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Existence and approximation of fixed point of a nonlinear mapping satisfying rational type contractive inequality condition in complex-valued Banach spaces

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Abstract

We prove the existence of a unique fixed point for a mapping satisfying a rational type contractive inequality condition in complex-valued Banach spaces. We approximate this fixed point via some fixed point iterative processes with high rate of convergence. We then prove that our results are valid in cone metric spaces with Banach algebras. Furthermore, our results are applied in finding the solutions of delay differential equations. Our results extend and generalize several known results in the literature.

Keywords And Phrases: Complex-valued Banach spaces; existence of a unique fixed point; approximate this fixed point; iterative processes; cone metric spaces with Banach algebras; delay differential equations.
MSC2010:47H09; 47H10; 49M05; 54H25.

1 Introduction

Whenever the existence of a fixed point of a mapping is established, one of the immediate challenges is how to compute the fixed point. One of the most effective methods developed by Mathematicians to resolve this problem is the fixed point iterative technique. Several authors have developed many iterative processes to approximate the fixed points of various types of mappings. See Mann [1] and Ishikawa [2]. Our audience may also consult [3, 4, 5, 6, 7] and a host of others in the literature for several but interesting iterative algorithms. The speed of convergence is a very important

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consideration in preferring an iterative process to another one. Fixed point theory has become an important tool which have been applied in the study of theoretical subjects, which are directly applicable in different applied fields of science. Other areas of applications includes optimization problems, control theory, economics and a host of others. In particular, it plays an important role in the investigation of existence of solutions to differential and integral equations, which direct the behaviour of several real life problems for which the existence of solution is critica. In 1922, Banach provided a general iterative method to construct a fixed point result and proved its uniqueness under linear contraction in complete metric spaces. This famous results of Banach have been generalized in several directions by many researchers (see [8], [9], [10]). These generalizations were made either by using the contractive condition, or, by imposing some additional conditions on the ambient space, or, by some topological approach. Some of these generalizations of metric spaces includes: rectangular metric spaces, pseudo metric spaces, partial metric spaces, G-metric spaces and cone metric spaces (see [11]).

The concept of complex valued metric spaces was introduced by Azam et al. [12] in 2011. They proved the existence of fixed point for a pair of mappings satisfying rational inequality. Their results is intended to define rational expressions which are meaningless in cone metric spaces. Several authors have obtained interesting and applicable results in complex valued metric spaces (see, e.g. [13], [14], [15], [16], [17], [12], [18], [19], [20]).

There is a practical and theoretical interests in approximating fixed points of several contractive type operators since there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator (see, e.g. [8], [21]). Since the introduction of the notion of complex-valued metric spaces by Azam *et al.* [12], most results obtained in literature by many authors are existential in nature (see [16], [12], [18], [20]). Consequently, there is a gap in the literature with respect to the approximation of the fixed point of several nonlinear mappings in this type of space. Recently, Okeke [21] exploited the idea of complex-valued metric spaces to define the concept of complex-valued Banach spaces and then initiated the idea of approximating the fixed point of nonlinear mappings in complex-valued Banach spaces. Interested readers should also see the recent results of Okeke and Abbas [22]. Recently, Okeke [5] introduced the Picard-Ishikawa hybrid iterative process and proved that this new iterative process converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor [23], Picard-Mann [24] and Picard-Krasnoselskii [6] in the sense of Berinde [8]. The author applied this new iterative process in finding the solution of delay differential equations in Banach spaces.

Interest in the study of delay differential equations stems out from the fact that several models in real life problems involve delay differential equations. For instance, delay models are common in many branches of biological modelling. They have been used for describing several aspects of infectious disease dynamics: primary infection, drug therapy and immune response, among others. Delays have also appeared in the study of chemostat models, circadian rhythms, epidemiology, the respiratory system, tumor growth and neural networks (see [6, 5]).

Our interest in the present paper is to prove the existence of a unique fixed point of a mapping satisfying rational type contractive inequality conditions in complex-valued Banach spaces. We approximate this fixed point via the Picard-Ishikawa hybrid iteration and other speedy iterative processes and prove that our results are valid in cone metric spaces with Banach algebras. Furthermore, we apply our results in finding solutions of delay differential equations. Our results extend and generalize several known results in the literature.

2 Preliminaries

For the rest of this paper, we will adopt the partial order defined on $\mathbb{C}$, the set of complex numbers in [12].

Definition 2.1. *([12]). Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{C}$,*

satisfies:
1. $0 \precsim d(x,y)$, *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
2. $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
3. $d(x,y) \precsim d(x,z) + d(z,y)$, *for all* $x, y, z \in X$.
Then d *is called a complex valued metric on* X, *and* (X,d) *is called a complex valued metric space.*

Recently, Okeke [21] defined a complex valued Banach space and proved some interesting fixed point theorems in the framework of complex valued Banach spaces.

Definition 2.2. *([21]). Let* E *be a linear space over a field* $\mathbb{K}$, *where* $\mathbb{K} = \mathbb{R}$ *(the set of real numbers) or* $\mathbb{C}$ *(the set of complex numbers). A complex valued norm on* E *is a complex valued function* $\|.\| : E \to \mathbb{C}$ *satisfying the following conditions:*
1. $\|x\| = 0$ *if and only if* $x = 0,\ x \in E$;
2. $\|kx\| = |k|.\|x\|$ *for all* $k \in \mathbb{K},\ x \in E$;
3. $\|x + y\| \precsim \|x\| + \|y\|$ *for all* $x, y \in E$.
A linear space with a complex valued norm defined on it is called a complex-valued normed linear space, denoted by $(E, \|.\|)$. *A point* $x \in E$ *is called an interior point of a set* $A \subseteq E$ *if there exist* $0 \prec r \in \mathbb{C}$ *such that*

$$B(x,r) = \{y \in E : \|x - y\| \prec r\} \subseteq A.$$

A point $x \in E$ *is called a limit point of the set* A *whenever for each* $0 \prec r \in \mathbb{C}$, *we have*

$$B(x,r) \cap (A \backslash E) \neq \emptyset.$$

The set A *is said to be open if each element of* A *is an interior point of* A. *A subset* $B \subseteq E$ *is said to be closed if it contains each of its limit point. The family*

$$F = \{B(x,r) : x \in E,\ 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ *on* E.
Suppose x_n *is a sequence in* E *and* $x \in E$. *If for all* $c \in \mathbb{C}$, *with* $0 \prec c$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$, $\|x_n - x_{n+m}\| \prec c$, *then* $\{x_n\}$ *is called a Cauchy sequence in* $(E, \|.\|)$. *If every Cauchy sequence is convergent in* $(E, \|.\|)$, *then* $(E, \|.\|)$ *is called a complex valued Banach space.*

Example 2.1. *([21]). Let* $E = \mathbb{C}$ *be the set of complex numbers. Define* $\|.\| : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *by*

$$\|z_1 - z_2\| = |x_1 - x_2| + i|y_1 - y_2| \quad \forall z_1, z_2 \in \mathbb{C},$$

where $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$. *Clearly,* $(\mathbb{C}, \|.\|)$ *is a complex valued normed linear space.*

Example 2.2. *([21]). Let* $(C[0, \frac{\pi}{2}], \|.\|_\infty)$ *be the space of all continuous complex-valued functions on a closed interval* $[0, \frac{\pi}{2}]$, *endowed with the Chebyshev norm*

$$\|x - y\|_\infty = \max_{t \in [0, \frac{\pi}{2}]} |x(t) - y(t)| e^{it},\ x, y \in C[0, \frac{\pi}{2}],\ t \in [0, \frac{\pi}{2}].$$

Then $(C[0, \frac{\pi}{2}], \|.\|_\infty)$ *is a complex valued Banach space, since the elements of* $C[0, \frac{\pi}{2}]$ *are continuous functions, and convergence with respect to the Chebyshev norm* $\|.\|_\infty$ *corresponds to uniform convergence. We can easily show that every Cauchy sequence of continuous functions converges to a continuous function, i.e. an element of the space* $C[0, \frac{\pi}{2}]$.

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2020, No. 1, pp. 707 - 717

Jaggi and Dass [25] established an extension of Banach's fixed point theorem by using the following rational type contractive inequality condition: Let (X, d) be a metric space and $T\colon X \to X$ a continuous mapping. For $\alpha,\ \gamma \in [0,1)$ with $\alpha + \gamma < 1$, we have

$$d(Tx,Ty) \leq \frac{\alpha d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \gamma d(x,y),\ \forall\ x,\ y \in X. \tag{2.1}$$

Indeed, Dass and Gupta [26] introduced the notion of rational type contractive inequality condition and followed by Jaggi [27]. However, the setbacks in the three articles above and some other ones in the literature were (i) that T must be continuous; and (ii) the restriction $\alpha + \gamma < 1$.
Both restrictive conditions have been removed from the article of Olatinwo and Ishola [10].

In 2013, Khan [24] introduced the Picard-Mann hybrid iterative process as follows. Let D be a nonempty convex subset of a normed space X and let $T : D \to D$ be a mapping. The Picard-Mann iterative process $\{m_n\}_{n=0}^{\infty}$ is defined by

$$\begin{cases} m_0 \in D, \\ m_{n+1} = Tz_n, \\ z_n = (1-\alpha_n)m_n + \alpha_n Tm_n,\ n = 0,1,2,\cdots, \end{cases} \tag{2.2}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$.
If $\alpha_n = 0$ in (2.2), then we obtain

$$m_{n+1} = Tm_n,\ n = 0,1,2,\cdots,\ m_0 \in X, \tag{Δ}$$

which is called the Picard iteration $\{m_n\}_{n=0}^{\infty}$.
Khan [24] proved that the iterative process defined in (2.2) converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde [8] for contractive mappings.

Let D be a nonempty convex subset of a normed space X and let $T : D \to D$ be a mapping. Okeke and Abbas [6] defined the Picard-Krasnoselskii hybrid iterative process $\{x_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = Ty_n, \\ y_n = (1-\lambda)x_n + \lambda Tx_n,\ n = 0,1,2,\cdots,\ \lambda \in [0,1]. \end{cases} \tag{2.3}$$

The authors proved that this new hybrid iteration process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes in the sense of Berinde [8]. The authors used this iterative process to find the solution of delay differential equations. Moreover, Okeke [21] proved that the Picard-Mann hybrid iterative process [24] have the same rate of convergence as the Picard-Krasnoselskii hybrid iterative process [6].

Recently, Okeke [5] introduced the Picard-Ishikawa hybrid iterative process $\{x_n\}_{n=0}^{\infty}$ as follows: for any fixed x_1 in D, construct the sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in D, \\ x_{n+1} = Tv_n, \\ v_n = (1-\alpha_n)x_n + \alpha_n Tu_n \\ u_n = (1-\beta_n)x_n + \beta_n Tx_n,\ n = 0,1,2,\cdots, \end{cases} \tag{2.4}$$

where $\{\alpha_n\},\ \{\beta_n\} \subset [0,1]$. The author proved that this new hybrid iterative process converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, Noor [23], Picard-Mann [24] and Picard-Krasnoselskii [6] iterative processes in the sense of Berinde [8]. The author applied this new iteration process in finding the solution of delay differential equations in Banach spaces.

Following the results of Das and Debata [28] we now modify iterative scheme (2.4) to involve three mappings satisfying a new contractive condition of rational type. We shall employ the resulting

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VOL. 2020, NO. 1, PP. 707 - 717

iterative scheme in proving a convergence result for three mappings in complex-valued Banach spaces in a subsequent section in this paper.

$$\begin{cases} x_0 \in D, \\ x_{n+1} = T_1 v_n, \\ v_n = (1-\alpha_n)x_n + \alpha_n T_2 u_n \\ u_n = (1-\beta_n)x_n + \beta_n T_3 x_n, \ n = 0, 1, 2, \cdots, \end{cases} \tag{2.5}$$

where $\{\alpha_n\}, \ \{\beta_n\} \subset [0,1]$ and $T_1, \ T_2$ and T_3 are three mappings. The iterative process in (2.5) shall be called the *generalized Picard-Ishikawa hybrid iterative process.*

We shall employ the following contractive condition to establish our results:
Let (X,d) be a metric space and $T\colon X \to X$ a mapping. There exist $\alpha, \ k, \ q, \ r, \ \nu, \ \mu,$ $\eta \in \mathbb{R}^+$ and $\gamma \in [0,1)$ such that $\forall \ x, \ y \in X$, we have

$$d(Tx,Ty) \leq \alpha \frac{[k + [d(x,Tx)]^\nu][d(y,Ty)]^r[d(y,Tx)]^q}{\mu d(y,Tx) + \eta d(x,Ty) + d(x,y)} + \gamma d(x,y), \tag{$\nabla\nabla$}$$

with $\mu d(y,Tx) + \eta d(x,Ty) + d(x,y) > 0$.

Remark 2.1. *(i) In* $(\nabla\nabla)$, T *is not necessarily continuous and only* $\gamma \in [0,1)$ *while* $\alpha \geq 0$.
(ii) If $\nu = 1$ *in* $(\nabla\nabla)$, *then we obtain one of the contractive conditions in the article [10] (which in itself is more general than several others in the literature including those rational type contractive conditions mentioned earlier in the present article).*

Our interest in the present paper is to prove the existence of a unique fixed point of the mapping satisfying rational type contractive inequality condition $(\nabla\nabla)$ in complex-valued Banach spaces. We approximate this fixed point via the Picard-Ishikawa hybrid iterative process and other faster fixed point iterations defined above. Furthermore, we apply our results in finding solutions of delay differential equations.

The following lemmas will be needed in this study.

Lemma 2.1. *[29] Let* $\{s_n\}_{n=0}^\infty$ *be a sequence of positive real numbers including zero satisfying*

$$s_{n+1} \leq (1-\mu_n)s_n. \tag{2.6}$$

If $\{\mu_n\}_{n=0}^\infty \subset (0,1)$ *and* $\sum_{n=0}^\infty \mu_n = \infty$, *then* $\lim_{n\to\infty} s_n = 0$.

Suppose that

$$F_T = \{x^\star \in E \mid Tx^\star = x^\star\}$$

is the fixed point set of T.

3 Existence and approximation of fixed points in complex-valued Banach spaces

In this section, we prove the existence of a unique fixed point of mappings satisfying contractive condition $(\nabla\nabla)$ in complex-valued Banach spaces. We also approximate this fixed point via faster fixed point iterative procedures. We start with the following remark:

Remark 3.1. *Since the metric is induced by the norm, then we have* $d(x,y) = \|x-y\|, \ x, \ y \in X$, *whenever we are working in the normed, or, Banach space setting.*

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2020, No. 1, pp. 707 - 717

So, in the complex-valued Banach space setting, we have the following variant of $(\nabla\nabla)$ which shall be employed in proving our results: Let $(E, \|.\|)$ be a complex-valued Banach space and $T\colon E \to E$ a mapping. There exist $\alpha,\ k,\ q,\ r,\ \nu,\ \mu,\ \eta \in \mathbb{R}^+$ and $\gamma \in [0,1)$ such that $\forall\ x,\ y \in E$, we have

$$\|Tx - Ty\| \precsim \alpha \frac{[k + [\|x - Tx\|]^{\nu}][\|y - Ty\|]^r[\|y - Tx\|]^q}{\mu\|y - Tx\| + \eta\|x - Ty\| + \|x - y\|} + \gamma\|x - y\|, \tag{$\star\nabla$}$$

with $\mu\|y - Tx\| + \eta\|x - Ty\| + \|x - y\| > 0$.

Theorem 3.1. *Let $(E, \|.\|)$ be a complex-valued Banach space and $T\colon E \to E$ a mapping satisfying $(\star\nabla)$. For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty} \subset E$ defined by (Δ) be the Picard iteration associated to T. Then, T has a unique fixed point.*

Proof. For $x_0 \in E$, $\{x_n\}_{n=0}^{\infty} \subset E$ defined by $x_{n+1} = Tx_n,\ n = 0, 1, 2, \cdots$, is the Picard iteration associated to T. So, using $(\star\nabla)$, we have

$$\begin{aligned}\|x_n - x_{n+1}\| &= \|Tx_{n-1} - Tx_n\| \\ &\precsim \alpha \tfrac{[k+[\|x_{n-1}-Tx_{n-1}\|]^{\nu}][\|x_n-Tx_n\|]^r[\|x_n-Tx_{n-1}\|]^q}{\mu\|x_n-Tx_{n-1}\|+\eta\|x_{n-1}-Tx_n\|+\|x_{n-1}-x_n\|} + \gamma\|x_{n-1} - x_n\| \\ &= \gamma\|x_{n-1} - x_n\|,\end{aligned}$$

and from which we have that

$$\|x_n - x_{n+1}\| \precsim \gamma\|x_{n-1} - x_n\| \precsim \gamma^2\|x_{n-2} - x_{n-1}\| \precsim \cdots \precsim \gamma^n\|x_0 - x_1\|. \tag{3.1}$$

Now, for $\ell \in \mathbb{N}$, using (3.1) inductively in the repeated application of the triangle inequality yields

$$\|x_n - x_{n+\ell}\| \precsim \sum_{j=n}^{n+\ell-1} \|x_j - x_{j+1}\| \precsim \sum_{j=n}^{n+\ell-1} \beta^j\|x_0 - x_1\| \precsim \frac{\gamma^n(1-\gamma^{\ell})}{1-\gamma}\|x_0 - x_1\|, \tag{3.2}$$

and we have from (3.2) that

$$|\|x_n - x_{n+\ell}\|| \precsim \frac{\gamma^n(1-\gamma^{\ell})}{1-\gamma}|\|x_0 - x_1\|| \to 0 \text{ as } n \to \infty. \tag{3.3}$$

Thus, we have from (3.3) that the sequence $\{x_n\}$ is a Cauchy sequence in E. Since $(E, \|.\|)$ is complete, then $\{x_n\}$ converges to some point $v \in E$. That is, $\lim\limits_{n\to\infty} x_n = v$.

We now show that $Tv = v$. Suppose not. That is, we have $|\|Tv - v\|| > 0$. Then, by the contractive condition $(\star\nabla)$, we have

$$\begin{aligned}\|v - Tv\| &\precsim \|v - x_{n+1}\| + \|x_{n+1} - Tv\| = \|Tx_n - Tv\| + \|v - x_{n+1}\| \\ &\precsim \alpha \tfrac{[k+[\|x_n-Tx_n\|]^{\nu}][\|v-Tv\|]^r[\|v-Tx_n\|]^q}{\mu\|v-Tx_n\|+\eta\|x_n-Tv\|+\|x_n-v\|} + \gamma\|x_n - v\| + \|v - x_{n+1}\| \\ &= \alpha \tfrac{[k+[\|x_n-x_{n+1}\|]^{\nu}][\|v-Tv\|]^r[\|v-x_{n+1}\|]^q}{\mu\|v-x_{n+1}\|+\eta\|x_n-Tv\|+\|x_n-v\|} + \gamma\|x_n - v\| + \|v - x_{n+1}\|.\end{aligned} \tag{3.4}$$

From (3.4), we have that $|\|v - Tv\|| \to 0$ as $n \to \infty$ since $|\|x_n - x_{n+1}\|| \to 0$ as $n \to \infty$,
$|\|v - x_{n+1}\|| \to 0$ as $n \to \infty$, $|\|x_n - v\|| \to 0$ as $n \to \infty$ and
$|\|x_n - Tv\|| \to |\|v - Tv\||$ as $n \to \infty$.
So, $|\|v - Tv\|| \precsim 0$ is a contradiction since norm is nonnegative. Hence,
$|\|v - Tv\|| = 0 \iff Tv = v$.

Next, we now prove that $v \in F_T$ is unique. To this end, assume that there exists another $w \in F_T$ such that $Tv = v,\ Tw = w,\ v \neq w,\ |\|v - w\|| > 0$. Then, by $(\star\nabla)$ again, we have

$$\begin{aligned}0 < |\|v - w\|| = |\|Tv - Tw\|| &\precsim \alpha \tfrac{[k+[|\|v-Tv\||]^{\nu}][|\|w-Tw\||]^r[|\|w-Tv\||]^q}{\mu|\|w-Tv\||+\eta|\|v-Tw\||+|\|v-w\||} + \gamma|\|v - w\|| \\ &= \gamma|\|v - w\||,\end{aligned}$$

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2020, No. 1, pp. 707 - 717

from which we obtain that $(1-\gamma)|||v-w||| \precsim 0$.
Since $1-\gamma > 0,\ \gamma \in [0,1)$, we have $|||v-w||| \precsim 0$ (which is a contradiction).
Hence, we obtain $|||v-w||| = 0 \iff w = v$,
thus, proving the uniqueness of the fixed point of T. □ □

Remark 3.2. *We remark that Theorem 3.1 generalizes and extends the Banach contraction mapping principle in [8], since the contractive condition* $(\star\nabla)$ *reduces to the Banach's contractive condition if* $\alpha = 0$. *Theorem 3.1 is also an extension of Theorem 6 in Olatinwo and Ishola [10].*

Next, we prove the following results for three mappings in complex-valued Banach spaces.

Theorem 3.2. *Let D be a nonempty closed convex subset of a complex-valued Banach space $(E, \|.\|)$ and $T_j\colon D \to D$ $(j = 1,2,3)$ three mappings satisfying $(\star\nabla)$, $\forall\ x,\ y \in D$. For $x_0 \in D$, let $\{x_n\}_{n=0}^{\infty} \subset D$ be the generalized Picard-Ishikawa hybrid iterative process defined by (2.5), with $\alpha_n,\ \beta_n \in [0,1]$ such that and $\sum_{k=0}^{\infty} \alpha_k = \infty$. Then, $\{x_n\}_{n=0}^{\infty}$converges strongly to a unique common fixed point $x^\star$ of $T_1,\ T_2$ and T_3.*

Proof. Recall that the contractive condition $(\star\nabla)$ guarantees the existence and the uniqueness of the fixed point p of T. We now show that $\{x_n\}_{n=0}^{\infty}$ defined in (2.5) converges strongly to a unique common fixed point $x^\star$ of $T_1,\ T_2$ and T_3 as $n \to \infty$. We now use the iteration defined in (2.5) and the fact that each of the mappings $T_1,\ T_2,\ T_3$ satisfies the contractive condition $(\star\nabla)$: Therefore, we have

$$\begin{aligned}\|x_{n+1}-x^\star\| &= \|T_1v_n - T_1x^\star\| \\ &\precsim \alpha\frac{[k+[\|v_n-T_1v_n\|]^\nu][\|x^\star-T_1x^\star\|]^r[\|x^\star-T_1v_n\|]^q}{\mu\|x^\star-T_1v_n\|+\eta\|v_n-T_1x^\star\|+\|v_n-x^\star\|} + \gamma\|v_n-x^\star\| \\ &= \gamma\|v_n-x^\star\|. \end{aligned} \tag{3.5}$$

Next, we have

$$\begin{aligned}\|v_n-x^\star\| &= \|(1-\alpha_n)x_n+\alpha_nT_2u_n-x^\star\| \\ &\precsim (1-\alpha_n)\|x_n-x^\star\|+\alpha_n\|T_2u_n-T_2x^\star\| \\ &\precsim (1-\alpha_n)\|x_n-x^\star\|+\alpha_n\left[\alpha\frac{[k+[\|u_n-T_2u_n\|]^\nu][\|x^\star-T_2x^\star\|]^r[\|x^\star-T_2u_n\|]^q}{\mu\|x^\star-T_2u_n\|+\eta\|u_n-T_2x^\star\|+\|u_n-x^\star\|}+\gamma\|u_n-x^\star\|\right] \\ &= (1-\alpha_n)\|x_n-x^\star\|+\alpha_n\gamma\|u_n-x^\star\|. \end{aligned} \tag{3.6}$$

Also, we obtain

$$\begin{aligned}\|u_n-x^\star\| &= \|(1-\beta_n)x_n+\beta_nT_3x_n-x^\star\| \\ &\precsim (1-\beta_n)\|x_n-x^\star\|+\beta_n\|T_3x_n-T_3x^\star\| \\ &\precsim (1-\beta_n)\|x_n-x^\star\|+\beta_n\left[\alpha\frac{[k+[\|u_n-T_3u_n\|]^\nu][\|x^\star-T_3x^\star\|]^r[\|x^\star-T_3u_n\|]^q}{\mu\|x^\star-T_3u_n\|+\eta\|u_n-T_3x^\star\|+\|u_n-x^\star\|}+\gamma\|u_n-x^\star\|\right] \\ &= [1-\beta_n(1-\gamma)]\|x_n-x^\star\|. \end{aligned} \tag{3.7}$$

Using (3.7) in (3.6) gives

$$\|v_n-x^\star\| \precsim \{1-\alpha_n(1-\gamma[1-\beta_n(1-\gamma)])\}\,\|x_n-x^\star\| = g_n\|x_n-x^\star\|. \tag{3.8}$$

Using (3.8) in (3.5) yields

$$\|x_{n+1}-x^\star\| \precsim \gamma g_n\|x_n-x^\star\|, \tag{3.9}$$

where

$$g_n = 1-\alpha_n(1-\gamma[1-\beta_n(1-\gamma)]) \precsim e^{-\alpha_n(1-\gamma[1-\beta_n(1-\gamma)])}.$$

We have from (3.9) that

$$\begin{aligned}\|x_{n+1}-x^\star\| &\precsim \gamma g_n\|x_n-x^\star\| \\ &\precsim \gamma^2 g_ng_{n-1}\|x_{n-1}-x^\star\| \precsim \cdots \precsim \gamma^{n+1}g_ng_{n-1}\cdots g_1g_0\|x_0-x^\star\|, \end{aligned} \tag{3.10}$$

from which it follows that

$$\|x_{n+1} - x^\star\| \leq \|x_0 - x^\star\|\gamma^{n+1}e^{-(1-\gamma[1-\beta_n(1-\gamma)])\sum_{k=0}^{n}\alpha_k}. \tag{3.11}$$

Therefore, we obtain from (3.11) that

$$|||x_{n+1} - x^\star||| \leq |||x_0 - x^\star|||\gamma^{n+1}e^{-(1-\gamma[1-\beta_n(1-\gamma)])\sum_{k=0}^{n}\alpha_k} \to 0 \text{ as } n \to \infty, \tag{3.12}$$

that is, $\lim_{n\to\infty} \|x_n - x^\star\| = 0$.
Hence, the new generalized Picard-Ishikawa hybrid iterative process $\{x_n\}_{n=0}^{\infty}$ defined in (2.5) converges strongly to a unique common fixed point $x^\star$ of T_1, T_2 and T_3 as $n \to \infty$. □

□

Next, we establish the following Corollaries of Theorem 3.2.

Corollary 3.3. *Let D be a nonempty closed convex subset of a complex-valued Banach space $(E, \|.\|)$ and $T\colon D \to D$ a mapping satisfying $(\star\nabla)$, $\forall\, x,\ y \in D$. For $x_0 \in D$, let $\{x_n\}_{n=0}^{\infty} \subset D$ be the Picard-Ishikawa hybrid iterative process defined by (2.4), with $\alpha_n,\ \beta_n \in [0,1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point $x^\star$ of T.*

Proof. Corollary 3.3 is established if $T_1 = T_2 = T_3 = T$ in the statement of Theorem 3.2 and its proof above. □ □

Corollary 3.4. *Let D be a nonempty closed convex subset of a complex-valued Banach space $(E, \|.\|)$ and $T\colon D \to D$ a mapping satisfying $(\star\nabla)$, $\forall\, x,\ y \in D$. For $m_0 \in D$, let $\{m_n\}_{n=0}^{\infty} \subset D$ be the Picard-Mann hybrid iterative process defined by (2.2), with $\alpha_n \in [0,1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, $\{m_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point $x^\star$ of T.*

Corollary 3.5. *Let D be a nonempty closed convex subset of a complex-valued Banach space $(E, \|.\|)$ and $T\colon D \to D$ a mapping satisfying $(\star\nabla)$, $\forall\, x,\ y \in D$. For $x_0 \in D$, let $\{x_n\}_{n=0}^{\infty} \subset D$ be the Picard-Krasnoselskii hybrid iterative process defined by (2.3), with $\lambda \in [0,1]$ such that $\sum_{k=0}^{\infty}\alpha_k = \infty$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to a unique fixed point $x^\star$ of T.*

4 Results in cone metric spaces with Banach algebras

Suppose $\mathcal{A}$ denote a real Banch algebra. This means that $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for each $x, y, z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):
(i) $(xy)z = x(yz)$;
(ii) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
(iv) $\|xy\| \leq \|x\|\|y\|$.

In the present paper, we assume that $\mathcal{A}$ has a unit; i.e. a multiplicative identity e such that $ex = xe = x$ for each $x \in \mathcal{A}$. The inverse of x is denoted by x^{-1} (see, e.g. Rudin [30]).

The work of Öztürk and Başarir [31] generalized the concept of cone metric spaces introduced by Huang and Zhang [11] by replacing a Bancach space with a Banach algebra $\mathcal{A}$ in cone metric spaces. They called this new concept BA-cone metric spaces. Abbas *et al.* [13] proved that complex valued metric spaces introduced in [12] is a BA-cone metric space, that is a cone metric space over a solid cone in commutative division Banach algebra $\mathcal{A}$ (see, [13], [31]). Readers interested in further studies in cone metric spaces may see Olaleru *et al.* [32] and the references therein.

Perhaps unaware of the work of Öztürk and Başarir [31], in 2013 Liu and Xu [33] introduced the concept of cone metric spaces with Banach algebras, by replacing Banach spaces with Banach algebras as the underlying space of cone metric spaces. They proved that fixed point theorems in the setting of cone metric spaces with Banach algebras are more useful than the standard results in cone metric spaces and that results in cone metric spaces with Banach algebras cannot be reduced to corresponding results in cone metric spaces. .

Example 4.1. *([33]). Let* $\mathcal{A} = M_n(\mathbb{R}) = \{a = (a_{ij})_{n\times n} | a_{ij} \in \mathbb{R} \ \text{ for all } \ 1 \le i,j \le n\}$ *be the algebra of all* n*-square real matrices, and define the norm*

$$\|a\| = \sum_{1\le i,j\le n} |a_{ij}|.$$

Then $\mathcal{A}$ *is a real Banach algebra with the unit* e*, the identity matrix.*

Let $\mathcal{P} = \{a \in \mathcal{A} | a_{ij} \ge 0 \ \text{ for all } \ 1 \le i,j \le n\}$*. Then* $\mathcal{P} \subset \mathcal{A}$ *is a normal cone with normal constant* $M = 1$*.*

Let $X = M_n(\mathbb{R})$*, and define the metric* $d : X \times X \to \mathcal{A}$ *by*

$$d(x,y) = d((x_{ij})_{n\times n}, (y_{ij})_{n\times n}) = (|x_{ij} - y_{ij}|)_{n\times n} \in \mathcal{A}.$$

Then (X,d) *is a cone metric space with a Banach algebra* $\mathcal{A}$*.*

Example 4.2. *([33]). Let* $\mathcal{A} = \ell^1 = \{a = (a_n)_{n\ge 0} | \sum_{n=0}^{\infty} |a_n| < \infty\}$ *with convolution as multiplication:*

$$ab = (a_n)_{n\ge 0}(b_n)_{n\ge 0} = \left(\sum_{i+j=n} a_i b_j\right)_{n\ge 0}.$$

Thus $\mathcal{A}$ *is a Banach algebra. The unit* e *is* $(1,0,0,\cdots)$*.*

Let $\mathcal{P} = \{a = (a_n)_{n\ge 0} \in \mathcal{A} | a_n \ge 0 \ \text{ for all } \ n \ge 0\}$*, which is a normal cone in* $\mathcal{A}$*. And let* $X = \ell^1$ *with the metric* $d : X \times X \to \mathcal{A}$ *defined by*

$$d(x,y) = d((x_n)_{n\ge 0}, (y_n)_{n\ge 0}) = (|x_n - y_n|)_{n\ge 0}.$$

Then (X,d) *is a cone metric space with* $\mathcal{A}$*.*

Inspired by the results above, we now prove that results in complex-valued Banach spaces involving mappings satisfying mapping satisfying rational type contractive inequality condition are true in the context of cone metric spaces with Banach algebras. Moreover, we show that our results cannot be deduced in cone metric spaces.

Theorem 4.1. *Let* D *be a nonempty closed convex subset of a complete cone metric space with Banach algebras* $(\mathcal{A}, \|.\|)$ *and* $T\colon D \to D$ *a mapping satisfying* $(\star\nabla)$*,* $\forall\ x,\ y \in D$*. For* $x_0 \in D$*, let* $\{x_n\}_{n=0}^{\infty} \subset D$ *be generated by the Picard-Ishikawa hybrid iterative process defined in (2.4), with* $\{\alpha_n\},\ \{\beta_n\} \in [0,1]$ *such that* $\sum_{k=0}^{\infty} \alpha_k = \infty$*. Then,* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to a unique fixed point* $x^\star$ *of* T*.*

Proof. The proof of Theorem 4.1 follows similar lines of argument as in the proof of Theorem 3.2. □

□

Remark 4.1. *Observe that the result of Theorem 4.1 was proved for mappings satisfying rational type contractive inequality condition. The result cannot be reduced to some corresponding results in cone metric spaces and this shows that our result is valid in cone metric spaces with Banach algebras.*

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