International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2019, No. 2, pp. 644 - 656

New Subclasses of Analytic Functions with Respect to Symmetric and Conjugate Points Defined by Extended Sălăgean Derivative Operator

Hamzat J.O. [1*] and Adeyemo A. A. [2]

1*. Department of Mathematics, University of Lagos, Nigeria.
2. Department of Mathematics and Statistics, The Oke-Ogun Polytechnic, Saki, Oyo State.
* Corresponding author's E-mail : jhamzat@unilag.edu.ng

Article Info
Received: 06 October 2019 Revised: 12 December 2019
Accepted: 13 February 2020 Available online: 24 February 2020

Abstract

In the present paper, the authors study certain subclasses of analytic p-valent functions $f(z)^\tau$ defined by $T^*(A,B,\omega,\alpha,\lambda,\tau,l)$ and $T^c(A,B,\omega,\alpha,\lambda,\tau,l)$ with respect to other points in the open unit disk $D=\{z:|z|<1\}$. Bounds on the first two coefficients $|a_{p+1}|$ and $|a_{p+1}|$ and their connections with the Fekete-Szego functional $|a_{p+2}-\mu a_{p+1}^2|$ were obtained while several other corollaries follow as simple consequences.

Keywords: Analytic, univalent, starlike, Convex, coefficient inequalities, Fekete-Szego functional.
MSC2010: 30C45.

1 Introduction

Let $T(\omega)$ be the class of functions which are analytic and univalent in the open unit disk $D=\{z:|z|<1\}$ and of the form

$$\rho(z)=(z-\omega)+\sum_{k=1}^{\infty}a_k(z-\omega)^k \tag{1.1}$$

satisfying the conditions

$$\rho(0)=0,\ \ |\rho(z)|>1,\ z\in D.$$

Also, let $S(\omega)$ denote the class of functions h which are analytic and univalent in the open unit disk D and normalized with $h(\omega)=0$ and $h'(\omega)-1=0$ such that

$$h(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k. \tag{1.2}$$

http://ijmao.unilag.edu.ng/article/view/453

where ω is an arbitrary fixed point in D.
It is noted here that several authors have studied the subclasses ω-starlike and ω-convex of analytic functions $h(z)$ and several interesting results were obtained.
The aforementioned subclasses were defined respectively, as

$$ST(\omega) = S^*(\omega) = \left\{ h(z) \in S(\omega) : \Re\left\{ \frac{(z-\omega)h'(z)}{h(z)} \right\} \right\} > 0, \quad z \in D$$

and

$$CV(\omega) = S^c(\omega) = \left\{ h(z) \in S(\omega) : \Re\left\{ 1 + \frac{(z-\omega)h''(z)}{h(z)} \right\} \right\} > 0, \quad z \in D$$

where ω is an arbitrary fixed point in D (see Acu and Owa [1], Kanas and Ronning [2], Oladipo [3] just to mention but few).
However, Wald [4] established that if $p(\omega) \subset p($class of Carathoedory functions $)$ and $p(z)$ is given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k (z-\omega)^k \tag{1.3}$$

then,

$$|p_k| \leq \frac{2}{(1+d)(1-d)^k}, \quad k \geq 1 \ \ and \ \ |\omega| = d$$

$(see$ $[1,3,5,6])$. Also, we note that if A and B are arbitrary fixed integers and $-1 \leq B < A \leq 1$, then we have

$$|p_k| \leq \frac{A-B}{(1+d)(1-d)^k}, \quad k \geq 1 \ \ and \ \ |\omega| = d \tag{1.4}$$

(See also [7]).
Now, let $S_p(\omega)$ be the class of p-valent functions f(z) such that

$$f(z) = (z-\omega)^p + \sum_{k=1}^{\infty} a_{p+k}(z-\omega)^{p+k}, \quad p, k \in N. \tag{1.5}$$

Suppose that for real τ $(\tau > 0)$, we write

$$f(z)^\tau = \left[(z-\omega)^p + \sum_{k=1}^{\infty} a_{p+k}(z-\omega)^{p+k} \right]^\tau. \tag{1.6}$$

Then from (1.6), we have that

$$f(z)^\tau = \left[(z-\omega)^p + a_{p+1}(z-\omega)^{p+1} + a_{p+2}(z-\omega)^{p+2} + a_{p+3}(z-\omega)^{p+3} + ... \right]^\tau$$
$$= (z-\omega)^{\tau p} \left[1 + a_{p+1}(z-\omega) + a_{p+2}(z-\omega)^2 + a_{p+3}(z-\omega)^3 + ... \right]^\tau.$$

Further expansion gives

$$f(z)^\tau = (z-\omega)^{\tau p} \left[1 + \tau \left[a_{p+1}(z-\omega) + a_{p+2}(z-\omega)^2 + a_{p+3}(z-\omega)^3 + ... \right] \right.$$
$$\left. + \frac{\tau(\tau-1)}{2!} \left[a_{p+1}(z-\omega) + a_{p+2}(z-\omega)^2 a_{p+3}(z-\omega)^3 + ... \right]^2 \right.$$

$$+\frac{\tau(\tau-1)(\tau-2)}{3!}\left[a_{p+1}(z-\omega)+a_{p+2}(z-\omega)^2+a_{p+3}(z-\omega)^3+...\right]^3+...\Bigg].$$

This later yield

$$=(z-\omega)^{\tau p}\left[1+\sum_{t=1}^{\infty}\tau_t\Big(a_{p+1}(z-\omega)+a_{p+2}(z-\omega)^2+a_{p+3}(z-\omega)^3+...\Big)^t\right], \tag{1.7}$$

where

$$\tau_1=\tau,\ \tau_2=\frac{\tau(\tau-1)}{2!},\ \tau_3=\frac{\tau(\tau-1)(\tau-2)}{3!},\ ...\ .$$

Hence,

$$f(z)^\tau=(z-\omega)^{\tau p}+\sum_{k=1}^{\infty}c_{p+k}(\tau)(z-\omega)^{\tau p+k}. \tag{1.8}$$

where

$$c_{p+1}=\tau_1a_{p+1},\ \ c_{p+2}=\tau_1a_{p+2}+\tau_2a_{p+1}^2,\ \ c_{p+3}=\tau_1a_{p+3}+2\tau_2a_{p+1}a_{p+2}+\tau_3a_{p+1}^3,\ \ ....$$

(see [8, 9] among others).
Using Aouf et al derivative operator discussed in [10], we write for functions $f(z)^\tau\in S_p(\omega,\tau)$ as follows:

$$I^m_{\omega,p}(\lambda,l)f(z)^\tau=\left(\frac{1+\lambda(\tau p-1)+l}{1+l}\right)^m(z-\omega)^{\tau p}\ +\ \sum_{k=1}^{\infty}\left(\frac{1+\lambda(\tau p+k-1)+l}{1+l}\right)^m c_{p+k}(\tau)(z-\omega)^{\tau p+k}. \tag{1.9}$$

In view of (1.9), we give the following definitions.
Definition 1.1:
(i) Let $S^*_s(\omega,\tau)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions (1.8) and satisfying the condition:

$$\Re\left\{\frac{(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'}{I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau}\right\}>0, z\in D.$$

This class of functions shall be called ω-starlike with respect to symmetric points and ω is an arbitrary fixed point in D.
(ii) Let $S^*_c(\omega,\tau)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\Re\left\{\frac{(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'}{I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{\overline{I^m_{\omega,p}(\lambda,l)f(z)^\tau}}}\right\}>0, z\in D.$$

This class of functions shall be called ω-starlike with respect to conjugate points.
(iii) Let $S^c_s(\omega,\tau)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\Re\left\{\frac{\Big[(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'\Big]'}{\Big[I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau\Big]'}\right\}>0, z\in D.$$

This class is called ω-convex with respect to symmetric points and ω is an arbitrary fixed point in D.

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2019, No. 2, pp. 644 - 656

(iv) Let $S_c^c(\omega,\tau)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\Re\left\{\frac{\left[(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'\right]'}{\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}\right]'}\right\}>0, z\in D.$$

This class is called ω-convex with respect to conjugate points.
The above subclasses of analytic functions can also be defined in terms of subordination as follow:
Definition 1.2:
(i) Let $S_s^*(A,B,\omega,\lambda,\tau,l)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'}{I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\quad -1\le B<A\le 1, z\in D$$

and ω is an arbitrary point in D.
(ii) Let $S_c^*(A,B,\omega,\lambda,\tau,l)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'}{I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\quad -1\le B<A\le 1, z\in D$$

and ω is an arbitrary fixed point in D.
(iii) Let $S_s^c(A,B,\omega,\lambda,\tau,l)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\frac{\left[(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'\right]'}{\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau\right]'}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\ -1\le B<A\le 1,\ z\in D$$

and ω is an arbitrary fixed point in D.
(iv) Let $S_c^c(A,B,\omega,\lambda,\tau,l)$ denote the subclass of $S_p(\omega,\tau)$ consisting of functions given by (1.8) and satisfying the condition:

$$\frac{\left[(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'\right]'}{\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}\right]'}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},\ -1\le B<A\le 1,\ z\in D$$

and ω is an arbitrary fixed point in D.
Also the authors wish to introduce respectively, the classes $T^*(A,B,\alpha,\omega,\lambda,\tau,l)$ and $T^c(A,B,\alpha,\omega,\lambda,\tau,l)$ which consist of analytic function $f(z)^\tau$ of the form (1.8) and satisfying

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'+2\alpha(z-\omega)^2[I^m_{\omega,p}(\lambda,l)f(z)^\tau]''}{(1-\alpha)\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau\right]+\alpha(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau-I^m_{\omega,p}(\lambda,l)f(-z)^\tau]'}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},$$

$$-1\le B<A\le 1,\ z\in D$$

and

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]'+2\alpha(z-\omega)^2[I^m_{\omega,p}(\lambda,l)f(z)^\tau]''}{(1-\alpha)\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}\right]+\alpha(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau+\overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}]'}\prec\frac{1+A(z-\omega)}{1+B(z-\omega)},$$

$$-1 \le B < A \le 1,\ z \in D$$

where ω is an arbitrary fixed point in D (see [6, 11]). With various choices of α, τ and ω, these classes of functions could give rise to those classes earlier defined.
By the definition of subordination, it follows that $u \in T^*(A, B, \alpha, \omega, \lambda, \tau, l)$ if and only if

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]' + 2\alpha(z-\omega)^2[I^m_{\omega,p}(\lambda,l)f(z)^\tau]''}{(1-\alpha)\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau - I^m_{\omega,p}(\lambda,l)f(-z)^\tau\right] + \alpha(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau - I^m_{\omega,p}(\lambda,l)f(-z)^\tau]'} = \frac{1+Ah\big(\omega(z)\big)}{1+Bh\big(\omega(z)\big)}$$

$$= p(z), \quad z, h \in D. \tag{1.10}$$

Also, $u \in T^c(A, B, \alpha, \omega, \lambda, \tau, l)$ if and only if

$$\frac{2(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau]' + 2\alpha(z-\omega)^2[I^m_{\omega,p}(\lambda,l)f(z)^\tau]''}{(1-\alpha)\left[I^m_{\omega,p}(\lambda,l)f(z)^\tau + \overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)}}\right] + \alpha(z-\omega)[I^m_{\omega,p}(\lambda,l)f(z)^\tau + \overline{I^m_{\omega,p}(\lambda,l)\overline{f(z)^\tau}}]'} = \frac{1+Ah\big(\omega(z)\big)}{1+Bh\big(\omega(z)\big)}$$

$$= p(z), \quad z, h \in D, \tag{1.11}$$

where

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k (z-\omega)^k$$

and

$$\big|p(z)\big| \le \frac{A-B}{(1+d)(1-d)^k}, \quad k \ge 1, \ \ |\omega| = d. \tag{1.12}$$

2 Coefficient Inequalities

Theorem 2.1: Let $f \in T^*(A, B, \alpha, \omega, \lambda, \tau, l)$, then for $0 \le \alpha \le 1,\ -1 \le B < A \le 1,\ \lambda \ge 0,\ l \ge 0,\ \tau, p \in N$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$\big|a_{p+1}\big| \le \frac{2(A-B)(\tau p)b_1}{\delta_1 \tau_1 b_2 \Phi_0 (1-d^2)},$$

$$\big|a_{p+2}\big| \le \frac{2(\tau p)(A-B)b_1\Big[\delta_1\tau_1^2 b_2^2 \Phi_0^2\big(\delta_1(1+d) + (\tau p)g_1(A-B)^2\big) - 2(\tau p)\delta_2\tau_2 b_1 b_3 \Phi_1 (A-B)^2\Big]}{\delta_1^2 \delta_2 \tau_1^3 b_2^2 b_3 \Phi_0^2 \Phi_1 (1-d^2)^2}$$

and for real μ

$$\left|a_{p+2} - \mu a_{p+1}^2\right|$$

$$\le \frac{2(\tau p)b_1(A-B)\left\{\delta_1\tau_1^2 b_2^2\Phi_0^2\Big(\delta_1(1+d) + (\tau p)g_1(A-B)\Big) - 2(\tau p)\delta_2\tau_2 b_1 b_2 \Phi_1(A-B)\right\} - 4\mu(\tau p)^2\delta_2\tau_1 b_1^2 b_3\Phi_1(A-B)^2}{\delta_1^2\delta_2\tau_1^3 b_2^2 b_3 \Phi_0^2\Phi_1(1-d^2)^2}$$

where

$$b_1 = [1+\alpha(\tau p - 1)],\ b_2 = [1+\alpha\tau p],\ b_3 = [1+\alpha(\tau p+1)], b_4 = [1+\alpha(\tau p+2)],\ ...$$
$$g_1 = [1-(-1)^{\tau p+1}],\ g_2 = [1-(-1)^{\tau p+2}],\ g_3 = [1-(-1)^{\tau p+3}],\ ...$$
$$\delta_1 = [2(\tau p+2) - \tau p g_1],\ \ \delta_2 = [2(\tau p+4) - \tau p g_2],\ \delta_3 = [3(\tau p) - \tau p g_3],\ ...$$

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2019, No. 2, pp. 644 - 656

and

$$\Phi_j = \left(\frac{1+\lambda(\tau p+j)+l}{1+\lambda(\tau p-1)+l}\right)^m \quad j=0,1,2,\ldots\ .$$

Proof: From (1.3) and (1.10), we have that

$$1+\sum_{k=1}^{\infty}\frac{(\tau p+k)[1+\alpha(\tau p+k-1)]}{(\tau p)[1+\alpha(\tau p-1)]}\left(\frac{1+\lambda(\tau p+k-1)+l}{1+\alpha(\tau p-1)+l}\right)^m c_{p+k}(z-\omega)^k$$

$$\left(1+\sum_{k=1}^{\infty}\frac{[1-(-1)^{\tau p+k}][1+\alpha(\tau p+k-1)]}{2[1+\alpha(\tau p-1)]}\left(\frac{1+\lambda(\tau p+k-1)+l}{1+\alpha(\tau p-1)+l}\right)^m c_{p+k}(z-\omega)^k\right)\cdot\left(1+p_1(z-\omega)+p_2(z-\omega)^2+p_3(z-\omega)^3+\right. \tag{2.1}$$

It is easily verified from (2.1) that

$$\delta_1 b_2\Phi_0 c_{p+1} = 2(\tau p)b_1p_1 \tag{2.2}$$

and

$$\delta_1\delta 2b_3\Phi_1 c_{p+2} = 2(\tau p)b_1p_1\left(\delta_1p_2+(\tau p)g_1p_1^2\right). \tag{2.3}$$

But we recall that $c_{p+1}=\tau_1a_{p+1}$ and $c_{p+2}=\tau_1a_{p+2}+\tau_2a_{p+1}^2$ then

$$\delta_1b_2\Phi_0\tau_1a_{p+1} = 2(\tau p)b_1p_1 \tag{2.4}$$

and

$$\delta_1^2\delta_2\tau_1^3b_2^2b_3\Phi_0^2\Phi_1a_{p+2} = 2(\tau p)b_1\left[\delta_1\tau_1^2b_2^2\Phi_0^2\left(\delta_1p_2+(\tau p)g_1p_1^2\right)-2\delta_2\tau_2(\tau p)b_1b_3\Phi_1p_1^2\right] \tag{2.5}$$

where

$$b_1=[1+\alpha(\tau p-1)],\ b_2=[1+\alpha\tau p],\ b_3=[1+\alpha(\tau p+1)], b_4=[1+\alpha(\tau p+2)],\ \ldots$$

$$g_1=[1-(-1)^{\tau p+1}],\ g_2=[1-(-1)^{\tau p+2}],\ g_3=[1-(-1)^{\tau p+3}],\ \ldots$$

$$\delta_1=[2(\tau p+2)-\tau pg_1],\ \ \delta_2=[2(\tau p+4)-\tau pg_2],\ \delta_3=[3(\tau p)-\tau pg_3],\ \ldots$$

and

$$\Phi_j = \left(\frac{1+\lambda(\tau p+j)+l}{1+\lambda(\tau p-1)+l}\right)^m \quad j=0,1,2,\ldots\ .$$

Using (1.12) in (2.4) and (2.5), then we obtain the required bounds on a_{p+1} and a_{p+2}.
Now appealing to (2.4) and (2.5), we can write that for real μ

$$a_{p+2}-\mu a_{p+1}^2 = \frac{2(\tau p)b_1\left[\delta_1\tau_1^2b_2^2\Phi_0^2\left(\delta_1p_2+(\tau p)g_1p_1^2\right)-2\delta_2\tau_2(\tau p)b_1b_3\Phi_1p_1^2\right]}{\delta_1^2\delta_2\tau_1^3b_2^2b_3\Phi_0^2\Phi_1}-\mu\frac{4(\tau p)^2b_1^2p_1^2}{\delta_1^2b_2^2\Phi_0^2}. \tag{2.6}$$

Applying (1.12) in (2.6) we obtain the desired bound on the Fekete-Szego functional $\left|a_{p+2}-\mu a_{p+1}^2\right|$ and this completes the proof of Theorem 2.1.
Now suppose that $p=1$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.2: Let $f\in T^*(A,B,\alpha,d,\lambda,1,l)$ then for $0\le\alpha\le1,\quad -1\le B<A\le1,\ \lambda\ge0,\quad l\ge0$ and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$.

$$|a_2| \le \frac{(A-B)}{2(1-d^2)(1+\alpha)}\left(\frac{1+l}{1+\lambda+l}\right)^m,$$

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2019, No. 2, pp. 644 - 656

$$|a_3| \leq \frac{(A-B)(1+d)}{2(1+2\alpha)(1-d^2)^2}\Big(\frac{1+l}{1+2\lambda+l}\Big)^m$$

and

$$|a_3-\mu a_2^2|$$

$$\leq \frac{(A-B)\Big[2(1+\alpha)^2(1+d)\Big(\frac{1+\lambda+l}{1+l}\Big)^{2m}-\mu(A-B)(1+2\alpha)\Big(\frac{1+2\lambda+l}{1+l}\Big)^m\Big]}{4(1+\alpha)^2(1+2\alpha)(1-d^2)^2}\Big(\frac{1+l}{1+\lambda+l}\Big)^{2m}\Big(\frac{1+l}{1+2\lambda+l}\Big)^m.$$

Corollary 2.3: Let $f \in T^*(A,B,\alpha,d,1,1,0)$ then for $0 \leq \alpha \leq 1$, and $-1 \leq B < A \leq 1$

$$|a_2| \leq \frac{2(A-B)}{4(1-d^2)(1+\alpha)}\Big(\frac{1}{2}\Big)^m,$$

$$|a_3| \leq \frac{(A-B)(1+d)}{2(1+2\alpha)(1-d^2)^2}\Big(\frac{1}{3}\Big)^m$$

and

$$|a_3-\mu a_2^2|$$

$$\leq \frac{(A-B)\Big[2(1+\alpha)^2(1+d)\Big(2\Big)^{2m}-\mu(A-B)(1+2\alpha)\Big(3\Big)^m\Big]}{4(1+\alpha)^2(1+2\alpha)(1-d^2)^2}\Big(\frac{1}{2}\Big)^{2m}\Big(\frac{1}{3}\Big)^m.$$

Corollary 2.4: Let $f \in T^*(A,B,\alpha,d,1,1,0)$ then for $0 \leq \alpha \leq 1$, $-1 \leq B < A \leq 1$ and $m=1$

$$|a_2| \leq \frac{(A-B)}{4(1-d^2)(1+\alpha)},$$

$$|a_3| \leq \frac{(A-B)(1+d)}{6(1+2\alpha)(1-d^2)^2}$$

and

$$|a_3-\mu a_2^2| \leq \frac{(A-B)\Big[8(1+\alpha)^2(1+d)-3\mu(A-B)(1+2\alpha)\Big]}{48(1+\alpha)^2(1+2\alpha)(1-d^2)^2}.$$

Suppose that $m=p=1$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.5: Let $f \in T^*(A,B,0,0,1,1,0)$ then for $-1 \leq B < A \leq 1$

$$|a_2| \leq \frac{(A-B)}{4},$$

$$|a_3| \leq \frac{(A-B)}{6}$$

and

$$|a_3-\mu a_2^2| \leq \frac{(A-B)\Big[8-3\mu(A-B)\Big]}{48}.$$

Suppose that $m=p=1$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.6: Let $f \in T^*(A,B,1,0,1,1,0)$ then for $-1 \leq B < A \leq 1$

$$|a_2| \leq \frac{(A-B)}{8},$$

$$|a_3| \leq \frac{(A-B)}{18}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\Big[32 - 9\mu(A-B)\Big]}{3.4.48}.$$

Suppose that $m = p = 1$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.7: Let $f \in T^*(1, -1, 0, 0, 1, 1, 0)$. then

$$|a_2| \leq \frac{1}{2},$$

$$|a_3| \leq \frac{1}{3}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{\Big[4 - 3\mu\Big]}{12}.$$

Suppose that $m = p = 1$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.8: Let $f \in T^*(1, -1, 1, 0, 1, 1, 0)$. then

$$|a_2| \leq \frac{1}{4},$$

$$|a_3| \leq \frac{1}{9}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{\Big[16 - 9\mu\Big]}{144}.$$

Suppose that $p = 1$ and $m = 0$ in Theorem 2.1, then we obtain the following corollary.
Corollary 2.9: Let $f \in T^*(A, B, \alpha, d, \lambda, 1, l)$ then for $0 \leq \alpha \leq 1$, $\quad -1 \leq B < A \leq 1$, $\lambda \geq 0$, and $l \geq 0$

$$|a_2| \leq \frac{2(A-B)}{4(1-d^2)(1+\alpha)},$$

$$|a_3| \leq \frac{(A-B)(1+d)}{2(1+2\alpha)(1-d^2)^2}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\Big[2(1+\alpha)^2(1+d) - \mu(A-B)(1+2\alpha)\Big]}{4(1+\alpha)^2(1+2\alpha)(1-d^2)^2}.$$

Corollary 2.10: Let $f \in T^*(A, B, \alpha, 0, \lambda, 1, l)$ then for $0 \leq \alpha \leq 1$, $\quad -1 \leq B < A \leq 1$, $\lambda \geq 0$, $l \geq 0$ and $m = 0$

$$|a_2| \leq \frac{(A-B)}{2(1+\alpha)},$$

$$|a_3| \leq \frac{(A-B)}{2(1+2\alpha)}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\Big[2(1+\alpha)^2 - \mu(A-B)(1+2\alpha)\Big]}{4(1+\alpha)^2(1+2\alpha)}.$$

Corollary 2.11: Let $f \in T^*(1,-1,\alpha,0,\lambda,1,l)$ then for $0 \leq \alpha \leq 1$ and $m=0$

$$|a_2| \leq \frac{1}{(1+\alpha)},$$

$$|a_3| \leq \frac{1}{(1+2\alpha)}$$

and

$$|a_3 - \mu a_2^2| \leq \frac{\Big[(1+\alpha)^2 - \mu(1+2\alpha)\Big]}{(1+\alpha)^2(1+2\alpha)}.$$

Corollary 2.12: Let $f \in T^*(1,-1,0,0,\lambda,1,l)$ then for $0 \leq \alpha \leq 1$, $\quad -1 \leq B < A \leq 1$, $\lambda \geq 0$, $l \geq 0$ and $m = 0$

$$|a_2| \leq 1,$$

$$|a_3| \leq 1$$

and

$$|a_3 - \mu a_2^2| \leq |1-\mu|.$$

Theorem 2.13: Let $f \in T^c(A,B,\alpha,\omega,\lambda,\tau,l)$, then for $0 \leq \alpha \leq 1$, $-1 \leq B < A \leq 1$, $\lambda \geq 0$, $l \geq 0$, $\tau, p \in N$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$\left|a_{p+1}\right| \leq \frac{(A-B)(\tau p)b_1}{\tau_1 b_2 \Phi_0 (1-d^2)},$$

$$\left|a_{p+2}\right| \leq \frac{(\tau p)(A-B)b_1\Big[\tau_1^2 b_2^2 \Phi_0^2 \big((1+d) + (\tau p)(A-B)\big) - 2(A-B)(\tau p)\tau_2 b_1 b_3 \Phi_1\Big]}{\tau_1^3 b_2^2 b_3 \Phi_0^2 \Phi_1 (1-d^2)^2}$$

and for real μ

$$\left|a_{p+2} - \mu a_{p+1}^2\right|$$

$$\leq \frac{(\tau p)b_1(A-B)\Big\{\tau_1^2 b_2^2 \Phi_0^2 \Big((1+d) + (\tau p)(A-B)\Big) - 2(A-B)(\tau p)\tau_2 b_1 b_3 \Phi_1\Big\} - 2\mu(\tau p)^2 \tau_1 b_1^2 b_3 \Phi_1 (A-B)^2}{2\tau_1^3 b_2^2 b_3 \Phi_0^2 \Phi_1 (1-d^2)^2}$$

where

$$b_1 = [1+\alpha(\tau p - 1)],\ b_2 = [1+\alpha\tau p],\ b_3 = [1+\alpha(\tau p+1)], b_4 = [1+\alpha(\tau p+2)],\ \ldots$$

and

$$\Phi_j = \left(\frac{1+\lambda(\tau p + j) + l}{1+\lambda(\tau p - 1) + l}\right)^m \quad j = 0,1,2,\ldots\ .$$

Proof: From (1.3) and (1.11), we have that

$$1 + \sum_{k=1}^{\infty} \frac{(\tau p + k)[1+\alpha(\tau p + k - 1)]}{(\tau p)[1+\alpha(\tau p - 1)]} \left(\frac{1+\lambda(\tau p + k - 1) + l}{1+\alpha(\tau p - 1) + l}\right)^m c_{p+k}(z-\omega)^k$$

$$\left(1 + \sum_{k=1}^{\infty} \frac{[1+\alpha(\tau p + k - 1)]}{[1+\alpha(\tau p - 1)]} \left(\frac{1+\lambda(\tau p + k - 1) + l}{1+\alpha(\tau p - 1) + l}\right)^m c_{p+k}(z-\omega)^k\right) \cdot \Big(1 + p_1(z-\omega) + p_2(z-\omega)^2 + p_3(z-\omega)^3 + \ldots\Big). \tag{2.7}$$

Equating the like powers of $(z-\omega)$ and $(z-\omega)^2$ in (2.7), we obtain

$$b_2\Phi_0\tau_1 a_{p+1} = (\tau p)b_1 p_1 \tag{2.8}$$

and

$$2\tau_1^3 b_2^2 b_3 \Phi_0^2 \Phi_1 a_{p+2} = (\tau p)b_1\Big[\tau_1^2 b_2^2 \Phi_0^2\Big(p_2 + (\tau p)p_1^2\Big) - 2\tau_2(\tau p)b_1 b_3\Phi_1 p_1^2\Big] \tag{2.9}$$

where

$$b_1 = [1+\alpha(\tau p - 1)],\ b_2 = [1+\alpha\tau p],\ b_3 = [1+\alpha(\tau p+1)], b_4 = [1+\alpha(\tau p+2)],\ ...$$

and

$$\Phi_j = \left(\frac{1+\lambda(\tau p + j) + l}{1+\lambda(\tau p - 1) + l}\right)^m \quad j = 0, 1, 2, ... \ .$$

Using (1.12) in (2.8) and (2.9), then we obtain the required bounds on a_{p+1} and a_{p+2}.
Now appealing to (2.8) and (2.9), and for real μ we have that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{(\tau p)b_1\Big[\tau_1^2 b_2^2 \Phi_0^2\Big(p_2 + (\tau p)p_1^2\Big) - 2\tau_2(\tau p)b_1 b_3 \Phi_1 p_1^2\Big]}{2\tau_1^3 b_2^2 b_3 \Phi_0^2 \Phi_1} - \mu\frac{(\tau p)^2 b_1^2 p_1^2}{b_2^2 \Phi_0^2 \tau_1^2}. \tag{2.10}$$

Applying (1.12) in (2.10) we obtain the desired bound on the Fekete-Szego functional $\big|a_{p+2} - \mu a_{p+1}^2\big|$ and this completes the proof of Theorem 2.13.
Now suppose that $p = 1$ in Theorem 2.13, then we obtain the following corollary.
Corollary 2.14: Let $f \in T^c(A, B, \alpha, d, \lambda, 1, l)$, then for $0 \le \alpha \le 1$, $\lambda \ge 0$, ≥ 0 and $m \in N_0$

$$\big|a_2\big| \le \frac{(A-B)}{(1-d^2)(1+\alpha)}\Big(\frac{1+l}{1+\lambda+l}\Big)^m,$$

$$\big|a_3\big| \le \frac{(A-B)[(1+d)+(A-B)]}{2(1-d^2)^2(1+2\alpha)}\Big(\frac{1+l}{1+2\lambda+l}\Big)^m$$

and for real μ

$$\Big|a_3 - \mu a_2^2\Big|$$

$$\le \frac{(A-B)\Big[(1+\alpha)^2[(1+d)+(A-B)]\Big(\frac{1+\lambda+l}{1+l}\Big)^{2m} - 2\mu(A-B)(1+2\alpha)\Big(\frac{1+2\lambda+l}{1+l}\Big)^m\Big]}{2(1-d^2)^2(1+\alpha)^2(1+2\alpha)}\Big(\frac{1+l}{1+\lambda+l}\Big)^{2m}\Big(\frac{1+l}{1+2\lambda+l}\Big)^m.$$

Corollary 2.15: Let $f \in T^c(A, B, \alpha, d, 1, 1, 0)$, then for $0 \le \alpha \le 1$, and $m \in N_0$

$$\big|a_2\big| \le \frac{(A-B)}{2^m(1-d^2)(1+\alpha)},$$

$$\big|a_3\big| \le \frac{(A-B)[(1+d)+(A-B)]}{2.3^m(1-d^2)^2(1+2\alpha)}$$

and for real μ

$$\Big|a_3 - \mu a_2^2\Big| \le \frac{(A-B)\Big[(1+\alpha)^2[(1+d)+(A-B)]2^m - 2\mu(A-B)(1+2\alpha)3^m\Big]}{2^{2m+1}3^m(1-d^2)^2(1+\alpha)^2(1+2\alpha)}.$$

Corollary 2.16: Let $f \in T^c(A, B, \alpha, d, 1, 1, 0)$, then for $0 \le \alpha \le 1$, and $m = 1$

$$\big|a_2\big| \le \frac{(A-B)}{2(1-d^2)(1+\alpha)},$$

$$\left|a_3\right| \leq \frac{(A-B)[(1+d)+(A-B)]}{6(1-d^2)^2(1+2\alpha)}$$

and for real μ

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\Big[(1+\alpha)^2[(1+d)+(A-B)]-3\mu(A-B)(1+2\alpha)\Big]}{12(1-d^2)^2(1+\alpha)^2(1+2\alpha)}.$$

Corollary 2.17: Let $f \in T^c(A,B,\alpha,d,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m=0$

$$\left|a_2\right| \leq \frac{(A-B)}{(1-d^2)(1+\alpha)},$$

$$\left|a_3\right| \leq \frac{(A-B)[(1+d)+(A-B)]}{2(1-d^2)^2(1+2\alpha)}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\Big[(1+\alpha)^2[(1+d)+(A-B)]-2\mu(A-B)(1+2\alpha)\Big]}{2(1-d^2)^2(1+\alpha)^2(1+2\alpha)}.$$

Corollary 2.18: Let $f \in T^c(A,B,0,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m=1$

$$\left|a_2\right| \leq \frac{(A-B)}{2},$$

$$\left|a_3\right| \leq \frac{(A-B)[1+A-B]}{4}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \frac{(A-B)\Big[1+A-B]-3\mu(A-B)\Big]}{12}.$$

Corollary 2.19: Let $f \in T^c(1,-1,0,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m=0$

$$\left|a_2\right| \leq 2,$$

$$\left|a_3\right| \leq 3$$

and

$$\left|a_3-\mu a_2^2\right| \leq |3-4\mu|.$$

Corollary 2.20: Let $f \in T^c(1,-1,0,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m=1$

$$\left|a_2\right| \leq 1,$$

$$\left|a_3\right| \leq \frac{3}{2}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{2}\left|1-2\mu\right|.$$

Corollary 2.21: Let $f \in T^c(A,B,1,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m=1$

$$\left|a_2\right| \leq \frac{(A-B)}{4},$$

$$|a_3| \leq \frac{(A-B)[1+A-B]}{12}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)\Big[4[1+A-B] - 9\mu(A-B)\Big]}{48}.$$

Corollary 2.22: Let $f \in T^c(1,-1,1,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m = 1$

$$|a_2| \leq \frac{1}{2},$$

$$|a_3| \leq \frac{1}{2}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{1}{4}|2 - 3\mu|.$$

Corollary 2.23: Let $f \in T^c(A,B,1,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m = 0$

$$|a_2| \leq \frac{(A-B)}{2},$$

$$|a_3| \leq \frac{(A-B)[1+A-B]}{6}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{(A-B)\Big[4[1+A-B] - 6\mu(A-B)\Big]}{8}.$$

Corollary 2.24: Let $f \in T^c(A,B,1,0,1,1,0)$, then for $0 \leq \alpha \leq 1$, and $m = 0$

$$|a_2| \leq 1,$$

$$|a_3| \leq 1$$

and

$$\left|a_3 - \mu a_2^2\right| \leq 3|1 - \mu|.$$

For relevant results on the classes of analytic functions with respect to other points (*see* [12, 13, 14]) among others.

References

[1] M. Acu and S.Owa, *On some subclasses of univalent functions*, J Ineq. Pure and Appl. Math., vol. 6, Issue 3. Article 70(2005), pp.1-14.

[2] S.Kanas, and F.Ronning, *Uniformly starlike and convex functions and other related classes of univalent functions*, Ann. Univ. Mariae Curie-Sklodowska section A, 53 (1999), pp.95-105.

[3] A. T. Oladipo, *On subclasses of analytic and Univalent Functions*, Advances in Applied Mathematical Analysis, 4(1)(2009), pp.87-93.

[4] J.K Wald, *On starlike functions*, Ph.D Thesis, University of Delaware, New Ark, Delaware (1978).

[5] A. T. Oladipo and D. Breaz, *On the family of Bazilevic functions*, Acta Universitatis Apulensis, 24(2010), 319-330.

[6] R.M. Goel and B.C. Mehok, *A subclass of starlike functions with respect to symmetric points*, Tamkang J. Math., 13(1), (1982), pp.11-24.

[7] J. O. Hamzat and O. J. Adeleke, *A new subclass of* $p-$ *valent functions defined by Aouf et al. derivative operator*, Int. J. Pure and Appl. Math., 101(2015), 33-41.

[8] J. O. Hamzat and M. T. Raji, *Bessel functions in the space of* $\lambda-$ *pseudo Starlike functions with respect to other pointss associated with modified sigmoid function*, Asian J. Math. and Compt. Research. 25(2018), 38-49.

[9] A. T. Oladipo and D. Breaz, *A brief study of certain class of Harmonic Functions of Bazilevic Type*, ISRN Math. Anal., 2013, Art. ID179856, 11 pages. http://dx.doi.org/10.1155/2013/179856, 2013.

[10] M.K Aouf, A.Shamandy, A.O Mostafa, and S.M Madian, *A subclass of M-W starlike functions*, Acta Universitatis Apulensis No 21 (2010), pp.135-142.

[11] C. Selvaraj and N. Vasanthi, *Subclasses of Analytic Functions with respect to symmetric and Conjugate points*, Tamkang J. Math.; 42(1),(2011), pp.87-94.

[12] A. T. Oladipo, *New Subclasses of Analytic functions with Respect to Other Points*, Int. J. Pure and Applied Math.; 75(1),(2012), pp.1-12.

[13] P. T. Ajai and M. Darus, *On coefficient problems of an operator with respect to symmetric point*, Fasciculi Mathematici, (2016), 147-155. DOI:10.1515/fascmath-2016-0022.

[14] S. O. Olatunji and A. T. Oladipo, *On a subfamilies of analytic and univalent functions with negative coefficient with respect to other points*, Bull. Math. Anal. Appl., vol. 3, Issue 2, (2011), 159-166.