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Numerical Solution of Linear Integral and Integro-Differential Equations Using Boubakar Collocation Method

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Abstract

This paper discusses a new collocation method based on Boubakar approximating polynomial for the solution of first order linear integral and integro-differential equations with initial condition. The integro-differential equations is converted into integral equations and later transformed to system of linear equations using standard collocation method. The linear equation is then solved using matrix inversion method. Three examples are given, numerical solutions show that the method is efficient in handling problems under consideration.

Keywords and phrases: integral equations, Fredholm-Volterra integro-differential equations, collocation method, Boubakar polynomial
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1 Introduction

Recently, attention has been focused on the development of efficient methods for the solution of integral and integro-differential equations, which have wider application in the model of heat and mass diffusion processes, electromagnetic theory and ocean circulating among others [1]. Some of the numerical methods for the solutions of integral and integro-differential equations developed in literature include: hybrid linear multistep method [2], Daftardar-Gejji and Jafari method [3], collocation method [1, 4, 5, 6], block by block method [7], shifted Chebyshev polynomial of the third kind [8], Chebyshev-Gelarkin method [9], differential transform [10], Adomian decomposition method [11], Chebyshev collocation method [12], Bernoulli matrix method [13], Berstein operational matrix [14], pseudospectral method using Legendre polynomial [15], to mention but few.

The aim of this paper is to develop a new collocation method for the solution of linear Fredholm Volterra integral equations

$$y(x) = f(x) + \lambda_1 \int_0^x k(x,t)\, y(t)\, dt + \lambda_2 \int_0^1 w(x,t)\, y(t)\, dt \tag{1.1}$$

and integro-differential equations

$$y'(x) = f(x) + \lambda_1 \int_0^x k(x,t)\, y(t)\, dt + \lambda_2 \int_0^1 w(x,t)\, y(t)\, dt,\ \ y(0) = y_0 \tag{1.2}$$

$a \leq x \leq b$, $y(x)$ is the unknown function, $f(x)$ and $k(x,t), w(x,t)$ are given continuous real valued functions, using the approximate solution

$$y_n(x) = \mathbf{B}(x)\, \mathbf{A} \tag{1.3}$$

where $\mathbf{B}(x)$ is the Boubakar interpolating polynomial with the recursive formula

$$B_{n+1}(x) = xB_n(x) - B_{n-1}(x), n \geq 2 \tag{1.4}$$

where $\mathbf{B}(x) = \begin{bmatrix} B_0(x) & B_1(x) & \cdots & B_N(x) \end{bmatrix}_{1\times(N+1)}$, $B_0(x) = 1,\ B_1(x) = x,\ B_2(x) = x^2 + 2$ and $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T_{(N+1)\times 1}$ are constants to be determined

2 Methodology

In this section, we discuss the steps in developing the new collocation method. We first convert the integro-differential equation to integral equation, hence, (1.2) can be written in the form

$$y(x) = y_0 + \int_0^x f(t)\, dt + \lambda_1 \int_0^x \left[\int_0^x k(x,t)\, y(t)\, dt\right] dx + \lambda_2 \int_0^x \left[\int_0^1 w(x,t)\, y(t)\, dt\right] dx \tag{2.1}$$

we substitute (1.3) into (2.1) to give

$$\mathbf{w}(x)\, \mathbf{A} = g(x) \tag{2.2}$$

where

$$\mathbf{w}(x) = \left[\mathbf{B}(x) - \lambda_1 \int_0^x \left[\int_0^x k(x,t)\, \mathbf{B}(t)\, dt\right] dx - \lambda_2 \int_0^x \left[\int_0^1 w(x,t)\, \mathbf{B}(t)\, dt\right] dx\right]_{1\times(N+1)}$$

$$g(x) = y_0 + \int_0^x f(t)\, dt$$

We then collocate (2.2) at

$$x_i = a + \frac{(b-a)\, i}{N} \tag{2.3}$$

to give

$$\mathbf{w}(x_i)\, \mathbf{A} = \mathbf{g}(x_i) \tag{2.4}$$

with the dimension $\mathbf{w}(x_i) = (N+1) \times (N+1)$, $\mathbf{g}(x_i) = (N+1) \times 1$. We then solve for the constants in (2.4) using matrix inversion method and substitute the result into the approximate solution (1.3) to get the numerical solution

3 Convergence Analysis

In this section, we establish the convergence of the new method. We substitute the approximate solution into (1.2)

$$y_N(x) = y_0 + \int_0^x f(t)\,dt + \lambda_1 \int_0^x \left[\int_0^x k(x,t)\, y_N(t)\,dt\right] dx + \lambda_2 \int_0^x \left[\int_0^1 w(x,t)\, y_N(t)\,dt\right] dx \quad (3.1)$$

Substract (1.2) from (3.1) and using

$$E_N(x) = y_N(x) - y(x) \quad (3.2)$$

then

$$|E_N(x)| \le |\lambda_1| \left|\int_0^x \left[\int_0^x k(x,t)\, E_N(t)\,dt\right] dx\right| + |\lambda_2| \left|\int_0^x \left[\int_0^1 w(x,t)\, E_N(t)\,dt\right] dx\right|$$

hence

$$\frac{\|E_N(x_i)\|_\infty}{\|E_N(t)\|_\infty} \le |\lambda_1| \left|\int_0^{x_i} \left[\int_0^{x_i} k(x,t)\,dt\right] dx_i\right| + |\lambda_2| \left|\int_0^{x_i} \left[\int_0^1 w(x_i,t)\,dt\right] dx_i\right|$$

hence, the method converges

4 Numerical Examples

In this section, we solve three examples to test the efficiency of the new method. All the results are presented in tables, all computations are done with the aid of program written in MATLAB (2015a) and run on a PC. abs-e$_N = |y_N(x) - y(x)|$ use in the tables refers to the absolute error for N, $y_N(x)$ is the numerical solution and $y(x)$ is the exact solution

Example 4.1. *[5] Consider the linear Volterra Fredholm integro-differential equation*

$$y'(x) = -f(x) - \left[\int_0^x x^2 t y(t)\,dt + \int_0^1 x(x-t)\, y(t)\,dt\right], \quad y(0) = 1, 0 \le x \le 1 \quad (4.1)$$

$$f(x) = \sin x - x^2 \cos x - x^3 \sin x + x^2 - x^2 \sin(1) + x\cos(1) + x\sin(1) - x$$

Integrating (4.1) from 0 to x gives

$$y(x) = y(0) - \int_0^x f(t)\,dt - \int_0^x \left(\int_0^x x^2 t y(t)\,dt\right) dx - \int_0^x \left(\int_0^1 x(x-t)\, y(t)\,dt\right) dt \quad (4.2)$$

We solve this problem using N=5 and 10 but we use N =5 for illustration. Let the approximate solution (1.3) reduces to the form $\mathbf{B}(x) = \begin{bmatrix} B_0(x) & B_1(x) & B_2(x) & B_3(x) & B_4(x) & B_5(x) \end{bmatrix}$ *and* $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}^T$. *Substituting into (4.2), (2.2) reduces to*

$$\mathbf{w}(x) = \mathbf{B}(x) + \int_0^x \left(\int_0^x x^2 t \mathbf{B}(t)\,dt\right) dx + \int_0^x \left(\int_0^1 x(x-t)\,\mathbf{B}(t)\,dt\right) dt$$

and

$$g(x) = 1 - \int_0^x f(t)\,dt$$

We then collocate at

$$x_i = \begin{bmatrix} 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \end{bmatrix}$$

then, the parameters in (2.4) give

$$\mathbf{w}\left(x_i\right)=\begin{bmatrix} 1 & 0 & \frac{-2}{3} & 0 & \frac{8}{15} & 0 \\ \frac{186131}{187500} & \frac{54751}{281250} & \frac{-14221861}{23625000} & \frac{-12133487}{337500000} & \frac{593103319}{1687500000} & \frac{251288807}{585937500} \\ \frac{46048}{46875} & \frac{54032}{140625} & \frac{-2419331}{5906250} & \frac{-6125971}{10546875} & \frac{-39323803}{421875000} & \frac{728904659}{1757812500} \\ \frac{61861}{62500} & \frac{18081}{31250} & \frac{-244609}{2625000} & \frac{-6628041}{12500000} & \frac{-93557237}{187500000} & \frac{-5622906}{48828125} \\ \frac{48911}{46875} & \frac{111548}{140625} & \frac{1018852}{2953125} & \frac{-812251}{10546875} & \frac{-38755979}{105468750} & \frac{-145726559}{292968750} \\ \frac{71}{60} & \frac{19}{18} & \frac{6911}{7560} & \frac{793}{861} & \frac{21667}{21600} & \frac{931}{900} \end{bmatrix}$$

and

$$\mathbf{g}\left(x_i\right)=\left[\begin{array}{cccccc} 1.0 & 0.974707 & 0.909465 & 0.824486 & 0.747276 & 0.712805 \end{array}\right]^T$$

Solving for the unknown constants $\mathbf{A}$ *in* (2.4) *using matrix inversion gives*

$$\mathbf{A}=\left[\begin{array}{cccccc} 0.812324 & -0.0035197 & -0.274497 & -0.00236426 & 0.00877066 & -0.000411614 \end{array}\right]^T$$

We then substitute $\mathbf{A}$ *into the approximate solution to give the numerical solution*

$$\begin{aligned} y_5(x) &= -\left(3.9618\times10^{-3}\right)x^5+\left(4.6046\times10^{-2}\right)x^4-\left(2.2857\times10^{-3}\right)x^3 \\ &\quad -0.49945x^2-\left(4.8104\times10^{-5}\right)x+1 \end{aligned}$$

Following the steps discussed earlier, we solve for N=10 to obtain

$$\begin{aligned} y_{10}(x) &= -\left(2.4083\times10^{-7}\right)x^{10}-\left(1.1041\times10^{-7}\right)x^9+\left(2.4971\times10^{-5}\right)x^8 \\ &\quad -\left(1.5683\times10^{-7}\right)x^7-\left(1.3888\times10^{-3}\right)x^6-\left(3.6103\times10^{-8}\right)x^5 \\ &\quad +\left(4.1667\times10^{-2}\right)x^4-\left(1.3907\times10^{-9}\right)x^3-0.5x^2-\left(3.9712\times10^{-12}\right)x+1 \end{aligned}$$

Evaluation of the numerical solution at some selected points is given is Table 1 with the comparison with the exact solution. Table 1 shows that as N is increasing, the error is decreasing, hence the accuracy is increasing. This shows that the method converges

Table 1: Comparison of exact and numerical solutions for (4.1)

	Exact	Present method			
x_i	$\cos(x_i)$	$N=5$	abs-e_5	$N=10$	abs-e_{10}
0.2	0.980066577841	0.980066576138695	1.7025e-09	0.980066577840337	9.0503e-13
0.4	0.921060994002	0.921060989870354	4.1325e-09	0.921060994002161	7.2389e-13
0.6	0.825335614909	0.825335611967181	2.9425e-09	0.825335614909179	4.9911e-13
0.8	0.696706709347	0.696706714328259	4.9811e-09	0.696706709346956	2.0923e-13
1.0	0.540302305868	0.540302337522625	3.1654e-08	0.540302305868027	1.1295e-13

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Example 4.2. *[5], Consider linear Volterra Fredholm integro-differential equation*

$$y'(x) = 2e^{x} - 2 + \int_0^x y(t)\,dt + \int_0^1 y(t)\,dt,\ \ y(0) = 0, 0 \le x \le 1 \tag{4.3}$$

Writing in the integral form gives

$$y(x) = \int_0^x \left(2e^{t} - 2\right) dt + \int_0^x \left(\int_0^x y(t)\,dt\right) dx + \int_0^x \left(\int_0^1 y(t)\,dt\right) dx \tag{4.4}$$

We the solve (4.4) for N=5 and 10, the numerical solutions are:

$$y_5(x) = 0.076\,43x^5 + 0.123\,31x^4 + 0.524\,x^3 + 0.994\,01x^2 + 1.000\,5x + 1.387\,8 \times 10^{-17}$$

$$\begin{aligned} y_{10}(x) \;=\; & \left(4.794\,3 \times 10^{-6}\right) x^{10} + \left(1.945\,3 \times 10^{-5}\right) x^{9} + \left(2.060\,9 \times 10^{-4}\right) x^{8} \\ & + \left(1.382 \times 10^{-3}\right) x^{7} + \left(8.337\,3 \times 10^{-3}\right) x^{6} + \left(4.166\,5 \times 10^{-2}\right) x^{5} \\ & + 0.166\,67x^4 + 0.5x^3 + x^2 + x + 2.273\,7 \times 10^{-13} \end{aligned}$$

Comparison of the exact solution and the numerical solution is given in Table 2. It is equally shown that the method is efficient and convergent

Table 2: Comparison of exact and numerical solutions for (4.3)

	exact	Present method			
x_i	$x_i e^{x_i}$	N=5	abs-e_5	N=10	abs-e_{10}
0.2	0.24428055163203	0.2442824737318	1.9221e-06	0.244280551632	7.8606e-13
0.4	0.59672987905650	0.5967338938959	4.0148e-06	0.596729879057	1.3940e-12
0.6	1.09327128023431	1.0932775362937	6.2561e-06	1.093271280236	2.0199e-12
0.8	1.78043274279397	1.7804414752188	8.7324e-06	1.780432742796	2.5655e-12
1.0	2.71828182845905	2.7182934980576	1.1670e-05	2.718281828462	2.9569e-12

Example 4.3. *[17] Consider the integral equation*

$$y(x) = e^{x} - 1 - x + \int_0^x y(t)\,dt + \int_0^1 xy(t)\,dt,\ \ 0 \le x \le 1 \tag{4.5}$$

We solve this problem using N=5 and 10, the numerical solutions give

$$\begin{aligned} y_5(x) \;=\; & \left(7.673\,1 \times 10^{-2}\right) x^5 + 0.122\,56x^4 + 0.524\,68x^3 + 0.993\,76x^2 + 1.000\,6x \\ & + 6.938\,9 \times 10^{-18} \end{aligned}$$

$$\begin{aligned} y_{10}(x) \;=\; & \left(4.798\,7 \times 10^{-6}\right) x^{10} + \left(1.943\,1 \times 10^{-5}\right) x^{9} + \left(2.061\,4 \times 10^{-4}\right) x^{8} \\ & + \left(1.382\,0 \times 10^{-3}\right) x^{7} + \left(8.337\,4 \times 10^{-3}\right) x^{6} + \left(4.166\,5 \times 10^{-2}\right) x^{5} \\ & + 0.166\,67x^4 + 0.5x^3 + 1.0x^2 + x - 5.684\,3 \times 10^{-14} \end{aligned}$$

Evaluation of numerical solution at some selected points is given in Table 3. It is shown clearly that the method is convergent and efficient in handling the problem

Table 3: Comparison of the exact and numerical solution for (4.5)

	exact	present method			
x_i	$x_i e^{x_i}$	N=5	abs-e_5	N=10	abs-e_{10}
0.2	0.24428055163203	0.2442872626330	6.7110e-06	0.24428055163263	5.9857e-13
0.4	0.59672987905650	0.5967418857557	1.2007e-05	0.59672987905751	1.0064e-12
0.6	1.09327128023431	1.0932909320262	1.9652e-05	1.09327128023544	1.1332e-12
0.8	1.78043274279397	1.7804605583613	2.7816e-05	1.78043274279503	1.0507e-12
1.0	2.71828182845905	2.7183224860943	4.0658e-05	2.71828182845989	8.3978e-13

5 Conclusion

We have developed a new collocation method for the solution of first order Fredholm Volterra integral and integro-differential equations. The method used in this study has lesser computational burden and easier to code when compared with the existing methods. Numerical solutions as shown in the tables confirm that the method is efficient in handling problems under consideration. Moreover, the method shows good stability, which is measured by the difference between the maximum and the minimum errors.

Conflicts of interest

The authors declared that there are no conflicts of interest regarding the publication of this paper

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