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Elliptic Gradient estimates for the heat equation on a weighted manifold with time-dependent metrics and potentials

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Abstract

In this paper, we get some elliptic type gradient estimates on positive solutions to the heat equation on a weighted Riemannian manifold with time dependent metrics and potentials. The geometry of the space in terms of curvature bounds play crucial role in determining the estimates. The gradient estimates derived are useful in proving the classical Harnack inequalities, Liouville type theorems, heat kernel bounds, e.t.c. As an example, we discuss Liouville principle on bounded positive solution. Indeed, each gradient estimate obtained is equivalent to saying bounded weighted harmonic function is a constant.

Keywords: Measure spaces, Ricci flow, heat kernel, gradient estimates, maximum principle.
MSC2010:53C44, 58J60

1 Introduction

In recent years, there have been many interesting results relating to elliptic gradient estimates, Harnack inequalities and Liouville type theorems on smooth metric measure spaces. This is due to the fact that these estimates have found numerous applications in the fields of Geometric Analysis and Partial Differential Equations among others. The pioneering paper [1] by Li and Yau gives more insight into the concept of gradient estimates in the field of differential equations. The Li-Yau gradient estimates were used to get classical Harnack principle and several heat kernel estimates. Hamilton [2] was motivated by [1] and then derived global elliptic gradient estimates for linear heat equation. Hamilton obtained a matrix Harnack estimate which can be used to prove some monotonicity formulas of Perelman type [3]. Later, some logarithmic correction term was added by Souplet and Zhang [4] to obtain localized elliptic gradient estimates on positive solutions to the linear heat equation defined on compact manifold.

The main aim of this paper is to derive some local elliptic gradient estimates for smooth positive solutions $u = u(x,t)$ to the weighted nonlinear heat equation

$$\frac{\partial u}{\partial t} = \Delta_f u, \quad t > 0, \tag{1.1}$$

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defined on a smooth metric measure space whose underlying time dependent metric, $g(t)$ and potential, $f(t)$, are being deformed by the action of the super Perelman-Ricci flow

$$\frac{\partial g}{\partial t} + 2Ric_f \geq 0 \tag{1.2}$$

$$\frac{\partial f}{\partial t} - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial g}{\partial t}\right) = 0, \tag{1.3}$$

where Tr stands for the metric trace on $(0,2)$-tensors and $\frac{\partial}{\partial t}$ is the partial derivative with respect to t. (Refer to Section 2 for other notations).

The gradient estimates obtained in this article are related to Hamilton type estimates for the heat equation on a compact manifold and Souplet-Zhang type estimates for the heat equation on noncompact manifolds. Hamilton type, Souplet-Zhang type and Li-Yau type gradient estimates are of fundamental roles in estimating smooth solutions to parabolic partial differential equations on compact and noncompact manifolds, either with static or evolving metrics. These estimates are widely known for their usefulness as highlighted above. They have been used in conjuction with the so-called Perelman entropy in several contexts. It is therefore natural to investigate whether or not these estimates are available for weighted manifolds equipped with time dependent metrics and potentials. This paper therefore addresses this issue in affirmative by providing the conditions needed on the generalized curvature and the potentials under super Perelman-Ricci flow.

1.1 Smooth metric measure space

A Smooth metric measure space (or weighted manifold) is usually denoted by $(M, g, e^{-f}dv)$, for an n-dimensional complete Riemannian manifold (M, g), volume measure on M (denoted by dv) and $C^\infty(M)$ function f. The triple (M, g, f) will be simply referred to as a weighted manifold throughout this paper. Associated with (M, g, f) is the so called Witten Laplacian or f-Laplacian denoted by $\Delta_f := \Delta - \nabla f \cdot \nabla$, where Δ is the usual Laplacian on (M, g) and ∇ is the connection with respect to metric g. A natural extension of Ricci curvature tensor (Ric), which arises on the space, is the ∞- Bakry-Émery tensor and defined by $Ric_f := Ric + \nabla^2 f$, where $\nabla^2 f$ is the Hessian of f. This tensor defines gradient Ricci solitons, $Ric_f = \sigma g$, $\sigma \in \mathbb{R}$, as some special solutions to Hamilton-Ricci flow [5]. A gradient Ricci soliton is shrinking when σ is positive, steady when σ is zero and expanding when σ is negative. Gradient Ricci solitons are of fundamental importance in understanding the natures of Hamilton Ricci flows singularities and in the final resolution of Poincaré conjecture [3, 6].

1.2 The Perelman-Ricci flow

Given the pair (M, g), a n-dimensional complete compact Riemannian manifold without boundary $(n \geq 2)$. The Hamilton-Ricci flow can be defined as the deformation of a family of metrics $g(t)$ by the following quasilinear parabolic equation

$$\begin{aligned} \frac{\partial g}{\partial t} &= -2Ric \\ g(0) &= g_0, \end{aligned} \tag{1.4}$$

where $t \in [0, T]$, $0 < T < T_{max}$ i.e., T_{max} is the maximal existence time after which the flow blows up and singularities occur. The Hamilton-Ricci flow theory emanated from the seminal paper [5]. It has since been widely studied [7, 8], and applied to solve some problems in geometry, topology and physics [3, 9].

A super Perelman-Ricci flow is defined by a complete manifold M without boundary, endowed with time dependent family of Riemannian metrics $g(t)$ and smooth functions, $f(t)$, $\{(M, g(t), f(t)) :$

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$t \in [0, T]\}$, satisfying

$$\begin{aligned} &\frac{\partial g}{\partial t} + 2Ric_f \geq 0 \\ &\frac{\partial f}{\partial t} - \frac{1}{2}\mathrm{Tr}\left(\frac{\partial g}{\partial t}\right) = 0. \end{aligned} \tag{1.5}$$

Replacing the inequality sign in (1.5) by equality we have the Perelman-Ricci flow

$$\frac{1}{2}\frac{\partial g}{\partial t} + Ric_f = 0, \tag{1.6}$$

as was introduced in [3] as a gradient flow of an energy functional defined by $\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2)e^{-f}d\mu$ on the condition that the measure, $d\mu := e^{-f}dv$, remains static in time, whereas $f(t)$ satisfies the heat type equation (compare with second equation in (1.5))

$$\frac{\partial f}{\partial t} = -R - \Delta f. \tag{1.7}$$

Clearly, the Hamilton-Ricci flow can be recovered from (1.6)–(1.7) by pulling back $(g(t), f(t))$ with a family of diffeomorphism generated by the gradient of f [3].

Owing to the strong relationship between the heat equation and the Hamilton-Ricci flow on one hand and due to the physical and geometric implications yielded when the two equations are coupled, many researchers have shown keen interests in the field. The coupling was studied in [10–15] and different gradient estimates were obtained for the heat equation, which consequently yield various estimates on the fundamental solution with respect to time dependent metrics. Recently, Wu [16] derived some elliptic type gradient estimate for (1.1) on static weighted manifold. Therefore it is natural to ask if one can obtain similar estimates when (1.1) and (1.2) are coupled. Motivated by [14–16], we investigate the geometric conditions under which one can obtain space only gradient estimates on positive solution to (1.1) when the underlying metric and potential are being evolved along the flow (1.2). Thus, this paper extends Hamilton Harnack inequalities on positive solutions to the weighted heat equation on time dependent metric measure space. Note that several attempts have been made in this direction by S. Li and X-D. Li [17–19] via Perelman $\mathcal{W}$-entropy, yielding many interesting results.

1.3 Main results

Fix a point x_0 on (M, g, f) and denote by $r(x, t)$ or $d(x, x_0; t)$, a distance function from x_0 to x with respect to $g(t)$. Define a compact space $\mathcal{B}_{R,T}$ by
$\mathcal{B}_{R,T} \equiv \{(x, t) : d(x, x_0; t) \leq R,\ 0 \leq t \leq T\} \subset M \times [0, T]$, for $R, T > 0$.

The first result is Souplet-Zhang type gradient estimate with the assumption that ∞- Bakry-Émery tensor is bounded uniformly locally.

Theorem 1.1. *Given a complete weighted manifold (M, g, f) of dimension $n \geq 2$ with $|Ric_f|_{g(t)} \leq (n-1)kg$ for some $k \geq 0$ and $\{(M, g(t), f(t)) : t \in [0, T]\}$ a complete solution to the super Perelman-Ricci flow (1.2). Suppose $u(x, t) \leq A$, $A > 0$, is a positive bounded solution to the weighted heat equation (1.1) in $\mathcal{B}_{R,T}$. Then the following estimate*

$$\frac{|\nabla u|}{u} \leq C\Big(\sqrt{\frac{1 + |\alpha|}{R}} + \frac{1}{\sqrt{T}} + \sqrt{k}\Big)\Big(1 + \ln\frac{A}{u}\Big) \tag{1.8}$$

holds for all (x, t) in $\mathcal{B}_{R/2,T/2}$ with $t \neq 0$, where C is a positive constant depending only on the dimension of M and $\alpha := \max_{\{x|d(x,x_0;t)=1,0\leq t\leq T\}} \Delta_f r(x, t)$.

Moreover, if $Ric_f \geq 0$, then there exists constants $C_1, C_2 > 0$ which depend only on n such that

$$\frac{|\nabla u|}{u} \leq C_1 \frac{1 + \sqrt{|\alpha|}}{t^{\frac{1}{4}}}\Big(1 + e^{C_2 D} + \ln\frac{u(x, 2t)}{u(x, t)}\Big) \tag{1.9}$$

for all $(x, t) \in M \times (0, T]$, where $D = \sup_{y\in B(x,\sqrt{t})} |f(y)|$.

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Remark 1.2. *Assuming that f is a constant, the above estimate in (1.8) recovers the estimate in (2.3) of [7, Theorem 2.2] for the heat equation along Hamilton-Ricci flow. Estimate (1.8) can also be compared with [22, Theorem 1] for heat equation under backward Ricci flow. It also extends gradient estimates on stationary manifold contained in [16]. Hence, Theorem 1.1 generalizes and improves the results in the aforementioned references.*

The next theorem is a global version of estimate (1.8). This has been established previously in [2, 14, 15].

Theorem 1.3. *Given a complete weighted manifold (M, g, f) of dimension $n \geq 2$ with $\{(M, g(t), f(t)) : t \in [0, T]\}$ evolving by the super Perelman-Ricci flow (1.2). Suppose $u \leq A$ is a positive bounded solution to (1.1) in $M \times [0, T]$. Then the following estimate*

$$\frac{|\nabla u|}{u} \leq \sqrt{\frac{1}{t} \ln \frac{A}{u}} \tag{1.10}$$

holds for $A = \sup_{M \times [0,T]} u(x, t) : (x, t) \in M \times [0, T]$. Moreover, there holds the following interpolation inequality for $\delta > 0$, $x, y \in M$

$$u(x, t) \leq C_3 u^{1/(1+\delta)}(y, t) A^{\delta/(1+\delta)} \exp\Big(C_4 \frac{d^2(x, y, t)}{t}\Big), \tag{1.11}$$

where $t \in (0, T]$ and $C_3, C_4 > 0$ are constant depending on $\delta > 0$.

Remark 1.4. *Note that local estimate (1.8) and global estimate (1.10) cannot replace each other, as (1.10) involves neither curvature assumption nor dimension dependent constant.*

The last result of this paper gives Hamilton type gradient estimate.

Theorem 1.5. *Given a complete weighted manifold (M, g, f) with $Ric_f \geq -(n-1)k, k > 0$. Let $\{(M, g(t), f(t)) : t \in [0, T]\}$ be a complete solution to the super Perelman-Ricci flow (1.2). Suppose $u = u(x, t)$ is a positive bounded solution to (1.1) in $\mathcal{B}_{R,T}$. Then for all x and $t > 0$*

$$\frac{|\nabla u|}{u} \leq CD\Big(\frac{1}{R^2} + \frac{|\alpha|}{R} + \frac{1}{T} + k\Big)^{\frac{1}{2}} \tag{1.12}$$

where $C > 0$ is a universal constant and $D := \sup\{u(x, t) : (x, t) \in \mathcal{B}_{R,T}\}$.

The rest of this paper is arranged as follows. Two important Lemma are first presented in Section 2 and then the proofs of Theorems 1.1 and 1.3. Section 3 contains the discusion on Theorem 1.5, while some remarks are made on Liouville type results in the last section.

2 Proof of Theorems 1.1 and 1.3

The section presents the proofs of the first two theorems. Meanwhile two important results to be applied are first stated in forms of lemmas

2.1 Fundamental results

The two lemmas below are fundamental to the proof of Theorem 1.1.

Lemma 2.1. *Let $\{(M, g(t), f(t)) : t \in [0, T]\}$ be a complete solution to the super Perelman Ricci flow (1.2). Suppose $u \leq A$, for some positive constant A, is a positive solution to (1.1). Then the function $w = |\nabla \ln(1 - h)|^2$, where $h = \ln u/A \leq 0$ verifies*

$$\Big(\Delta_f - \frac{\partial}{\partial t}\Big) w \geq \frac{2h}{1-h} \nabla h \nabla w + 2(1 - h) w^2 \tag{2.1}$$

in $\mathcal{B}_{R,T}$.

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Proof. We adopt the convention that $h_i^2 = |\nabla h|^2, h_{ii} = \Delta h$ with repeated indices summed up in a local orthonormal system, while subscript t means partial derivative with respect to t. By the scaling $u \to \bar{u} = u/A$, we have $0 < \bar{u} \le 1$ and (1.1) then implies

$$\frac{\partial \bar{u}}{\partial t} = \Delta_f \bar{u}.$$

Let $h = \ln \bar{u} \le 0$ and $w = |\nabla h|^2/(1-h)^2$. A simple calculation yields

$$h_t = \Delta_f h + |\nabla h|^2.$$

Standard computation using super Perelman-Ricci flow (1.2) gives

$$(|\nabla h|^2)_t \le 2(R_{ij} + f_{ij})h_i h_j + 2h_i(h_t)_j. \tag{2.2}$$

and by the equation $h_t = \Delta_f h + |\nabla h|^2$ we have

$$(|\nabla h|^2)_t \le 2(R_{ij} + f_{ij})h_i h_j + 2h_i(\Delta_f h)_j + 2h_i(|\nabla h|^2)_j.$$

Then

$$\begin{aligned} w_t &= \frac{(|\nabla h|^2)_t}{(1-h)^2} + \frac{2|\nabla h|^2 h_t}{(1-h)^3} \\ &\le \frac{2(R_{ij} + f_{ij})h_i h_j}{(1-h)^2} + \frac{2h_i(\Delta_f h)_j}{(1-h)^2} + \frac{2h_i(|\nabla h|^2)_j}{(1-h)^2} + \frac{2h_i^2 \Delta_f h}{(1-h)^3} \\ &+ \frac{2h_i^2 h_j^2 h}{(1-h)^3}. \end{aligned} \tag{2.3}$$

Similarly,

$$w_j = \frac{2h_i h_{ij}}{(1-h)^2} + \frac{2h_i^2 h_j}{(1-h)^3}$$

and

$$\begin{aligned} \Delta w = w_{jj} &= \Big(\frac{2h_i h_{ij}}{(1-h)^2}\Big)_j + \Big(\frac{2h_i^2 h_j}{(1-h)^3}\Big)_j \\ &= \frac{2h_{ij}^2}{(1-h)^2} + \frac{2h_i h_{ijj}}{(1-h)^2} + \frac{8h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 h_{jj}}{(1-h)^3} + \frac{6h_i^2 h_j^2}{(1-h)^4}. \end{aligned}$$

Hence

$$\begin{aligned} \Delta_f w = \Delta w - \nabla f \nabla w &= w_{jj} - w_j f_j \\ &= \frac{2h_{ij}^2}{(1-h)^2} + \frac{2h_i h_{ijj}}{(1-h)^2} + \frac{8h_i h_{ij} h_j}{(1-h)^3} + \frac{2h_i^2 h_{jj}}{(1-h)^3} + \frac{6h_i^2 h_j^2}{(1-h)^4} \\ &\quad - \frac{2h_i h_{ij} f_j}{(1-h)^2} - \frac{2h_i^2 h_j f_j}{(1-h)^3}. \end{aligned} \tag{2.4}$$

Using the Ricci identity $h_{ijj} = h_{jji} + R_{ij}h_j$, we have

$$\frac{2h_i h_{ijj}}{(1-h)^2} - \frac{2h_i h_{ij} f_j}{(1-h)^2} = \frac{2h_i(\Delta_f h)_i}{(1-h)^2} + \frac{2(R_{ij} + f_{ij})h_i h_j}{(1-h)^2}. \tag{2.5}$$

Putting (2.3)-(2.5) together, we get

$$\begin{aligned} \Big(\Delta_f - \frac{\partial}{\partial t}\Big)w &\ge \Big(\frac{2h_{ij}^2}{(1-h)^2} + \frac{4h_i h_j h_{ij}}{(1-h)^3} + \frac{2h_i^2 h_j^2}{(1-h)^4}\Big) + \Big(\frac{4h_i h_j h_{ij}}{(1-h)^3} + \frac{4h_i^2 h_j^2}{(1-h)^4} \\ &\qquad - \frac{4h_i h_j h_{ij}}{(1-h)^2} - \frac{2h_i^2 h_j^2}{(1-h)^3}\Big) \\ &= \frac{2}{(1-h)^2}\Big(h_{ij} + \frac{h_i h_j}{(1-h)}\Big)^2 + \Big(\frac{2}{1-h} h_j w_j - 2h_j w_j + \frac{2h_i^2 h_j^2}{(1-h)^3}\Big) \\ &\ge \frac{2}{1-h} h_j w_j - h_j w_j + \frac{2h_i^2 h_j^2}{(1-h)^3}, \end{aligned}$$

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which implied the required inequality. Note that we have used the following identities

$$h_j w_j = \frac{2h_i h_{ij} h_j}{(1-h)^2} + \frac{2h_i^2 h_j^2}{(1-h)^3} \quad \text{and} \quad \frac{2}{(1-h)} h_j w_j = \frac{4h_i h_j h_{ij}}{(1-h)^3} + \frac{4h_i^2 h_j^2}{(1-h)^4}.$$

The second lemma gives some properties of a cut-off function that will be applied to get useful bounds in the space $\mathcal{B}_{R,T}$. The idea of cut-off functions was introduced by Li and Yau [1] and has become standard in this context, see for instance [4, 14, 20]. Only the statement of the lemma is therefore given here.

Lemma 2.2. *Let $\phi = \phi(r(x,t),t)$ be a cut-off function which is smoothly supported in $\mathcal{B}_{R,T}$ with the following properties [20, 21]: Fix $\tau \in (0,T]$, there exists a smooth function $\bar{\phi} : [0,\infty) \times [0,T] \to \mathbb{R}$ satisfying*

(i) $\phi = \bar{\phi}(r,t)$; $\bar{\phi}(r,t) = 1$ *in* $\mathcal{B}_{R/2,T/2}$, $0 \le \phi(r,t) \le 1$.

(ii) ϕ *is a radially decreasing function in spatial variables.*

(iii) $|\frac{\partial \bar{\phi}}{\partial r}| \frac{1}{\bar{\phi}^a} \le \frac{C_a}{R}$ *and* $|\frac{\partial^2 \bar{\phi}}{\partial r^2}| \frac{1}{\bar{\phi}^a} \le \frac{C_a}{R^2}$ *in* $[0,\infty) \times [0,T]$, *where* $0 < a < 1$ *for some constant* $C_a > 0$.

(iv) $|\frac{\partial \bar{\phi}}{\partial t}| \frac{1}{\bar{\phi}^{1/2}} \le \frac{C}{\tau}$ *in* $[0,\infty) \times [0,T]$ *for some* $C > 0$ *and* $\bar{\phi}(r,0) = 0$ $\forall r \in [0,\infty)$.

2.2 Proof of Theorem 1.1

Applying (2.1) of Lemma 2.1, a simple calculation yields

$$\begin{aligned}\Big(\Delta_f - \frac{\partial}{\partial t}\Big)(\phi w) \ge \frac{2h}{1-h}\Big[\nabla h \nabla(\phi w) - w \nabla h \nabla \phi\Big] + 2(1-h)\phi w^2 \\ +2\frac{\nabla \phi}{\phi}\nabla(\phi w) - 2\frac{|\nabla\phi|^2}{\phi} w + w\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi.\end{aligned} \tag{2.6}$$

Assume ϕw is maximal at the point (x_1, t_1) in $B_{R,T}$ for any fixed $\tau \in (0,T]$. Suppose that x_1 is not in the cut locus of M due to Calabi's argument [1]. Also assume $(\phi w)(x_1,t_1) > 0$, otherwise the result becomes trivial with $w(x,t) \le 0$ whenever $d(x,x_0) < R/2$. Then at the point (x_1,t_1) we have

$$\Delta_f(\phi w) \le 0, \quad (\phi w)_t \ge 0 \text{ and } \nabla(\phi w) = 0. \tag{2.7}$$

By (2.6) and (2.7) we deduce that

$$2(1-h)\phi w^2 \le \frac{2h}{1-h} w \nabla h \nabla \phi + 2\frac{|\nabla \phi|^2}{\phi} w - (\Delta_f \phi) w + \phi_t w \tag{2.8}$$

at (x_1,t_1).

Firstly, we consider the case $x \in B(x_0,1)$. In this case ϕ is a constant function in space direction in $B(x_0,R/2), R \ge 2$, based on the assumption. Thus at (x_1,t_1), we have from (2.8)

$$w \le \frac{1}{2}\frac{\phi_t}{\phi} \le \frac{C}{T}$$

at (x_1,t_1) since $1-h \ge 1$.

Since $\phi = 1$ when $d(x,x_0) < R/2$ due to assumption (i) on ϕ, we have

$$w(x,\tau) = (\phi w)(x,\tau) \le (\phi w)(x_1,t_1) \le w(x_1,t_1) \le \frac{C}{T}$$

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for all $(x,t) \in \mathcal{B}_{R/2,T}$ with $t \neq t_0$. By the definition of $w(x,\tau)$ and the fact that τ was chosen arbitrarily, we prove that

$$\frac{|\nabla h|}{1-h}(x,t) \leq \frac{C}{\sqrt{T}}$$

for all $(x,t) \in \mathcal{B}_{R/2,T}$. Then the required estimate follows since $h = \ln u/A$.

Secondly, we suppose $x_1 \notin B(x_0,1)$. Let $C > 0$ be a constant which depends on n only, though the value may vary from step to step. Following the argument in [16, 21] we have that

$$2\frac{|\nabla \phi|^2}{\phi} w \leq \frac{1}{8}\phi w^2 + \frac{C}{R^4} \tag{2.9}$$

and

$$\frac{2h}{1-h} w \nabla h \nabla \phi \leq (1-h)\phi w^2 + \frac{C}{R^4} \cdot \frac{h^4}{(1-h)^3}. \tag{2.10}$$

Now, using the properties of ϕ and the weighted Laplacian comparison theorem [22], $\Delta_f r(x,t) \leq \alpha + (n-1)k(R-1)$, where $r \geq 1$ in $B(x_0,R)$, $\alpha = \max_{\{x:d(x,x_0;t)=1, 0\leq t\leq T\}} \Delta_f r(x,t)$ at this point and $Ric_f \geq -(n-1)k, k \geq 0$:

$$\begin{aligned}
-\Delta_f \phi &= -(\phi_r \Delta_f r + \phi_{rr}|\nabla r|^2) \\
&\leq \Big(|\phi_r| \frac{1}{\phi^{1/2}} (|\alpha| + (n-1)k(r-1)) + |\phi_{rr}| \frac{1}{\phi^{1/2}} \Big) \phi^{1/2} \\
&\leq C_{1/2} \Big(\frac{|\alpha|}{R} + nk + \frac{1}{R^2} \Big) \phi^{1/2}.
\end{aligned}$$

Now ϕ_t is estimated as follows: For all $x \in B(y,R)$, let $\gamma : [0,a] \to M$ be a minimal geodesic connecting $x = \gamma(0)$ and $y = \gamma(a)$ at time $t \in [0,T]$. Then

$$\begin{aligned}
\partial_t r(x,t) = \partial_t \int_0^a |\gamma'(s)| ds &= -\int_0^a Ric_f(\gamma'(s), \gamma'(s)) ds \\
&\leq (n-1)kr \leq nkR.
\end{aligned}$$

Then apply the properties of ϕ as stated above

$$\phi_t \leq |\bar{\phi}_t| + |\phi_r||\partial_t r| \leq \frac{C_{1/2}\phi^{1/2}}{\tau} + C_{1/2}\phi^{1/2} nk.$$

Therefore

$$-(\Delta_f \phi) w \leq \frac{1}{8}\phi w^2 + \frac{C}{R^4} + \frac{C|\alpha|^2}{R^2} + Ck^2 \tag{2.11}$$

and

$$\phi_t w \leq \frac{1}{8}\phi w^2 + \frac{C}{\tau^2} + Ck^2. \tag{2.12}$$

By putting (2.9)–(2.12) into (2.8) and rearranging we obtain

$$2(1-h)\phi w^2 \leq \frac{1}{2}\phi w^2 + \frac{C}{R^4}\frac{h^4}{(1-h)^3} + \frac{C}{R^4} + \frac{C|\alpha|^2}{R^2} + Ck^2 + \frac{C}{\tau^2} \tag{2.13}$$

at (x_1,t_1). Since $1-h \geq 1$, clearly $h^4/(1-h)^4 \leq 1$ and then (2.13) implies

$$\phi w^2 \leq \frac{1}{2}\phi w^2 + \frac{C}{R^4} + \frac{C|\alpha|^2}{R^2} + Ck^2 + \frac{C}{\tau^2}$$

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at (x_1, t_1). It therefore follows for all $(x,t) \in \mathcal{B}_{R,T}$, that there holds

$$\begin{aligned}(\phi^2 w^2)(x,\tau) &\le (\phi^2 w^2)(x_1,t_1) \le (\phi w^2)(x_1,t_1)\\ &\le \frac{C}{R^4} + \frac{C|\alpha|^2}{R^2} + \frac{C}{\tau^2} + Ck^2.\end{aligned}$$

Since $\phi(x,\tau) = 1$ in $\mathcal{B}_{R/2,T}$, $w = |\nabla h|^2/(1-h)^2$ and that τ was arbitrarily chosen. Then

$$\frac{|\nabla h|}{1-h}(x,t) \le \frac{C}{R} + C\sqrt{\frac{|\alpha|}{R}} + \frac{C}{\sqrt{T}} + C\sqrt{k} \tag{2.14}$$

for all $(x,t) \in \mathcal{B}_{R/2,T}$ with $t \neq t_0$. Since $h = \ln u/A$ we obtain

$$\frac{|\nabla h|}{1-h}(x,t) = \Big(\frac{1}{1+\ln\frac{A}{u}} \cdot \frac{|\nabla u|}{u}\Big)(x,t) \tag{2.15}$$

because $h = \ln u/A$. Substituting (2.15) into (2.14) and rearranging leads to estimate (1.8).

To prove (1.9), we apply estimate (1.8) on $\mathcal{B}_{\sqrt{t},t/2}$. By the parabolic Moser's Harnack inequality [21, Theorem 3.1] one has

$$A := \sup_{\mathcal{Q}_{\sqrt{t},t/2}} u(x,t) \le \exp(C_5 e^{c_6} D) u(x,2t),$$

where $D = \sup_{y \in B(x,\sqrt{t})} |f(y)|$. Setting $k=0$, then (1.9) follows from (1.8) at once.

2.3 Proof of Theorem 1.3

Our argument follows the idea in [2, 15] summarised as follows. Consider the function $P(x,t) := t\dfrac{|\nabla u|^2}{u} - u\ln\dfrac{A}{u}$ on $M \times [0,T]$ and show that

$$\Big(\frac{\partial}{\partial t} - \Delta_f\Big)P(x,t) \le 0, \quad \text{for all } (x,t)$$

and

$$P(x,0) \le 0 \quad \text{for all} \quad x \in M.$$

The above assertion follows from routine calculation, which we show as follows

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(t\frac{|\nabla u|^2}{u}\Big) &\le \frac{|\nabla u|^2}{u} + t\Big(\frac{2Ric_f(\nabla u, \nabla u) + 2\nabla u \nabla u_t}{u} - \frac{\partial u}{\partial t}\frac{|\nabla u|^2}{u^2}\Big),\\
\frac{\partial}{\partial t}\Big(u\ln\frac{A}{u}\Big) &= \frac{\partial u}{\partial t}\ln\frac{A}{u} - \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t}\Big(\ln\frac{A}{u} - 1\Big),\\
\Delta_f\Big(t\frac{|\nabla u|^2}{u}\Big) &= t\Big(\frac{\Delta_f(|\nabla u|^2)}{u} - \frac{4\nabla\nabla u|\nabla u|^2}{u^2} - \frac{\Delta_f u|\nabla u|^2}{u^2} + \frac{2|\nabla u|^2|\nabla u|^2}{u^3}\Big),\\
\Delta_f\Big(u\ln\frac{A}{u}\Big) &= \Delta_f u\Big(\ln\frac{A}{u} - 1\Big) - \frac{|\nabla u|^2}{u}.
\end{aligned}$$

Putting the above equations together yields

$$\begin{aligned}
\Big(\frac{\partial}{\partial t} - \Delta_f\Big)P(x,t) &= \Big(\frac{\partial}{\partial t} - \Delta_f\Big)\Big(t\frac{|\nabla u|^2}{u} - u\ln\frac{A}{u}\Big)\\
&\le t\Big(\frac{2Ric_f(\nabla u,\nabla u) + 2\nabla u\nabla\Delta_f u - \Delta_f(|\nabla u|^2)}{u}\\
&\qquad - \frac{4\nabla\nabla u|\nabla u|^2}{u^2} + \frac{2|\nabla u|^2|\nabla u|^2}{u^3}\Big)
\end{aligned}$$

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since u solves the weighted equation (1.1). Applying the weighted Bochner formula

$$\Delta_f(|\nabla u|^2) = 2|\nabla\nabla u|^2 + 2\nabla u\nabla\Delta_f u + 2Ric_f(\nabla u, \nabla u) \tag{2.16}$$

gives

$$\Big(\frac{\partial}{\partial t} - \Delta_f\Big)P(x,t) \leq -\frac{2t}{u}\Big(\nabla\nabla u - \frac{\nabla u\cdot\nabla u}{u}\Big)^2 \leq 0.$$

Then application of the maximum principle yields $P(x,t) \leq 0$ for $(x,t) \in M\times[0,T]$ from where the desired estimate (1.10) follows.

To prove estimate (1.11). We set $\mathcal{Z}(x,t) = \ln\frac{A}{u(x,t)}$ and by (1.10) we obtain

$$|\nabla\sqrt{\mathcal{Z}}| = \frac{1}{2}\Big|\frac{\nabla u}{u\sqrt{\mathcal{Z}}}\Big| \leq \frac{1}{2\sqrt{t}}.$$

Integrating along a minimising geodesic joinning two fixed points x and y gives

$$\sqrt{\ln\frac{A}{u(x,t)}} \leq \sqrt{\ln\frac{A}{u(y,t)}} + \frac{d(x,y,t)}{2\sqrt{t}}.$$

For any $\delta > 0$, we obtain

$$\begin{aligned}\ln\frac{A}{u(x,t)} &\leq \ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4t} + \sqrt{\ln\frac{A}{u(y,t)}}\cdot\frac{d(x,y,t)}{\sqrt{t}}\\ &\leq \ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4t} + \delta\ln\frac{A}{u(y,t)} + \frac{d^2(x,y,t)}{4\delta t}.\end{aligned}$$

Rearranging and exponentiating yield the desired interpolation inequality (1.10).

3 Proof Theorem 1.5

The method of proof of Theorem 1.5 is similar to that of Theorem 1.1, but it will be shown for completeness sake. Here the quantity $w = h\cdot|\nabla h|^2$ with $h = u^\varepsilon, 0 < \varepsilon < 1$ is used instead of $w = |\nabla\ln(1-h)|^2$ with $h = \ln u$ used in getting Souplet-Zhang estimate in Theorem 1.1.

Proof. Let $h = u^\varepsilon$, where $\varepsilon \in (0,1)$ is a constant to be chosen and $W = h\cdot|\nabla h|^2$. Direct computation implies $h_t = \varepsilon u^{\varepsilon-1}u_t$ and

$$\Big(\Delta_f - \frac{\partial}{\partial t}\Big)h = \frac{(\varepsilon-1)}{\varepsilon}\frac{|\nabla h|^2}{h}. \tag{3.1}$$

By the weighted Bochner formula (2.16) and (3.1), we obtain

$$\begin{aligned}\Big(\Delta_f - \frac{\partial}{\partial t}\Big)(|\nabla h|^2) &\geq 2|\nabla^2 h|^2 + 2\langle\nabla h, \nabla\Delta_f h\rangle - 2\langle\nabla h, \nabla h_t\rangle\\ &= 2|\nabla^2 h|^2 + 2\frac{\varepsilon-1}{\varepsilon}\nabla h\nabla\Big(\frac{|\nabla h|^2}{h}\Big).\end{aligned}$$

Similarly, by (2.2) derived from super Perelman-Ricci flow,

$$\begin{aligned}W_t &= h_t|\nabla h|^2 + h(|\nabla h|^2)_t\\ &\leq h_t|\nabla h|^2 + 2hRic_f(\nabla h, \nabla h) + 2h\nabla h\nabla h_t\end{aligned}$$

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and again by weighted Bochner formula (2.16)

$$\begin{aligned}\Delta_f W &= h\Delta_f(|\nabla h|^2) + |\nabla h|^2\Delta_f h + 2\nabla h\nabla(|\nabla h|^2)\\ &= 2h|\nabla^2 h|^2 + 2h\langle\nabla h, \nabla\Delta_f h\rangle + 2hRic_f(\nabla h,\nabla h) + |\nabla h|^2\Delta_f h + 2\nabla h\nabla(|\nabla h|^2).\end{aligned}$$

Hence,

$$\begin{aligned}\Big(\Delta_f - \frac{\partial}{\partial t}\Big)W &\geq 2h|\nabla^2 h|^2 + 2h\langle\nabla h, \nabla(\Delta_f h - h_t)\rangle + (\Delta_f - h_t)|\nabla h|^2 + 2\nabla h\nabla(|\nabla h|^2)\\ &= 2h|\nabla^2 h|^2 + 2\frac{\varepsilon-1}{\varepsilon}h\nabla h\nabla\Big(\frac{|\nabla h|^2}{h}\Big) + \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h} + 2\nabla h\nabla(|\nabla h|^2)\\ &= 2h|\nabla^2 h|^2 + 2\frac{\varepsilon-1}{\varepsilon}\nabla h\nabla(|\nabla h|^2) - \frac{\varepsilon-1}{\varepsilon}\frac{|\nabla h|^4}{h} + 2\nabla h\nabla(|\nabla h|^2)\\ &= 2\Big|\sqrt{h}|\nabla^2 h| + \frac{\nabla h\otimes\nabla h}{\sqrt{h}}\Big|^2 + 2\frac{\varepsilon-1}{\varepsilon}\nabla h\nabla(|\nabla h|^2) - \Big(\frac{\varepsilon-1}{\varepsilon}+2\Big)\frac{|\nabla h|^4}{h}\\ &\geq 2\frac{\varepsilon-1}{\varepsilon}\nabla h\nabla(|\nabla h|^2) - \Big(\frac{\varepsilon-1}{\varepsilon}+2\Big)\frac{|\nabla h|^4}{h}.\end{aligned}$$

Note that $\nabla h\cdot(h|\nabla h|^2) = |\nabla h|^2 + h\nabla h\nabla(|\nabla h|^2)$, which implies

$$\nabla h\nabla(|\nabla h|^2) = \frac{\nabla h}{h}\nabla(h|\nabla h|^2) - \frac{|\nabla h|^4}{h}.$$

Therefore

$$\begin{aligned}\Big(\Delta_f - \frac{\partial}{\partial t}\Big)W &\geq 2\frac{\varepsilon-1}{\varepsilon}\frac{\nabla h}{h}\nabla(h|\nabla h|^2) - \Big(3\frac{\varepsilon-1}{\varepsilon}+2\Big)\frac{|\nabla h|^4}{h}\\ &= 2\frac{\varepsilon-1}{\varepsilon}\frac{1}{h}\langle\nabla h,\nabla W\rangle - \Big(\frac{3(\varepsilon-1)+2\varepsilon}{\varepsilon}\Big)\frac{1}{h^3}W^2.\end{aligned}$$

Suppose we choose $\varepsilon = \frac{1}{3}$, we have

$$\Big(\Delta_f - \frac{\partial}{\partial t}\Big)W \geq -\frac{4}{h}\langle\nabla h,\nabla W\rangle + \frac{4}{h^3}W^2.$$

Now, follow the similar idea as in the proof of Theorem 1.1, using the same cut-off function ϕ satisfying properties $(i)-(iv)$ above.

$$\begin{aligned}\Big(\Delta_f - \frac{\partial}{\partial t}\Big)(W\phi) \geq\ & 2\frac{\varepsilon-1}{\varepsilon}\frac{1}{h}\Big[\nabla h\nabla(\phi W) - W\nabla h\nabla\phi\Big] - 2\frac{|\nabla\phi|^2}{\phi}W\\ &+ 2\frac{\nabla\phi}{\phi}\nabla(\phi W) - \frac{3(\varepsilon-1)+2\varepsilon}{\varepsilon}\frac{1}{h^3}\phi W^2 + W\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi.\end{aligned}$$

For a fixed $\tau\in(0,T)$, let (x_1,t_1) be a maximum point for ϕW in $\mathcal{B}_{R,T}$. It then follows that

$$\begin{aligned}-\frac{3(\varepsilon-1)+2\varepsilon}{\varepsilon}\phi W^2 &\leq -2\frac{\varepsilon-1}{\varepsilon}h^2 W\nabla h\nabla\phi + 2\frac{|\nabla\phi|^2}{\phi}h^3 W - h^3 W\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi\\ &= -2\frac{\varepsilon-1}{\varepsilon}h^{3/2}W^{3/2}\nabla\phi + 2\frac{|\nabla\phi|^2}{\phi}h^3 W - h^3 W\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi.\end{aligned}$$

Since $\varepsilon\in(0,1)$, $-2\frac{\varepsilon-1}{\varepsilon} > 0$ and $-\frac{3(\varepsilon-1)+2\varepsilon}{\varepsilon} > 0$. Therefore

$$\frac{3-\varepsilon}{\varepsilon}\phi W^2 \leq \frac{2(1-\varepsilon)}{\varepsilon}h^{3/2}W^{3/2}\nabla\phi + 2\frac{|\nabla\phi|^2}{\phi}h^3 W - h^3 W\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi. \tag{3.2}$$

We suppose $x \notin B(x_0, 1)$ and estimate each term of the right hand side of (3.1) as before. It follows from the Young's inequality, Cauchy-Schwarz inequality and properties of ϕ that:

$$\begin{aligned}\frac{2(1-\varepsilon)}{\varepsilon} h^{3/2} W^{3/2} \nabla\phi &= \frac{2(1-\varepsilon)}{\varepsilon} \phi^{3/4} W^{3/2} h^{3/2} \frac{\nabla\phi}{\phi^{3/4}} \\ &\leq \frac{1-\varepsilon}{\varepsilon}\phi W^2 + C(\varepsilon)(\sup_{\mathcal{B}_{R,T}}\{h\})^6 \frac{|\nabla\phi|^4}{\phi^3} \\ &\leq \frac{1-\varepsilon}{\varepsilon}\phi W^2 + \frac{C(\varepsilon)}{R^4}(\sup_{\mathcal{B}_{R,T}}\{h\})^6.\end{aligned}$$

and

$$\begin{aligned}2\frac{|\nabla\phi|^2}{\phi} h^3 W &\leq 2\frac{|\nabla\phi|^2}{\phi^{3/2}}\phi^{1/2} W(\sup_{\mathcal{B}_{R,T}}\{h\})^3 \\ &\leq \frac{1-\varepsilon}{\varepsilon}\phi W^2 + \Big(C(\varepsilon)\frac{|\nabla\phi|^2}{\phi^{3/2}}\Big)^2(\sup_{\mathcal{B}_{R,T}}\{h\})^6 \\ &\leq \frac{1-\varepsilon}{\varepsilon}\phi W^2 + \frac{C(\varepsilon)}{R^4}(\sup_{\mathcal{B}_{R,T}}\{h\})^6.\end{aligned}$$

Recall from previous explanation that

$$-\Delta_f\phi \leq C_{1/2}\Big(\frac{|\alpha|}{R} + nk + \frac{1}{R^2}\Big)\phi^{1/2} \quad \text{and} \quad \phi_t \leq \frac{C_{1/2}\phi^{1/2}}{\tau} + C_{1/2}\phi^{1/2} nk.$$

Hence, we have

$$\begin{aligned}-h^3 W\Big(\Delta_f - \frac{\partial}{\partial t}\Big)\phi &\leq C_{1/2} h^3 W\Big(\frac{|\alpha|}{R} + \frac{1}{R^2} + \frac{1}{\tau} + nk\Big)\phi^{1/2} \\ &\leq CW(\sup_{\mathcal{B}_{R,T}}\{h\})^3\Big(\frac{|\alpha|}{R} + \frac{1}{R^2} + \frac{1}{\tau} + k\Big)\phi^{1/2} \\ &\leq \frac{1-\varepsilon}{\varepsilon}\phi W^2 + C(\varepsilon)\Big(\frac{|\alpha|^2}{R^2} + \frac{1}{R^4} + \frac{1}{\tau^2} + k^2\Big)(\sup_{\mathcal{B}_{R,T}}\{h\})^6\end{aligned}$$

at (x_1, t_1) for some constant $C(\varepsilon) > 0$ depending on n and ε only. Putting these estimates into (3.2) yields

$$\frac{3-\varepsilon}{\varepsilon}\phi W^2 \leq \frac{3(1-\varepsilon)}{\varepsilon}\phi W^2 + C(\varepsilon)\Big(\frac{|\alpha|^2}{R^2} + \frac{1}{R^4} + \frac{1}{\tau^2} + k^2\Big)(\sup_{\mathcal{B}_{R,T}}\{h\})^6$$

at (x_1, t_1). It therefore follows that

$$\phi W^2 \leq C\Big(\frac{|\alpha|^2}{R^2} + \frac{1}{R^4} + \frac{1}{\tau^2} + k^2\Big)(\sup_{\mathcal{B}_{R,T}}\{h\})^6.$$

Then for all $(x, \tau) \in \mathcal{B}_{R/2,T/2}$, there holds

$$(\phi^2 W^2)(x,\tau) \leq (\phi^2 W^2)(x_1, t_1) \leq (\phi W^2)(x_1, t_1)$$

. Since $\phi(x,\tau) = 1$ when $d(x_1, x) \leq R/2$

$$\begin{aligned}W(x,\tau) &\leq \phi W(x_1, t_1) \\ &\leq C(\varepsilon)\Big(\frac{|\alpha|^2}{R^2} + \frac{1}{R^4} + \frac{1}{\tau^2} + k^2\Big)^{1/2}(\sup_{\mathcal{B}_{R,T}}\{h\})^3\end{aligned}$$

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for all $(x,t) \in \mathcal{B}_{R/2,T/2}$. Since $\tau \in (0,T]$ is arbitrary, $W = h \cdot |\nabla h|^2 = \varepsilon^2 h^3 \dfrac{|\nabla u|^2}{u^2}$ and $\sup_{\mathcal{B}_{R,T}}\{h\} = \sup_{\mathcal{B}_{R,T}}\{u^\varepsilon\} \leq (\sup_{\mathcal{B}_{R,T}}\{u\})^\varepsilon$, then we have

$$\frac{|\nabla u|^2}{u^{2-3\varepsilon}} \leq C(\varepsilon)(\sup_{\mathcal{B}_{R,T}}\{u\})^{3\varepsilon}\Big(\frac{|\alpha|^2}{R^2} + \frac{1}{R^4} + \frac{1}{T^2} + k^2\Big)^{1/2}$$

for all $(x,t) \in \mathcal{B}_{R/2,T/2}$. In conclusion, choosing $\varepsilon = \frac{1}{3}$ yields the required result.

4 Remarks on Liouville type theorem

Our estimates in Theorems 1.1 and 1.3 are obtained under the constrained super Perelman-Ricci flow. Indeed, they are extensions of Hamilton's [2] and Souplet-Zhang's estimates to the setting of smooth metric measure spaces with time dependent metrics and potentials. As in the classical setting, these estimates can be used to obtain sharp estimates on weighted heat kernel, Harnack inequality and Liouville principle. For example, we briefly highlight how Liouville principle can be obtained.

Letting $R \to +\infty$ and $t \to +\infty$ in estimates (1.8) and (1.12), respectively give

$$\frac{|\nabla u|}{u} \leq C\sqrt{k}\Big(1 + \ln \frac{A}{\inf u}\Big) \tag{4.1}$$

and

$$\frac{|\nabla u|}{u} \leq C\sqrt{k} \tag{4.2}$$

for a bounded positive solution u of $\Delta_f u = 0$ on smooth metric measure spaces with $Ric_f \geq -k$. In particular, if $Ric_f \geq 0$, each of the estimates in (4.1) and (4.2) implies that every bounded weighted harmonic function must be a constant.

Finally, we show in the last two propositions two approaches to obtaining triviality of solution to the heat equation with cutvature free conditions.

Proposition 4.1. *Suppose $\{(M, g(t), f(t)) : t \in [0,T]\}$ is a complete solution to the super Perelman-Ricci flow (1.2). Let $u = u(x,t)$ be a solution to the weighted heat equation (1.1) for $x \in M$ and $t > 0$, which is uniformly bounded. Then u is constant.*

Proof. Define a quantity $U(x,t) = u^2(x,t) + 2t|\nabla u(x,t)|^2$ for $t > 0$ and compute its time evolution $U_t = (u^2 + 2t|\nabla u|^2)_t$ (subscript t means partial derivative with respect to t). Reverting to (1.1) a direct computation gives

$$(u^2)_t = 2uu_t = 2u\Delta_f u = \Delta_f u^2 - 2|\nabla u|^2.$$

Reverting to (1.1), (1.2), (2.2) and (2.16)

$$(|\nabla u|^2)_t \leq 2Ric_f(\nabla u, \nabla u) + 2\nabla u \nabla \Delta_f u = \Delta_f(|\nabla u|^2) - 2|\nabla^2 u|^2.$$

Then by the last two expressions we arrive at

$$(u^2 + 2t|\nabla u|^2)_t \leq \Delta_f(u^2 + 2t|\nabla u|^2)$$

and so by the maximum principle we have $U_{max} \leq 0$ a.e., implying that $U(x,t) \leq U(x,0)$ and

$$u^2(x,t) + 2t|\nabla u(x,t)|^2 \leq u^2(x,0) \leq \sup_M\{u(x,0)\}^2 < +\infty$$

a.e., which then follows that $|\nabla u|^2 \leq c \sup_M\{u|_{t=0}\}^2/t$ (c is a finite positive number). Sending t to $+\infty$ yields $|\nabla u| \equiv 0$ $\forall x \in M, t > 0$, that is, u is constant.

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Proposition 4.2. *Suppose $\{(M, g(t), f(t)) : t \in [0, T]\}$ is a complete solution to the super Perelman-Ricci flow (1.2). Let $u = u(x, t)$ be a solution to the weighted heat equation (1.1) for $x \in M$ and $t > 0$, with uniformly bounded L^2-norm. Then u is constant.*

Proof. Here we use the so-called energy estimate. Define the energy

$$E_p(u(x,t)) = \int_M u^p(x,t) d\mu_{g(t)}, \quad p \geq 1.$$

Recall that the weighted measure $d\mu_{g(t)} = e^{-f(t)}dv$ is stationary in the sense that $f_t = \frac{1}{2}Tr\left(\frac{\partial g}{\partial t}\right)$ (see (1.2)) and $(d\mu_{g(t)})_t = 0$ (see [6, eq.(1.15)]). Note that heat is conserved in the sense that $E_1'(u) = 0$ (by differentiation and integration by parts) and $E_1(u) = E_1(u_0)$ $\forall t \geq 0$. Also a simple but standard computation (omitted here) shows that $E_p(u)$ is nonincreasing in time under the Perelman-Ricci flow.

Now consider the functional $F(u(x,t))$ defined by

$$F(u) = \int_M (u^2 + 2t|\nabla u|^2) d\mu.$$

Denote by $'$ and $''$, the first and second order derivatives with respect to t respectively. Note that $E_2(u) = \int_M u^2 d\mu$, $E_2'(u) = -2\int_M |\nabla u|^2 d\mu$ and

$$\begin{aligned} E_2''(u) &= -2\frac{d}{dt}\int_M |\nabla u|^2 d\mu = -2\int_M (|\nabla u|^2)_t d\mu \\ &\geq -4\int_M (Ric_f(\nabla u, \nabla u) + \nabla u \nabla \Delta_f u) d\mu \\ &= -2\int_M (\Delta_f(|\nabla u|^2) - 2|\nabla^2 u|^2) d\mu = 4\int_M |\nabla^2 u|^2 d\mu. \end{aligned}$$

by the compactness of $\{(M, g(t), f(t))\}$. Thus,
$F(u) = E_2(u) - tE_2'(u)$ and $F'(u) = -tE_2''(u) \leq -4t\int_M |\nabla^2 u|^2 d\mu$ showing that $F(u)$ is monotone increasing for $t \geq 0$. It then follows that

$$F(u(x,t)) \leq F(u(x,0)) = \int_M u^2(x,0) d\mu, \quad \forall t \geq 0$$

and then

$$\int_M |\nabla u(x,t)|^2 \leq \frac{c}{t}\|u(x,0)\|^2_{L^2(M)}.$$

With the assumption that the L^2-norm of u is uniformly bounded in time, one then concludes that u is constant in space since $|\nabla u| \equiv 0$ everywhere as $t \to +\infty$.

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Competing Financial Interests

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