximum Absolute Error
BUM (h=0.05)	$4.5E-9$
RKSM; [7] (n=10)	$9.2E-06$
Method in [8]	$5.0E-05$

RKSM=Reproducing Kernel Space Method [7]

Table 4 presents the approximate solution y_{app}, exact solution y_{ex} and the error Er obtained in Example 3, while Table 5 presents the maximum absolute error obtained using the BUM, methods in Ghazala and Hamood (2013) and Gamel (2007). The errors obtained show that the BUM is more accurate.

Problem 4. Consider the following fifth-order boundary value problem [16]

$$\begin{cases} y^{(v)}(x) - 4y'(x) = 0 \\ y(0) = 0, \quad y'(0) = 1, \\ y''(0) = 2, \quad y(1) = \exp(1)\sin(1), \\ y'(1) = \exp(1)(\sin(1) + \cos(1)) \end{cases}$$

Its exact solution is $y(x) = \exp(x)\sin(x)$.

Table 6: Error for Problem 4, $h = 0.1$

x	y_{ex}	y_{approx}	$Error$
0.1	0.1103329887302397	0.110332988730237573	$3.603E-14$
0.2	0.2426552685961388	0.2426552685962604	$1.216E-12$
0.3	0.3989105537863469	0.3989105537871326	$7.857E-12$
0.4	0.5809439007982372	0.5809439008010042	$2.767E-11$
0.5	0.7904390832854186	0.7904390832925986	$7.180E-11$
0.6	1.0288456664271048	1.0288456664426048	$1.550E-10$
0.7	1.2972951121731566	1.2972951122029466	$2.979E-10$
0.8	1.5965053411284735	1.5965053411812935	$5.282E-10$
0.9	1.9266733048537612	1.9266733049418612	$8.810E-10$
1.0	2.2873552885773614	2.2873552887172614	$1.399E-9$

The approximate solution is arrived at for $N = 10$.

Table 7: Maximum Error for Problem 4

N	BUM	N	Method in [16]
15	$5.50429E-11$	16	$6.2513351E-4$
30	$7.96047E-13$	32	$1.6224384E-4$
60	$1.21794E-14$	64	$4.4584274E-5$
120	$2.30832E-16$	128	$1.1444092E-5$

Unlike the method in [16], we recall from Remark 2.1 above that the Block Unification Method (BUM), is applied sequentially on block of intervals of the form $[x_n, x_{n+5}]$. Here $n = 0, 5, N-5$, where N, is the number of subintervals and must be a multiple of 5 in order to reach after an integer number of blocks which is the final point $x_N = b$. This can be seen in Table 7 and Table 9. This shows the superiority of the BUM in terms of the errors obtained.

Problem 5. Consider the following fifth-order boundary value problem [16]

$$\begin{cases} y^{(v)}(x) - \frac{(y'(x))^2}{(5+x)^3} = \frac{23}{(5+x)^5}, \quad 0 < x < 1, \\ y(0) = \ln(5), \quad y'(0) = \frac{1}{5}, \\ y''(0) = -\frac{1}{25}, \quad y(1) = \ln(6)\sin(1), \\ y'(1) = \frac{1}{6} \end{cases}$$

Its exact solution is $y(x) = \ln(x+5)$.

Table 8: Error for Problem 5, $h = 0.1$

x	y_{ex}	y_{approx}	$Error$
0.1	1.9240539730283548547	1.9240539730283548603	$5.628E-17$
0.2	1.6486586255873836254	1.6486586255873838149	$1.895E-15$
0.3	1.6677068205580885254	1.6677068205580897494	$1.224E-14$
0.4	1.6863989535702722145	1.6863989535702765285	$4.314E-14$
0.5	1.7047480922385372215	1.7047480922385491215	$1.119E-13$
0.6	1.7227665977413453515	1.7227665977413695315	$2.418E-13$
0.7	1.7404661748409693874	1.7404661748410158574	$4.647E-13$
0.8	1.7578579175531985845	1.7578579175532810645	$8.248E-13$
0.9	1.7749523509130516454	1.7749523509131893454	$1.377E-12$
1.0	1.7917594692302468545	1.7917594692304659545	$2.191E-12$

The approximate solution is arrived at for $N = 10$.

Table 9: Error for Problem 5

N	BUM	N	Method in [16]
15	$9.35911E-14$	16	$1.9752979E-4$
30	$1.42133E-15$	32	$4.3869019E-5$
60	$2.20392E-17$	64	$1.1444092E-5$
120	$4.16899E-19$	128	$2.5033951E-6$

Similarly as in Example 4, Table 8 shows the errors obtained for Example 5. In Table 9, at different values of N as above, it shows the maximum absolute errors obtained, in comparison with the BUM and the method of Pandey (2017). Again, this shows the superiority of the BUM in terms of the errors obtained.

Problem 6. Consider the following fifth-order boundary value problem [17].

$$\begin{cases} y^{(v)}(x) = y(x) - 15\,e^x - 10\,x\,e^x \\ y(0) = 0, \quad y(1) = 1, \\ y'(0) = 1, \quad y'(1) = -e, \\ y''(0) = 0. \end{cases}$$

Its exact solution is $y(x) = x(1-x)e^x$.

Table 10: Error for Problem 6

Method	Maximum Absolute error
BUM (h=0.002)	$2.98E-19$
BUM (h=0.001)	$4.66E-21$
Method in [17]	$6.35E-17$

Method in [17] is a nonsymmetric Generalized Jacobi PetrovGalerkin method (GJPGM) uses P_N, which is the space of all polynomials of degree less than or equal to N. The result was obtained for $N = 15$. It can be seen that the BUM performs well for very small values of h in Table 10.

5 Conclusion

Continuous block linear multistep method derived via the collocation approach is proposed. This block method is called the BUM. This method was applied to solve $y^{(v)} = f(x, y, y', \ldots, y^{(iv)})$ subject to appropriate boundary conditions numerically, directly without first reducing the problem to an equivalent first order system. Numerical experiments performed using the BUM shows the efficiency and accuracy in terms of the small global error produced. This gives an edge of this method over existing ones in the literature as seen in the numerical examples.

Competing of financial interest

The author(s) declares there are no competing financial interest.

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