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Lomax-Weibull Distribution with Properties and Applications in Lifetime Analysis

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Abstract

The paper introduces a new distribution called the Lomax-Weibull distribution using the competing risk approach of constructing lifetime distributions. Some structural and mathematical properties of the proposed lifetime distribution are considered. Parameter estimation of the Lomax Weibull distribution is obtained using maximum likelihood estimation. The applicability and flexibility of the new distribution in lifetime analysis is illustrated with the aid of two real life examples.

Keywords And Phrases: Competing risk approach, Lomax distribution, Weibull distribution, Lomax Weibull distribution, Moments, Entropy, Maximum likelihood estimation.
MSC2010: 60E05; 62-07

1 Introduction

Classical distributions such as Weibull and Lomax distributions ([22], [23] and [11]) have been known to be incapable of fitting data from complex systems with non-monotonic aging phenomena. This reason has led to construction of new distributions for some years using several approaches presented in [16]. One of the approaches involves mixing two or more lifetime distributions to obtain the n-fold competing risk models for modelling monotonic and nonmonotonic failure data whicih could be obtained from complex systems used in diverse scientific fields. Several researchers have developed lifetime distributions using the method of constructing competing risk models. For a series system with i^{th} (i=1,2,..,n) independent components following different distributions, the distribution of the system is given as;

$$F(x) = 1 - \prod_{i=1}^{n} S_i(x), \tag{1.1}$$

where $S_i(x)$ is the survival function of the distribution of the i^{th} component. New distributions have been proposed using equation (1.1) by researchers for $n = 2$. These distributions have been found useful in modeling lifetime data analysis. Examples are the Additive Weibull distribution ([24]), modified Weibull distribution ([19]) and log-logistic Weibull distribution ([15]). Other generalized and modified Lomax distributions such as Gumbel Lomax distribution ([21]), Poisson Lomax distribution ([2]), transformer Lomax distributions ([4]), Rayleigh Lomax distribution ([8]), transmuted Lomax distribution ([5]), McDonald Lomax distribution ([10]), gamma Lomax distribution ([7]), Odd Lindley-Lomax model ([3]) and power Lomax distribution ([17]) have been developed using different constructing methods discussed in [16].
The paper proposes a new lifetime distribution called the Lomax-Weibull (LW) distribution. The distribution can model monotonic and nonmonotonic failure properties of complex system. Its applications can be employed in engineering, finance and other scientific areas where lifetime analysis are needed. The flexibility and usefulness of the LW distribution in fitting lifetime data is presented in the paper.

The organization of the paper is presented as follows. Section 2 presents the construction of the LW distribution. Section 3 presents the quantile functions, momemts and mean deviattions of the LW distribution. In section 4, statistical properties such as order statistics, entropy measures and residual lifetimes for the new distribution are presented. Section 5 presents parameter estimation of the LW distribution. Section 6 presents applications of the LW distribution in comparison with some known distribution in literature using two lifetime data sets. In section 7, the conclusion of the paper is presented.

2 Construction of the Lomax-Weibull (LW) distribution

The section presents the construction of the LW distribution. Suppose a series system has two independent components, for which one component follows the Lomax distribution and the second component follows the Weibull distribution. Employing equation (1.1), the cdf of the LW distribution is given as;

$$F_{LW}(x; \alpha, \beta, \gamma, \lambda) = 1 - (1 + \beta x)^{-\alpha} e^{-\gamma x^{\lambda}}; x > 0, \alpha > 0, \beta > 0, \gamma > 0, \lambda > 0, \tag{2.2}$$

Differentiating equation (2.2) with respect to x, the probability density function (pdf) of the LW distribution is given as

$$f_{LW}(x) = (\alpha\beta + \gamma\lambda x^{\lambda-1}(1 + \beta x))(1 + \beta x)^{-(\alpha+1)} e^{-\gamma x^{\lambda}}. \tag{2.3}$$

The corresponding survival and hazard functions are given by

$$S_{LW}(x) = (1 + \beta x)^{-\alpha} e^{-\gamma x^{\lambda}} \tag{2.4}$$

and

$$h_{LW}(x) = (\alpha\beta + \gamma\lambda x^{\lambda-1}(1 + \beta x))(1 + \beta x)^{-1}. \tag{2.5}$$

where $S_{LW}(x)$ and $h_{LW}(x)$ denote the survival and hazard functions of the LW distribution.

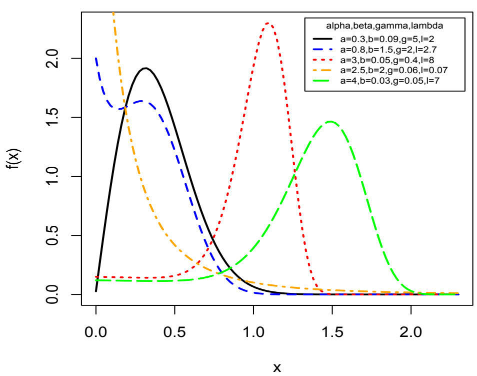

Figure 1: LW Density function

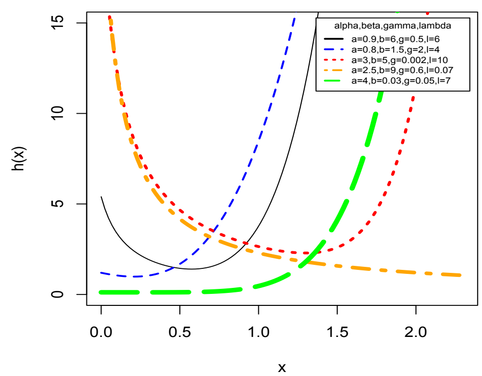

Figure 2: LW Hazard function

Figures (1) and (2) show the shapes of the density and hazard function at different combinations of parameter values of the LW distribution. It is seen in Figure (2) that the LW distribution exhibits different failure rates shapes that can be experienced in real lifetime events such as increasing, decreasing, $\mathcal{J}$-shaped and bathtub-shaped failure patterns.
It is necessary to expand the *pdf* of the LW distribution to allow for easy manipulation of some mathematical properties of the distribution. It is known that

$$(1+x)^{-n} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j-1}{j} x^j.$$

It follows that

$$(1+\beta x)^{-(\alpha+1)} = \textstyle\sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j}{j} \beta^j x^j.$$

Substituting the resulting expression into equation (3),

$$f_{LW}(x) = \sum_{j=0}^{\infty} \Omega_j (\alpha\beta x^j + \gamma\lambda x^{j+\lambda-1}(1+\beta x)) e^{-\gamma x^{\lambda}} \tag{2.6}$$

where $\Omega_j = (-1)^j \binom{\alpha+j}{j} \beta^j$. Equation (2.6) will be employed in the simplications of some of the properties of the LW distribution.

3 Quantile function, moments and mean deviations

3.1 Quantile function

If $q \in (0,1)$, then the quantile function of the LW distribution can be derived from the solution of

$$F_{LW}(x_q) = q. \tag{3.7}$$

Substituting equation (2.2) into equation (3.7), we obtain which results in;

$$(1+\beta x_q)^{-\alpha} e^{-\gamma x_q^{\lambda}} = 1-q,$$

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which, further, simplifies as

$$\alpha log(1+\beta x_q)+\gamma x_q^\lambda+log(1-q)=0. \tag{3.8}$$

The root, x_q, gives the unique solution of the nonlinear equation for every $q \in (0,1)$. Equation (3.8) can be used for random number generation at different parameter values of $\alpha > 0, \beta > 0, \gamma > 0$ and $\lambda > 0$ of LW distribution as presented in Table 1.

Table 1: Quantile values for the LW distribution for different parameter sets

q	$\alpha=0.7,\beta=0.3,$ $\gamma=0.1,\lambda=0.9$	$\alpha=2,\beta=0.5,$ $\gamma=1.4,\lambda=7$	$\alpha=0.5,\beta=1.5,$ $\gamma=1.8,\lambda=3$	$\alpha=0.4,\beta=5,$ $\gamma=0.6,\lambda=1$	$\alpha=8,\beta=0.4,$ $\gamma=0.6,\lambda=0.8$
1	0.3383	0.1082	0.1470	0.0437	0.0237
2	0.7648	0.2360	0.2886	0.1005	0.0525
3	1.2980	0.3882	0.4048	0.1751	0.0865
4	1.9778	0.5534	0.5034	0.2749	0.1268
5	2.8704	0.6878	0.5928	0.4109	0.1758
6	4.0933	0.7840	0.6796	0.6016	0.2376
7	5.8784	0.8603	0.7695	0.8815	0.3198
8	8.7773	0.9305	0.8715	1.3272	0.4404
9	14.6747	11.0085	1.0077	2.1852	0.6594

3.2 Raw moment

Let the r^{th} raw moment of variable X following the LW distribution be given as $E[X_{LW}^r]$, then it can be obtained by the relation given as

$$E[X_{LW}^r]=\int_0^\infty x^r f_{LW}(x)dx. \tag{3.9}$$

Substituting equation (2.3) into equation (3.9), we obtain

$$E[X_{LW}^r]=\textstyle\sum_{j=0}^\infty \Omega_j \int_0^\infty (\alpha\beta x^{r+j}+\gamma\lambda x^{r+j+\lambda-1}(1+\beta x))e^{-\gamma x^\lambda}dx$$

Letting $m=\gamma x^\lambda$, then

$$E[X_{LW}^r]=\sum_{j=0}^\infty \Omega_j\left[\frac{\alpha\beta\Gamma\left(\frac{r+j+1}{\lambda}\right)}{\lambda\gamma^{\frac{r+j+1}{\lambda}}}+\frac{\Gamma\left(\frac{r+j}{\lambda}+1\right)}{\gamma^{\frac{r+j}{\lambda}}}+\frac{\beta\Gamma\left(\frac{r+j+1}{\lambda}+1\right)}{\gamma^{\frac{r+j+1}{\lambda}}}\right] \tag{3.10}$$

gives the r^{th} raw moments of the LW distribution. Presented in Table 2 is the first four moments of the LW distribution, its kurtosis and skewness.

Table 2: Moments of LW distribution for different parameter values

Quantities	*Parameters*	$\gamma = 0.5,\lambda = 1.5$	$\gamma = 1.5,\lambda = 1.5$	$\gamma = 0.5,\lambda = 0.5$	$\gamma = 1,\lambda = 1$
μ	$\alpha = 1,\beta = 3$	0.5049	0.3404	0.2671	0.3856
μ_2		0.6187	0.2323	04145	0.4096
μ_3		1.1908	0.2305	2.1631	0.7952
μ_4		3.0263	0.2926	27.1320	2.3132
Kurtosis		10.3751	7.3686	212.1901	20.3575
Skewness		2.3292	1.8149	9.2990	3.2719
μ	$\alpha = 1.8,\beta = 0.7$	0.6819	0.4408	0.3464	0.5850
μ_2		0.8945	0.3323	0.5262	0.5850
μ_3		1.6993	0.3420	2.0546	1.1067
μ_4		4.1551	0.4379	17.3898	3.0055
Kurtosis		7.4083	5.7234	90.1875	13.7968
Skewness		1.7884	1.4400	6.1458	2.5734
μ	$\alpha = 1.8,\beta = 3$	0.2788	0.2205	0.1549	0.2280
μ_2		0.2170	0.1102	0.1104	0.1498
μ_3		0.3059	0.0874	0.2581	0.1980
μ_4		0.6253	0.0938	1.6617	0.4286
Kurtosis		18.9364	11.0092	203.1463	29.9329
Skewness		3.2286	2.3500	8.4389	3.8952
μ	$\alpha = 0.2,\beta = 0.7$	1.3005	0.6518	0.7653	0.9134
μ_2		2.5831	0.6348	3.4321	1.7088
μ_3		6.6415	0.7941	39.6819	6.8668
μ_4		20.5680	1.1921	882.7319	18.6640
Kurtosis		4.5898	4.5013	95.3207	9.6096
Skewness		1.1431	1.1100	6.8090	2.0892

Dimensional plots for skewness and kurtosis for the LW distribution are presented in Figures (3) and (4) for some parameter values.

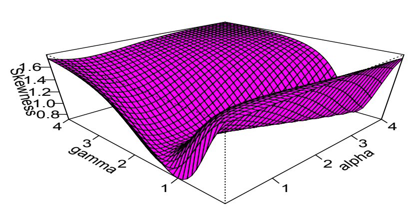

Figure 3: LW Skewness(α,0.6,γ,7)

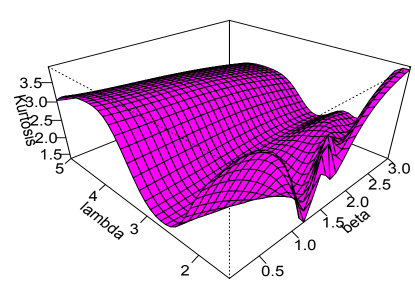

Figure 4: LW Kurtosis(0.08,β,0.06,λ)

3.3 Incomplete moment, conditional moment and moment generating function

The conditional moment for the LW distribution is derived using

$$E\left(X^r/X>t\right)=\tfrac{1}{S(t)}\int_t^\infty x^r f_{LW}(x).$$

This is given as

$$E\left(X^r/X>t\right)=\frac{1}{S(t)}\sum_{j=0}^{\infty}\Omega_j\left[\frac{\alpha\beta\Gamma_u\left(\frac{r+j+1}{\lambda},\gamma t^\lambda\right)}{\lambda\gamma^{\frac{r+j+1}{\lambda}}}+\frac{\Gamma_u\left(\frac{r+j}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j}{\lambda}}}+\frac{\beta\Gamma_u\left(\frac{r+j+1}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j+1}{\lambda}}}\right], \tag{3.11}$$

where $\Gamma_u\left(s,m\right)=\int_m^\infty x^{s-1}e^{-x}dx$.

The moment generating function of variable X following the LW distribution is given as

$$M_{X_{LW}}(t)=\int_0^\infty e^{tx}f_{LW}(x)dx=\sum_{j=0}^{\infty}\Omega_j\int_0^\infty\left(\alpha\beta x^j+\gamma\lambda x^{j+\lambda-1}(1+\beta x)\right)e^{tx-\gamma x^\lambda}dx \tag{3.12}$$

Substituting $e^{tx}=\sum_{k=0}^{\infty}\frac{(tx)^k}{k!}$ into equation (3.12), we obtain

$$M_{X_{LW}}(t)=\sum_{j=0}^{\infty}\Omega_j\sum_{k=0}^{\infty}\frac{t^k}{k!}\left(\frac{\alpha\beta\Gamma\left(\frac{k+j+1}{\lambda}\right)}{\lambda\gamma^{\frac{k+j+1}{\lambda}}}+\frac{\Gamma\left(\frac{k+j}{\lambda}+1\right)}{\gamma^{\frac{k+j}{\lambda}}}+\frac{\beta\Gamma\left(\frac{k+j+1}{\lambda}+1\right)}{\gamma^{\frac{k+j+1}{\lambda}}}\right) \tag{3.13}$$

The incomplete moment of variable X following the LW distribution can be obtained by the relation given as

$$I_{X_{LW}}(t)=\int_0^t x^r f_{LW}(x)dx.$$

This is obtained as

$$I_{X_{LW}}(t)=\sum_{j=0}^{\infty}\Omega_j\left[\frac{\alpha\beta\Gamma_l\left(\frac{r+j+1}{\lambda},\gamma t^\lambda\right)}{\lambda\gamma^{\frac{r+j+1}{\lambda}}}+\frac{\Gamma_l\left(\frac{r+j}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j}{\lambda}}}+\frac{\beta\Gamma_l\left(\frac{r+j+1}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j+1}{\lambda}}}\right], \tag{3.14}$$

3.4 Mean deviations

Mean deviations are measures of total variations of data in a given set from the mean and median of the data set. The mean deviations about the mean and median for any continuous lifetime distribution are defined as

$$MD(\mu)=\int_0^\infty|x-\mu|f(x)dx=2\mu F(\mu)-\int_0^\mu xf(x)dx$$

and

$$MD(M)=\int_0^\infty|x-M|f(x)dx=\mu-\int_0^M xf(x)dx.$$

where μ and M are the mean and median of the distribution respectively. The mean deviations about the mean and median for the LW distribution are given as

$$MD_{LW}(\mu)=-2\sum_{j=0}^{\infty}\Omega_j\left[\frac{\alpha\beta\Gamma_l\left(\frac{r+j+1}{\lambda},\gamma\mu^\lambda\right)}{\lambda\gamma^{\frac{r+j+1}{\lambda}}}+\frac{\Gamma_l\left(\frac{r+j}{\lambda}+1,\gamma\mu^\lambda\right)}{\gamma^{\frac{r+j}{\lambda}}}+\frac{\beta\Gamma_l\left(\frac{r+j+1}{\lambda}+1,\gamma\mu^\lambda\right)}{\gamma^{\frac{r+j+1}{\lambda}}}\right] + 2\mu\left[1-(1+\beta\mu)^{-\alpha}e^{-\gamma\mu^\lambda}\right], \tag{3.15}$$

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and

$$MD_{LW}(M) = -2\sum_{j=0}^{\infty}\Omega_j\left[\frac{\alpha\beta\Gamma_l\left(\frac{r+j+1}{\lambda},\gamma M^{\lambda}\right)}{\lambda\gamma^{\frac{r+j+1}{\lambda}}} + \frac{\Gamma_l\left(\frac{r+j}{\lambda}+1,\gamma M^{\lambda}\right)}{\gamma^{\frac{r+j}{\lambda}}} + \frac{\beta\Gamma_l\left(\frac{r+j+1}{\lambda}+1,\gamma M^{\lambda}\right)}{\gamma^{\frac{r+j+1}{\lambda}}}\right] + \mu. \quad (3.16)$$

where $\Gamma_l\left(s,m\right) = \int_0^m x^{s-1}e^{-x}dx$.

4 Order statistics, entropy measure and residual lifetimes

4.1 Order statistics

Let $X_{(r)}$ be the r^{th} order statistics for LW distribution. Suppose a random sample of size m be given as $X_1, X_2, ..., X_m$, then pdf of r^{th} order statistics for the class can be obtained from the general expression given by

$$f_{r:m}(x) = \frac{m!}{(r-1)!(m-r)!}[F_{LW}(x)]^{r-1}[1-F_{LW}(x)]^{m-r}f_{LW}(x).$$

Substituting the expansion of $[1-F_{LW}(x)]^{m-r}$ into the above expression, we obtain

$$f_{r:m}(x) = \frac{m!}{(r-1)!(m-r)!}\sum_{k=0}^{m-r}(-1)^k\binom{m-r+k}{k}[F_{LW}(x)]^{r+k-1}f_{LW}(x) \quad (4.17)$$

Substituting equations (2.2) and (2.3) into equation (4.17) and further simplification gives

$$f_{r:m}(x) = \frac{m!}{(r-1)!(m-r)!}\sum_{i=0}^{r+k-1}\sum_{k=0}^{m-r}(-1)^{i+k}\binom{m-r+k}{k}\binom{r+k-1}{i} \times \left(\alpha\beta+\gamma\lambda x^{\lambda-1}(1+\beta x)\right)(1+\beta x)^{-[(i+1)\alpha+1]}e^{-(i+1)\gamma x^{\lambda}}. \quad (4.18)$$

The corresponding cdf to equation (4.18) is given as

$$F_{r:m}(x) = \sum_{r=1}^{m}\sum_{k=0}^{m-r}\binom{m}{r}\binom{m-r}{k}(-1)^k\left[1-(1+\beta x)^{-\alpha}e^{-\gamma x^{\lambda}}\right]^{r+k}. \quad (4.19)$$

.
The associated p^{th} raw moment of the r^{th} order statistics for LW distribution is given as

$$E(X_{r:m}^p) = \frac{m!}{(r-1)!(m-r)!}\sum_{j=0}^{\infty}\sum_{i=0}^{r+k-1}\sum_{k=0}^{m-r}(-1)^{i+k}\binom{m-r+k}{k}\binom{r+k-1}{i} \binom{(i+1)\alpha+j}{j}\beta^j\left[\frac{\alpha\beta\Gamma\left(\frac{p+j+1}{\lambda}\right)}{\lambda[(i+1)\gamma]^{\frac{p+j+1}{\lambda}}} + \frac{\Gamma\left(\frac{p+j}{\lambda}+1\right)}{[(i+1)\gamma]^{\frac{p+j}{\lambda}}} + \frac{\beta\Gamma\left(\frac{p+j+1}{\lambda}+1\right)}{[(i+1)\gamma]^{\frac{p+j+1}{\lambda}}}\right]. \quad (4.20)$$

4.2 Measures of Entropies

Two measures of entropies commonly used in lifetime analysis are the Shannon and Renyi entropies. Respectively, they are defined as

$$\mathcal{I}_{\mathcal{R}}(\omega) = \frac{1}{(1-\omega)}log\int_0^{\infty}f^{\omega}(x)dx; \omega > 0, \omega \neq 1$$

and

$$\mathcal{H}_{\mathcal{S}}(f) = E[-log(f(X))] = -\int_0^\infty f(x)logf(x)dx.$$

The Renyi and Shannon entropies for the LW distribution are given as

$$\mathcal{I}_{\mathcal{R}}(\omega) = \frac{1}{(1-\omega)} log\left[\sum_{i=1}^{\omega}\sum_{j=0}^{\infty}\Omega_{i,j}(\gamma\lambda)^i(\alpha\beta)^{\omega-i}\frac{\Gamma\left(\frac{j+i(\lambda-1)+1}{\lambda}\right)}{\lambda(\omega\gamma)^{\frac{j+i(\lambda-1)+1}{\lambda}}}\right]. \tag{4.21}$$

and

$$\mathcal{H}_{\mathcal{S}}(f) = -log(\alpha) - log(\beta) + \sum_{i=1}^{\infty}\sum_{k=0}^{i}\binom{k}{i}\frac{(-1)^{i+2}}{i}\left(\frac{\gamma\lambda}{\alpha\beta}\right)^i E\left(X^{i(\lambda-1)+k}\right)$$
$$+ (\alpha+1)\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\beta^i E\left(X^i\right) + \gamma E\left(X^\lambda\right) \tag{4.22}$$

where $\Omega_{i,j} = \binom{\omega}{i}\binom{i-\omega(\alpha-1)}{j}\beta^j$. The quantities defined by $E\left(X^{i(\lambda-1)+k}\right)$, $E\left(X^i\right)$ and $E\left(X^\lambda\right)$ can be obtained for the Shannon entropy using equation (3.10).

4.3 Residual and reversed residual lifetimes

The r^{th} moment of the residual and reversed residual lifetime of random variable X following a LW distribution can be obtained from

$$m_r(t) = E[(X-t)^r / X > t] = \frac{1}{S(t)}\int_t^\infty (x-t)^r f(x)dx$$

and

$$M_r(t) = E[(t-X)^r / X \leq t] = \frac{1}{F(t)}\int_0^t (t-x)^r f(x)dx.$$

The resulting expressions are given as

$$m_r(t) = \frac{1}{(1+\beta t)^{-\alpha}e^{-\gamma t^\lambda}}\sum_{i=0}^{r}(-1)^i\binom{r}{i}t^i\sum_{j=0}^{\infty}\Omega_j$$
$$\left[\frac{\alpha\beta\Gamma_u\left(\frac{r+j+1-i}{\lambda},\gamma t^\lambda\right)}{\lambda\gamma^{\frac{r+j+1-i}{\lambda}}} + \frac{\Gamma_u\left(\frac{r+j-i}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j-i}{\lambda}}} + \frac{\beta\Gamma_u\left(\frac{r+j+1-i}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j+1-i}{\lambda}}}\right] \tag{4.23}$$

and

$$M_r(t) = \frac{1}{1-(1+\beta t)^{-\alpha}e^{-\gamma t^\lambda}}\sum_{i=0}^{r}(-1)^{r+i}\binom{r}{i}t^i\sum_{j=0}^{\infty}\Omega_j$$
$$\left[\frac{\alpha\beta\Gamma_l\left(\frac{r+j+1-i}{\lambda},\gamma t^\lambda\right)}{\lambda\gamma^{\frac{r+j+1-i}{\lambda}}} + \frac{\Gamma_l\left(\frac{r+j-i}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j-i}{\lambda}}} + \frac{\beta\Gamma_l\left(\frac{r+j+1-i}{\lambda}+1,\gamma t^\lambda\right)}{\gamma^{\frac{r+j+1-i}{\lambda}}}\right] \tag{4.24}$$

5 Parameter estimation of LW distribution

This section presents the maximum likelihood estimation for estimating parameters $\Omega = (\alpha, \beta, \gamma, \lambda)$ in the LW distribution. Suppose a random sample $x_1, x_2, ..., x_n$ is given, then the total log-likelihood function of the random sample is given as;

$$\Upsilon_n = \sum_{i=1}^{n} log(\alpha\beta + \gamma\lambda x_i^{\lambda-1}(1 + \beta x_i)) - (\alpha + 1)\sum_{i=1}^{n} log(1 + \beta x_i) - \sum_{i=1}^{n}\gamma x_i^{\lambda}. \quad (5.25)$$

Obtaining the components of the score function $\Lambda = (\frac{\partial\Upsilon}{\partial\alpha}, \frac{\partial\Upsilon}{\partial\beta}, \frac{\partial\Upsilon}{\partial\gamma}, \frac{\partial\Upsilon}{\partial\lambda}, \frac{\partial\Upsilon}{\partial p})$ which gives the first partial derivatives of (5.25) with respect to the four parameters, we have

$$\frac{\partial\Upsilon_n}{\partial\alpha} = \sum_{i=1}^{n} \frac{\beta}{\alpha\beta+\gamma\lambda x_i^{\lambda-1}(1+\beta x_i)} - (\alpha+1)\sum_{i=1}^{n} log(1+\beta x_i),$$

$$\frac{\partial\Upsilon_n}{\partial\beta} = \sum_{i=1}^{n} \frac{\alpha+\gamma\lambda x_i^{\lambda-1}}{\alpha\beta+\gamma\lambda x_i^{\lambda-1}(1+\beta x_i)} - \sum_{i=1}^{n} \frac{x_i}{(1+\beta x_i)},$$

$$\frac{\partial\Upsilon_n}{\partial\gamma} = \sum_{i=1}^{n} \frac{\lambda x_i^{\lambda-1}(1+\beta x_i)}{\alpha\beta+\gamma\lambda x_i^{\lambda-1}(1+\beta x_i)} - \sum_{i=1}^{n} x_i^{\lambda}$$

and

$$\frac{\partial\Upsilon_n}{\partial\lambda} = \sum_{i=1}^{n} \frac{\gamma x_i^{\lambda-1}(1+\beta x_i)+\gamma\lambda x_i^{\lambda-1}(1+\beta x_i)log(x_i)}{\alpha\beta+\gamma\lambda x_i^{\lambda-1}(1+\beta x_i)} - \sum_{i=1}^{n} x_i^{\lambda}.$$

The resulting derivatives given in $\Lambda_n(\Omega) = (\frac{\partial\Upsilon_n}{\partial\alpha}, \frac{\partial\Upsilon_n}{\partial\beta}, \frac{\partial\Upsilon_n}{\partial\epsilon}, \frac{\partial\Upsilon_n}{\partial\omega}, \frac{\partial\Upsilon_n}{\partial p})^T$ are equated to zero and are solved iteratively by numerical optimization method in R package to obtain $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\lambda}$. Hypothesis testing and interval estimation for $\Omega = (\alpha, \beta, \gamma, \lambda)$ is carried out by deriving the asymptotic confidence intervals using Fisher's information matrix. Under standard regularity condition, the asymptotic distribution of $\sqrt{n}(\hat{\Omega} - \Omega)$ is a multivariate normal distribution with mean 0 and variance-covariance matrix $J_n^{-1}(\Omega)$ with $J_n(\Omega) = \lim_{n\to\infty} n^{-1}I_n(\Omega)$ ([15]). These statistics are required for approximating confidence intervals for Ω. The total observed information matrix $I_n(\Omega)$ is given as

$$I_n(\Omega) = -\begin{pmatrix} \Lambda_{\alpha\alpha} & \Lambda_{\alpha\beta} & \Lambda_{\alpha\gamma} & \Lambda_{\alpha\lambda} \\ & \Lambda_{\beta\beta} & \Lambda_{\beta\gamma} & \Lambda_{\beta\lambda} \\ & & \Lambda_{\gamma\gamma} & \Lambda_{\gamma\lambda} \\ & & & \Lambda_{\lambda\lambda} \end{pmatrix}$$

$100(1-\Psi)\%$ confidence interval for α, β, γ and λ is given by $\hat{\alpha} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{J}_{\alpha\alpha}}$, $\hat{\beta} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{J}_{\beta\beta}}$, $\hat{\gamma} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{J}_{\gamma\gamma}}$ and $\hat{\lambda} \pm Z_{\frac{\Psi}{2}}\sqrt{\hat{J}_{\lambda\lambda}}$, where $Z_{\frac{\Psi}{2}}$ is the standard normal upper percentile.

6 Lifetime data analysis

This section pesents two real data sets to illustrate the usefulness and applicability of the Lomax-Weibull distribution in comparison with some modified Lomax distributions in literature. The first data set is 63 service times of aircraft windshield for 1000 hours obtained from [12] and have been analysed by several authors in literature, recently by [3]. The data set is given as;
0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

The second set consist of data obtained on the breaking stress of 66 carbon fibres of 50 mm length measured in GPa ([13]). This data set is obtained in the article, recently, published by [8] and is

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given as follows;
0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.5, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42 , 4.70, 4.90.

Some discrepancy tests for comparing the different distributions are employed to determine the distribution that best fits the data sets. They include the information criteria and goodness-of-tests. The AdequacyModel package in R software is used to obtain the estimates of the distributions, AIC, CAIC, BIC, HQIC, Cramer von Mises (W), Anderson darling (A) and Kolmogorov-Smirnov (KS) tests. The distributions to be compared include
(i) Kumaraswamy-Lomax (KwL) distribution ([6])

$$F(x) = 1 - (1 - (1 - (1 + \beta x)^{-\alpha})^{\gamma})^{\lambda}.$$

(ii) Gumbel-Lomax (GumbL) distribution ([20])

$$F(x) = e^{-\gamma((1+\beta x)^{\alpha}-1)^{-\frac{1}{\lambda}}}.$$

(iii) Gompertz-Lomax (GompL) distribution ([14])

$$F(x) = 1 - e^{-\frac{\lambda}{\gamma}(1-(1+\beta x)^{\gamma\alpha})}.$$

(iv) Weibull-Lomax (WL) distribution ([21])

$$F(x) = 1 - e^{-\gamma((1+\beta x)^{\alpha}-1)^{\lambda}}.$$

(v) Lomax distribution ([11])

$$F(x) = 1 - (1 + \beta x)^{\alpha}.$$

Table 3: Estimates and loglikelihood values for first data set

Distribution	α (std. error)	β (std. error)	γ (std. error)	λ (std. error)	-2loglik
LW	0.1229 (0.2198)	2.3118 (5.3825)	0.0963 (0.0669)	2.3029 (0.4495)	195.7284
KwL	7.3698 (7.8564)	0.0106 (0.0069)	1.6921 (0.2089)	21.3306 (26.9112)	202.1840
GumbL	92.2248 (75.1108)	0.0395 (0.0345)	3.7636 (0.7231)	3.8292 (1.0840)	201.3078
GompL	1.4491 (11.1928)	0.2412 (0.6081)	2.8045 (21.4862)	0.4989 (3.9267)	196.2728
WL	3.8634 (3.5585)	0.3463 (0.8994)	0.1236 (0.2747)	0.9165 (0.4210)	196.2340
Lomax	33.9859 (26.6808)	0.0143 (0.0114)	—	—	219.7980

Table 4: Information criteria and goodness-of-fit values for first data set

Distribution	AIC	CAIC	BIC	HQIC	W	A	KS (p-value)
LW	**203.7284**	**204.4180**	**212.3009**	**207.1000**	**0.0286**	**0.2036**	**0.0589 (0.9721)**
KwL	210.1840	210.8737	218.7566	213.5557	0.1286	0.7301	0.1196 (0.3035)
GumbL	209.3079	209.9975	217.8804	212.6795	0.1138	0.6865	0.1095 (0.4070)
GompL	204.2728	204.9124	212.8453	207.6444	0.0383	0.2550	0.0676 (0.9166)
WL	204.2340	204.9237	212.8065	207.6056	0.0354	0.2407	0.0662 (0.9285)
Lomax	219.7980	223.9979	228.0842	225.4837	0.1951	1.1807	0.2093 (0.0067)

The following approximate intervals; (0.1229 ± 0.4308), (2.3118 ± 10.5497), (0.0963 ± 0.1311) and (2.3029 ± 0.8810) give the 95% two-sided confidence intervals for α, β, γ and λ respectively for the first data set. At 5% level of significance, the critical value of KS test is 0.1713 which is used to determine the distribution that will poorly fit the data.

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Table 5: Estimates and loglikelihood values for second data set

Distribution	α (std. error)	β (std. error)	γ (std. error)	λ (std. error)	-2loglik
LW	0.7532 (14.6960)	0.0220 (0.4324)	0.0146 (0.0081)	3.7082 (0.4113)	171.0706
KwL	10.2457 (8.6902)	0.0231 (0.0193)	4.8211 (0.9110)	27.9891 (26.3127)	174.8294
GumbL	76.1144 (45.9723)	0.0465 (0.0288)	12.8927 (3.0099)	3.0042 (1.3000)	188.5374
GompL	1.0026 (0.5869)	0.9097 (0.4260)	4.7492 (2.5570)	0.0081 (0.0051)	171.3840
WL	3.1884 (2.6385)	0.3884 (0.2965)	0.0107 (0.0099)	1.8597 (0.9117)	171.5448
Lomax	44.8534 (27.3031)	0.0081 (0.0049)	—	—	267.2807

Table 6: Information criteria and goodness-of-fit values for second data set

Model	AIC	CAIC	BIC	HQIC	W	A	KS (p-value)
LW	**179.0707**	**179.7264**	**187.8293**	**182.5316**	**0.0632**	**0.3972**	**0.0790 (0.8050)**
KwL	182.8296	183.4853	191.5882	186.2905	0.1353	0.7263	0.0977 (0.5550)
GumbL	196.5373	197.1930	205.2959	199.9983	0.3171	1.7268	0.1441 (0.1289)
GompL	179.3840	180.0397	188.1426	182.8449	0.0674	0.4348	0.0847 (0.7314)
WL	179.5448	180.2005	188.3034	183.0057	0.0726	0.4665	0.0867 (0.7039)
Lomax	271.2807	271.4712	275.6600	273.0112	0.2490	1.3514	0.3584 (0.0000)

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The 95% two-sided confidence intervals for α, β, γ and λ for the second data set are (0.7532 ± 128.8042), (0.0220 ± 0.8475), (0.0146 ± 0.0159) and (3.7082 ± 0.8061) respectively with the critical value of KS test as 0.1674 at 5% level of significance.

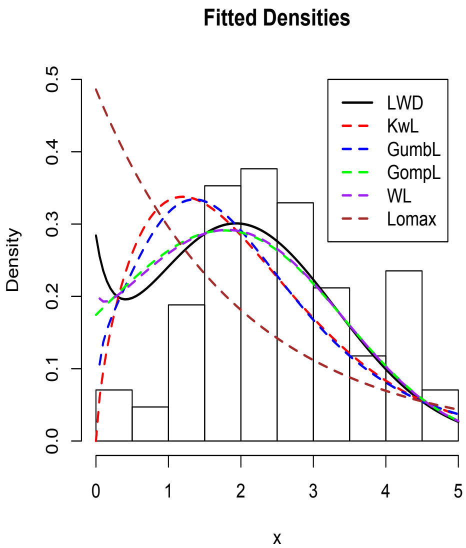

Figure 5: Densities for first data set

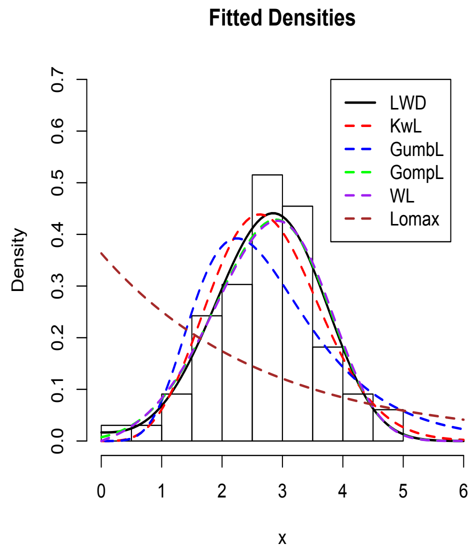

Figure 6: Densities for second data set

The superiority of the LW distribution oveer the other compared distributions is evidence in the values of the discrepancy criteria used. The LW distribution has the ***lowest values of AIC, CAIC, BIC, HQIC, W, A, KS and highest p-values for KS test*** in the two data sets. Hence, the LW distribution best fits the data sets than the other distribution in comparison with it.

7 Conclusion

The Lomax-Weibull (LW) distribution as a new four-parameter lifetime distribution has been introduced and investigated in this paper. The hazard function of the LW distribution exhibits very flexible property in handing monotonic and nonmonotonic lifetime data. Closed form expressions are obtained for the moments, quantile function, mean deviations, residual and reversed residual liifetimes, order statistics and its distribution as well as measures of entropies. Parameter estimates for the LW distribution are achieved by maximum likelihood estimation technique. Lastly, the applicability and usefulness of the LW distribution is illustrated using two lifetime data sets obtained from literature.

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