International Journal of Mathematical Sciences and
Optimization: Theory and Applications
Vol. 6, No. 2, pp. 885 - 900.
doi.org/10.6084/m9.figshare.13829129.

Spectral Invariants of Laplacians on Quaternionic Projective Spaces $\mathsf{P}^n(\mathbb{H})$ $(n \geq 1$

Richard Olu Awonusika

Department of Mathematical Sciences, Adekunle Ajasin University, Akungba Akoko, Ondo State, Nigeria.

Email Address: richard.awonusika@aaua.edu.ng

Article Info

Received: 22 December 2020 Revised: 01 February 2021
Accepted: 02 February 2021 Available online: 05 February 2021

Abstract

This paper presents explicit closed-form formulae for the determinants of the Laplacians on n-dimensional quaternionic projective spaces $\mathsf{P}^n(\mathbb{H})$ $(n \geq 1)$. The explicit calculation of this spectral invariant associated with $\mathsf{P}^n(\mathbb{H})$ is novel in the context of spectral theory of the Laplacian on Riemannian manifolds. Other spectral properties of the Laplacian on $\mathsf{P}^n(\mathbb{H})$ – the Minakshisundaram-Pleijel coefficients and Minakshisundaram-Pleijel zeta functions are also explicitly discussed.

Keywords and Phrases: Quaternionic projective space, Determinant of Laplacian, Minakshisundaram-Pleijel coefficients, Spectral zeta function.

MSC2010:33C45, 35A08, 35C05, 58J35, 58J50

1 Introduction

Let $\mathsf{M} = G/K$ be a N-dimensional $(N \geq 1)$ compact rank one symmetric space, K_N and Δ_N the associated heat kernel and Laplace-Beltrami operator (or simply Laplacian) respectively. Here the Lie group G is compact and K is the isotropy group of a point in M. Using the addition formula for the matrix coefficients (see, e.g., [1][Ch. IV]), it is not difficult to see that the heat kernel $\mathsf{K}_N(t,x,y)$ takes the form (see [2][Appendix A.1])

$$\mathsf{K}_N(t,\vartheta) = \frac{1}{\omega_N}\sum_{k=0}^{\infty} \Lambda_k^N \Psi_k^N(\vartheta) e^{-\lambda_k^N t}. \tag{1.0.1}$$

The numbers λ_k^N (with $k \geq 0$) are the numerically distinct eigenvalues of Δ_N, Λ_k^N is the dimension of the eigenspace associated with λ_k^N (i.e., the multiplicity of the eigenvalue λ_k^N), $\Psi_k^N(\vartheta)$ is the spherical function on M associated with the eigenvalue λ_k^N, ϑ is the geodesic distance between the points $x, y \in \mathsf{M}$ and $\omega_N = \mathrm{Vol}(\mathsf{M})$ is the volume of M. It is remarkable to know that the spherical functions $\Psi_k^N(\vartheta)$ on M can be explicitly given in terms of the normalised Jacobi polynomials $\mathsf{P}_k^{(\alpha,\beta)}(\cos\vartheta) :=$

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$P_k^{(\alpha,\beta)}(\cos\vartheta)/P_k^{(\alpha,\beta)}(1)$ (with $k \geq 0; \alpha, \beta > -1$). For thorough discussions on heat kernels in Riemannian manifolds, see [3–11].

Examples of rank one compact symmetric spaces include the sphere $\mathbb{S}^n = \mathsf{SO}(n+1)/\mathsf{SO}(n)$, the real projective space $\mathsf{P}^n(\mathbb{R}) = \mathsf{SO}(n+1)/\mathsf{O}(n)$, the complex projective space $\mathsf{P}^n(\mathbb{C}) = \mathsf{SU}(n+1)/\mathsf{S}(\mathsf{U}(n)\times\mathsf{U}(1))$ (of real dimension $2n$), the quaternionic projective space $\mathsf{P}^n(\mathbb{H}) = \mathsf{Sp}(n+1)/\mathsf{Sp}(n)\times\mathsf{Sp}(1)$ (of real dimension $4n$) and the Cayley projective plane $\mathsf{P}^2(\mathrm{Cay}) = \mathsf{F}_4/\mathsf{Spin}(9)$ (of real dimension 16). See the monographs [9,12,13,15,16] for related material on Lie groups and symmetric spaces.

This paper specialises M to the real $4n$-dimensional ($n \geq 1$) quaternionic projective space $\mathsf{P}^n(\mathbb{H})$. Recall that the symplectic group $\mathsf{Sp}(n)$ is the set of $n \times n$ matrices with entries in the quaternions $\mathbb{H}$, where $\mathbb{H}$ consists of element $x + iy + jz + kw$ with $x, y, z, w \in \mathbb{R}$; i.e., $\mathbb{H} \cong \mathbb{R}^4$. Projective spaces are useful in: coding theory, number theory, design theory, physics, combinatorics, computer vision modelling, computer graphics, and extremal combinatorial problems ([17]). The quaternionic projective space is a space in which the coordinates lie in the ring of quaternions $\mathbb{H}$. The quaternionic projective line $\mathsf{P}^1(\mathbb{H})$ is homeomorphic to the sphere $\mathbb{S}^4$. In a group theoretic description, $\mathsf{P}^n(\mathbb{H})$ is the orbit space of $\mathbb{H}^{n+1}\setminus\{(0,\ldots,0)\}$ by the action of $\mathbb{H}^\times$, where $\mathbb{H}^\times$ is the multiplicative group of nonzero quaternions. If we first project onto the unit sphere inside $\mathbb{H}^{n+1}$, then one may also consider $\mathsf{P}^n(\mathbb{H})$ as the orbit space of $\mathbb{S}^{4n+3}$ by the action of $\mathsf{Sp}(1)$, the group of unit quaternions; under this consideration, the sphere $\mathbb{S}^{4n+3}$ then becomes a principal $\mathsf{Sp}(1)$-bundle over $\mathsf{P}^n(\mathbb{H})$ (called a generalised Hopf fibration):

$$\mathsf{Sp}(1) \to \mathbb{S}^{4n+3} \to \mathsf{P}^n(\mathbb{H}).$$

For further discussion on the geometry of quaternionic manifolds, see [18].

In this special case of $\mathsf{M} = \mathsf{P}^n(\mathbb{H})$, the Laplacian Δ_N on $\mathsf{P}^n(\mathbb{H})$ has eigenvalues given explicitly by [1] $\lambda_k^n = (k(k+2n+1) : k \geq 0)$ each of which with multiplicity

$$\Lambda_k^n = \frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\Gamma(2n+2)\Gamma(2n)\Gamma(k+1)\Gamma(k+2)}, \qquad k \geq 0. \tag{1.0.2}$$

Corresponding to the eigenvalue λ_k^n are the (normalised) eigenfunctions (called spherical functions) Ψ_k^n given explicitly by the normalised Jacobi polynomials

$$\Psi_k^n(\vartheta) := \mathsf{P}_k^{(2n-1,1)}(\cos\vartheta) = \frac{P_k^{(2n-1,1)}(\cos\vartheta)}{P_k^{(2n-1,1)}(1)}, \tag{1.0.3}$$

where $P_k^{(\alpha,\beta)} = P_k^{(\alpha,\beta)}(t)$ (with integer $k \geq 0$ and real $\alpha, \beta > -1$) is the Jacobi polynomial with

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)\Gamma(k+1)}. \tag{1.0.4}$$

For further treatment of Jacobi polynomials, the interested reader is referred to [19–23].

It is now straight forward to see that the heat kernel on $\mathsf{P}^n(\mathbb{H})$ admits the spectral series representation

$$\begin{aligned}
\mathsf{K}_N(t,\vartheta) &= \frac{1}{\omega_n}\sum_{k=0}^{\infty}\Lambda_k^n\Psi_k^n e^{-\lambda_k^n t} \\
&= \sum_{k=0}^{\infty}\frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\omega_n\Gamma(2n)\Gamma(2n+2)\Gamma(k+1)\Gamma(k+2)}\mathsf{P}_k^{(2n-1,1)}(\cos\vartheta)e^{-k(k+2n+1)t},
\end{aligned} \tag{1.0.5}$$

[1]Note that $\mathsf{P}^n(\mathbb{H})$ as a real manifold has dimension $4n$. For the sake of convenience of notation here and in future we slightly abuse notation and denote its associated spectral and geometric data by $\Delta_n, \lambda_k^n, \Lambda_k^n, \omega_n, a_k^n, \Psi_k^n, \mathsf{Z}_n, \mathsf{K}_n, \zeta_n$, instead of $\Delta_{4n}, \lambda_k^{4n}, \Lambda_k^{4n}, \omega_{4n}, a_k^{4n}, \Psi_k^{4n}, \mathsf{Z}_{4n}, \mathsf{K}_{4n}, \zeta_{4n}$, respectively.

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Vol. 6, No. 2, pp. 885 - 900.
doi.org/10.6084/m9.figshare.13829129.

where $\omega_n = \mathrm{Vol}(\mathsf{P}^n(\mathbb{H})) = (4\pi)^{2n}/\Gamma(2n+2)$.

The heat trace $\mathsf{Z}_n(t)$ associated with $\mathsf{P}^n(\mathbb{H})$ takes the form ([24])

$$\mathsf{Z}_n(t) = \sum_{k=0}^{\infty} \frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\Gamma(2n)\Gamma(2n+2)\Gamma(k+1)\Gamma(k+2)} e^{-k(k+2n+1)t}. \tag{1.0.6}$$

It is well-known ([3], [24]) that the heat trace $\mathsf{Z}_n(t)$ satisfies the asymptotic expansion

$$\begin{aligned} \mathsf{Z}_n(t) &\sim \frac{\omega_n e^{\frac{(2n+1)^2}{4}t}}{(4\pi t)^{2n}} \left[1 + \sum_{k=1}^{\infty} u_k^n t^k\right], \qquad t \searrow 0 \\ &\sim \frac{1}{(4\pi t)^{2n}} \sum_{k=0}^{\infty} a_k^n t^k, \qquad t \searrow 0, \end{aligned} \tag{1.0.7}$$

where

$$a_k^n = \omega_n \sum_{j=0}^{k} \frac{\left[(2n+1)^2/4\right]^{k-j}}{(k-j)!} u_j^n, \tag{1.0.8}$$

for some coefficients u_k^n $(k \geq 0)$.

As a result the Minakshisundaram-Pleijel zeta function $\zeta_n(s)$ has the formulation ([3])

$$\zeta_n(s) = \sum_{k=1}^{\infty} \frac{\Lambda_k^n}{[\lambda_k^n]^s} = \sum_{k=1}^{\infty} \frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\Gamma(2n+2)\Gamma(2n)\Gamma(k+1)\Gamma(k+2)[k(k+2n+1)]^s}. \quad \mathrm{Re}\, s > 2n. \tag{1.0.9}$$

It follows immediately that ζ_n can be analytically continued to a meromorphic function on $\mathbb{C}$ with simple poles located at the points $s = 2n, 2n-1, \ldots, 2$. The residues of ζ_n at these poles and their relation to the heat coefficients a_k^n will be discussed in Section 3.

The Minakshisundaram-Pleijel zeta functions are useful tools in the computation of the determinants of the Laplacians on Riemannian manifolds. For the applications of the spectral zeta functions in this regard, see, e.g., [25–44]. Other important applications of spectral zeta functions are in analytic number theory, harmonic analysis on symmetric spaces and quantum field theory ([45–50]).

The set of all surfaces, with associated varying metrics, and with determinants of associated Laplacians are useful tools in the study of modern quantum geometry of strings. The determinant can therefore be considered as a function of the metric on surfaces and its extreme values can as well be estimated. For the unification of this problem by the determinant of the Laplacian, see [39]. For the determinants of Laplacians on spheres, see [30,31,36,38,43,51]. See [52] for a recent survey on the explicit description of spectral invariants of Laplacians on spheres. The spectral invariants of the Laplacians on the complex projective spaces and Cayley projective plane are considered in [53] and [54] respectively. In [2] and [24], we introduced and studied a new class of heat coefficients, namely, the Maclaurin heat coefficients (i.e., the coefficients appearing in the Maclaurin expansion of the heat kernel) in terms of the classical and generalised Minakshisundaram-Pleijel coefficients, when $\mathsf{M} = \mathsf{P}^n(\mathbb{C})$ and $\mathsf{M} = \mathsf{P}^n(\mathbb{H})$ $(n \geq 1)$ respectively. Remarkable asymptotic expansions for the Maclaurin spectral functions were established. We also introduced and constructed new zeta functions associated with these Maclaurin heat coefficients (generalised Minakshisundaram-Pleijel zeta functions), and it was interesting that these generalised zeta functions could be explicitly understood in terms of the classical (Minakshisundaram-Pleijel) zeta functions. To the best of our knowledge, the determinant of the Laplacian on the quaternionic projective space $\mathsf{P}^n(\mathbb{H})$ has not appeared in the literature. It is the purpose of this paper to determine explicitly the determinants of the Laplacians on the quaternionic projective spaces $\mathsf{P}^n(\mathbb{H})$ $(n \geq 1)$:

$$-\log\det\Delta_n = \frac{d}{ds}\zeta_n(s)\bigg|_{s=0} = \lim_{s\searrow 0} \frac{\zeta_n(s) - \zeta_n(0)}{s}. \tag{1.0.10}$$

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2 The Heat Coefficients a_k^n Associated with $\mathsf{P}^n(\mathbb{H})$

In this section, we present results on the heat coefficients a_k^n associated with $\mathsf{P}^n(\mathbb{H})$ as calculated explicitly in [24] following Cahn and Wolf ([55]). The idea is to first express the heat trace $\mathsf{Z}_n(t)$ associated with $\mathsf{P}^n(\mathbb{H})$ purely in terms of a Jacobi theta function and its higher order derivatives, and then employ the asymptotics of the Jacobi theta function (as $t \searrow 0$) in the spirit of [56]. Here, for completeness, we present the proof as given in [24].

Theorem 2.1 ([24]) *The heat trace $\mathsf{Z}_n(t)$ associated with $\mathsf{P}^n(\mathbb{H})$ (given by (1.0.6)) admits the Minakshisundaram-Pleijel asymptotic expansion*

$$\mathsf{Z}_n(t) \sim \frac{1}{(4\pi t)^{2n}} \sum_{k=0}^{\infty} a_k^n t^k \qquad (\text{as } t \searrow 0), \tag{2.0.1}$$

where the heat coefficients a_k^n are given by

$$\begin{aligned} a_k^n &= \omega_n \sum_{j=0}^{k} \frac{(2n+1)^{2k-2j} u_j^n}{4^{k-j}(k-j)!}, \qquad u_k^n = \mathscr{A}_{2n-1-k}^n \frac{(2n-k-1)!}{(2n-1)!}, \ 0 \le k \le 2n-1; \\ a_k^n &= \omega_n \sum_{j=0}^{k} \frac{(2n+1)^{2k-2j} u_j^n}{4^{k-j}(k-j)!}, \qquad u_k^n = \sum_{\ell=0}^{2n-1} \frac{(-1)^\ell \mathscr{A}_\ell^n}{(k-2n)!} \frac{\mathsf{B}_{k+\ell-2n}}{(2n-1)!}, \ k \ge 2n; \end{aligned} \tag{2.0.2}$$

with $u_0^n = 1$ and $a_0^n = \omega_n$. Here the coefficient $\mathscr{A}_m^n$ is defined by

$$\left[\eta^2 - \left(\frac{2n-1}{2}\right)^2\right] \prod_{j=1/2}^{\frac{2n-3}{2}} \left[\eta^2 - j^2\right]^2 = \sum_{m=0}^{2n-1} \mathscr{A}_m^n \eta^{2m} \tag{2.0.3}$$

and

$$\mathsf{B}_m = \frac{(-1)^m \left(1 - 2^{-2m-1}\right)}{(m+1)} B_{2m+2}, \tag{2.0.4}$$

where B_m is the well-known mth Bernoulli number.

Proof 2.1 *Writing the multiplicity Λ_k^n in a polynomial form we have*

$$\begin{aligned} \Lambda_k^n &= \frac{(2k+2n+1)\Gamma(k+2n)\Gamma(k+2n+1)}{\Gamma(2n+2)\Gamma(2n)\Gamma(k+1)\Gamma(k+2)} = \frac{(k+2n)(2k+2n+1)}{2n(2n+1)(k+1)} \prod_{j=1}^{2n-1} \left(\frac{k+j}{j}\right)^2 \\ &= \frac{2\left(k+\frac{2n+1}{2}\right)\left[\left(k+\frac{2n+1}{2}\right)^2 - \left(\frac{2n-1}{2}\right)^2\right]}{(2n+1)!(2n-1)!} \prod_{j=1/2}^{\frac{2n-3}{2}} \left[\left(k+\frac{2n+1}{2}\right)^2 - j^2\right]^2 \\ &= \frac{2}{(2n+1)!(2n-1)!} \sum_{m=0}^{2n-1} \mathscr{A}_m^n \left(k+\frac{2n+1}{2}\right)^{2m+1}. \end{aligned} \tag{2.0.5}$$

Using the multiplicity (2.0.5) in the trace formula (1.0.6) we have

$$\begin{aligned} \mathsf{Z}_n(t) &= \frac{2e^{\frac{(2n+1)^2}{4}t}}{(2n+1)!(2n-1)!} \sum_{k=0}^{\infty} \sum_{m=0}^{2n-1} \mathscr{A}_m^n \left(k+\frac{2n+1}{2}\right)^{2m+1} e^{-\left(k+\frac{2n+1}{2}\right)^2 t} \\ &= \frac{e^{\frac{(2n+1)^2}{4}t}}{(2n+1)!(2n-1)!} \sum_{m=0}^{2n-1} \mathscr{A}_m^n 2 \sum_{s=(2n+1)/2}^{\infty} s^{2m+1} e^{-s^2 t}. \end{aligned} \tag{2.0.6}$$

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doi.org/10.6084/m9.figshare.13829129.

It is interesting to see that (for $n \geq 1$),

$$\sum_{m=0}^{2n-1} \mathscr{A}_m^n 2 \sum_{s=(2n+1)/2}^{\infty} s^{2m+1} e^{-s^2 t} = \sum_{m=0}^{2n-1} \mathscr{A}_m^n 2 \sum_{s=1/2}^{\infty} s^{2m+1} e^{-s^2 t}$$
$$= \sum_{m=0}^{2n-1} (-1)^m \mathscr{A}_m^n \theta^{(m)}(t), \tag{2.0.7}$$

where the Jacobi theta function θ is given by

$$\theta(t) = \sum_{j=0}^{\infty} (2j+1) e^{-(j+\frac{1}{2})^2 t} \backsimeq \frac{1}{t} + \sum_{j=0}^{\infty} \mathsf{B}_j t^j, \quad \textit{(see, e.g., [56])}, \tag{2.0.8}$$

with $\mathsf{B}_j = \mathsf{B}_j/j!$. We see, for example ($n = 1$), that

$$\mathsf{Z}_1(t) = \frac{e^{\frac{9}{4}t}}{6} \left[-\frac{1}{4} 2 \sum_{s=3/2}^{\infty} s e^{-s^2 t} + 2 \sum_{s=3/2}^{\infty} s^3 e^{-s^2 t} \right]$$
$$= \frac{e^{\frac{9}{4}t}}{6} \left[-\frac{1}{4} 2 \sum_{s=1/2}^{\infty} s e^{-s^2 t} + 2 \sum_{s=1/2}^{\infty} s^3 e^{-s^2 t} \right]$$
$$= \frac{e^{\frac{9}{4}t}}{6} \left[-\theta'(t) - \frac{1}{4}\theta(t) \right]. \tag{2.0.9}$$

(The higher cases $n \geq 2$ follow similarly.) Thus

$$\mathsf{Z}_n(t) = \frac{e^{\frac{(2n+1)^2}{4}t}}{(2n+1)!(2n-1)!} \sum_{m=0}^{2n-1} (-1)^m \mathscr{A}_m^n \theta^{(m)}(t). \tag{2.0.10}$$

In general, we have

$$\theta^{(p)}(t) \backsimeq \frac{(-1)^p p!}{t^{p+1}} + \sum_{j=p}^{\infty} \mathsf{B}_j^p t^{j-p}, \qquad p \geq 1, \tag{2.0.11}$$

where $\mathsf{B}_j^p = \mathsf{B}_j/(j-p)!$. Substituting the generalised Jacobi theta function (2.0.11) in the trace formula (2.0.10), we have

$$\mathsf{Z}_n(t) \backsimeq \frac{e^{\frac{(2n+1)^2}{4}t}}{(2n+1)!(2n-1)!} \left[\frac{(2n-1)!}{t^{2n}} + \sum_{\ell=0}^{2n-2} \frac{\mathscr{A}_\ell^n \ell!}{t^{\ell+1}} + \sum_{\ell=0}^{2n-1} \widetilde{\mathscr{A}_\ell^n} \sum_{j=\ell}^{\infty} \mathsf{B}_j^\ell t^{j-\ell} \right] \tag{2.0.12}$$

(where $\widetilde{\mathscr{A}_\ell^n} = (-1)^\ell \mathscr{A}_\ell^n$), which after further simplification gives

$$\mathsf{Z}_n(t) \backsimeq \frac{\omega_n e^{\frac{(2n+1)^2}{4}t}}{(4\pi t)^{2n}} \left[1 + \sum_{\ell=0}^{2n-2} \mathsf{A}_\ell^n t^{2n-\ell-1} + \sum_{\ell=0}^{2n-1} \widetilde{\mathsf{A}}_\ell^n \sum_{j=\ell}^{\infty} \mathsf{B}_j^\ell t^{2n+j-\ell} \right]$$
$$= \frac{1}{(4\pi t)^{2n}} \sum_{k=0}^{\infty} a_k^n t^k, \tag{2.0.13}$$

where $\mathsf{A}_\ell^n = \mathscr{A}_\ell^n \ell!/(2n-1)!$, $\widetilde{\mathsf{A}}_\ell^n = \widetilde{\mathscr{A}_\ell^n}/(2n-1)!$. Consequently, we obtain the heat coefficients a_k^n given in (2.0.2) as required.

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The first few coefficients $(\mathscr{A}_m^n : 0 \le m \le n-1)$ are illustrated in Table 1.

Table 1: The coefficients $(\mathscr{A}_m^n : 0 \le m \le 2n-1)$.

$\mathscr{A}_0^1$	$\mathscr{A}_1^1$	$\mathscr{A}_0^2$	$\mathscr{A}_1^2$	$\mathscr{A}_2^2$	$\mathscr{A}_3^2$	$\mathscr{A}_0^3$	$\mathscr{A}_1^3$	$\mathscr{A}_2^3$	$\mathscr{A}_3^3$	$\mathscr{A}_4^3$	$\mathscr{A}_5^3$
$-\frac{1}{4}$	1	$-\frac{9}{64}$	$\frac{19}{16}$	$-\frac{11}{4}$	1	$-\frac{2025}{1024}$	$\frac{4581}{256}$	$-\frac{1665}{32}$	$\frac{313}{8}$	$-\frac{45}{4}$	1

Example 2.2 *We give explicit computations of the heat coefficients a_k^n for the special cases $n = 1, 2, 3$.*

- *($n=1$) Clearly, $\mathsf{P}^1(\mathbb{H})$ is the sphere $\mathbb{S}^4$. It is easily seen in this case that*

$$\mathsf{Z}_1(t) = \frac{e^{\frac{9}{4}t}}{6}\left[-\theta'(t) - \frac{1}{4}\theta(t)\right]. \tag{2.0.14}$$

Upon substituting the theta function (2.0.11) we have

$$\mathsf{Z}_1(t) \backsimeq \frac{\omega_1 e^{\frac{9}{4}t}}{(4\pi t)^2}\left[1 - \frac{t}{4} + \sum_{j=2}^{\infty} u_j^1 t^j\right], \tag{2.0.15}$$

where

$$u_0^1 = 1,\ u_1^1 = -\frac{1}{4},\ u_k^1 = -\left(\mathsf{B}_{k-1}^1 + \frac{1}{4}\mathsf{B}_{k-2}^0\right),\ k \ge 2. \tag{2.0.16}$$

Thus the Minakshisundaram-Pleijel heat coefficients a_k^1 given by

$$a_k^1 = \omega_1 \sum_{j=0}^{k} \frac{\left(\frac{9}{4}\right)^{k-j}}{(k-j)!} u_j^1, \qquad k \ge 0, \tag{2.0.17}$$

follow immediately.

- *($n=2$) It is clear that*

$$\mathsf{Z}_2(t) = \frac{e^{\frac{25}{4}t}}{5!3!}\left[-\theta'''(t) - \frac{11}{4}\theta''(t) - \frac{19}{16}\theta'(t) - \frac{9}{64}\theta(t)\right], \tag{2.0.18}$$

and consequently, we have

$$\mathsf{Z}_2(t) \backsimeq \frac{\omega_2 e^{\frac{25}{4}t}}{(4\pi t)^4}\left[1 - \frac{11t}{12} + \frac{19t^2}{96} - \frac{3t^3}{128} + \sum_{j=4}^{\infty} u_j^2 t^j\right], \tag{2.0.19}$$

where

$$\begin{aligned} &u_0^2 = 1,\ u_1^2 = -\frac{11}{12},\ u_2^2 = \frac{19}{96},\ u_3^2 = -\frac{3}{128}, \\ &u_k^2 = -\frac{1}{6}\left(\mathsf{B}_{k-1}^3 + \frac{11}{4}\mathsf{B}_{k-2}^2 + \frac{19}{16}\mathsf{B}_{k-3}^1 + \frac{9}{64}\mathsf{B}_{k-4}^0\right)\ k \ge 4. \end{aligned} \tag{2.0.20}$$

Hence, we obtain the heat coefficients a_k^2 given by

$$a_k^2 = \omega_2 \sum_{j=0}^{k} \frac{(25/4)^{k-j} u_j^2}{(k-j)!}, \qquad k \ge 0. \tag{2.0.21}$$

- $(n=3)$ *Indeed we have*

$$\mathsf{Z}_3(t) = \frac{e^{\frac{49}{4}t}}{7!5!}\left[-\theta^{(5)}(t) - \frac{45}{4}\theta^{(4)}(t) - \frac{313}{8}\theta'''(t) - \frac{1665}{32}\theta''(t) - \frac{4581}{256}\theta'(t) - \frac{2025}{1024}\theta(t)\right], \tag{2.0.22}$$

from which we obtain the asymptotic formula

$$\mathsf{Z}_3(t) \backsimeq \frac{\omega_3 e^{\frac{49}{4}t}}{(4\pi t)^6}\left[1 - \frac{9t}{4} + \frac{313t^2}{160} - \frac{111t^3}{128} + \frac{4581t^4}{30720} - \frac{135t^5}{8192} + \sum_{j=6}^{\infty} u_j^3 t^j\right], \tag{2.0.23}$$

where

$$u_0^3 = 1,\ u_1^3 = -\frac{9}{4},\ u_2^3 = \frac{313}{160},\ u_3^3 = -\frac{111}{128},\ u_4^3 = \frac{4581}{30720},\ u_5^3 = -\frac{135}{8192},$$
$$u_k^3 = -\frac{1}{120}\left(\mathsf{B}_{k-1}^5 + \frac{45}{4}\mathsf{B}_{k-2}^4 + \frac{313}{8}\mathsf{B}_{k-3}^3 + \frac{1665}{32}\mathsf{B}_{k-4}^2 + \frac{4581}{256}\mathsf{B}_{k-5}^1 + \frac{2025}{1024}\mathsf{B}_{k-6}^0\right)\ k \geq 6. \tag{2.0.24}$$

As a result, we obtain the Minakshisundaram-Pleijel heat coefficients a_k^3 given by

$$a_k^3 = \omega_3 \sum_{j=0}^{k} \frac{(49/4)^{k-j} u_j^3}{(k-j)!}, \qquad k \geq 0. \tag{2.0.25}$$

3 Description of Heat Coefficients a_k^n By Residues of Zeta Functions ζ_n

In this section, we relate the heat coefficients a_k^n to the residues of the zeta functions ζ_n. That is, we show that the Minakshisundaram-Pleijel formula [24][(1.8)] also holds for the special case of the quaternionic projective space $\mathsf{P}^n(\mathbb{H})$. This is done by first expressing the Minakshisundaram-Pleijel zeta function ζ_n in terms of the Hurwitz zeta function $\zeta(z,\cdot)$ $(z \in \mathbb{C})$ of number theory.

By definition, the Hurwitz zeta function $\zeta(z,a)$ is defined, for $a > 0$, by (see [51][Sec. 2.2], [57])

$$\zeta(z,a) = \sum_{j=0}^{\infty} \frac{1}{(j+a)^z}, \qquad \mathrm{Re}\, z > 1. \tag{3.0.1}$$

The Hurwitz zeta function $\zeta(z,\cdot)$ can be analytically continued to a meromorphic function on all of $\mathbb{C}$ with its only (simple) pole located at $z=1$ and the residue at this pole is one.

The following theorem is given in [24] without a proof. Here we present a proof of the result.

Theorem 3.1 ([24]) *The Minakshisundaram-Pleijel zeta function ζ_n can be written in terms of the Hurwitz zeta function as*

$$\zeta_n(s) = \frac{2}{(2n+1)!(2n-1)!}\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}\left(\frac{2n+1}{2}\right)^{2m}\sum_{\ell=0}^{2n-1}\mathscr{A}_\ell^n \zeta\left(2(s-\ell+m)-1, \frac{2n+3}{2}\right). \tag{3.0.2}$$

Moreover, $\zeta_n(s)$ $(\mathrm{Re}\, s > 2n)$ has a meromorphic continuation on the whole complex plane with at most simple poles at $s = 2n-k$ $(0 \leq k \leq 2n-1)$ with residues given by

$$\mathrm{Res}\,\zeta_n(s)\Big|_{s=2n-k} = \frac{1}{(2n+1)!(2n-1)!}\sum_{\ell=0}^{k}\mathscr{A}_{\ell+2n-k-1}^n \frac{\Gamma(2n-k+\ell)}{\ell!\Gamma(2n-k)}\left(\frac{2n+1}{2}\right)^{2\ell}. \tag{3.0.3}$$

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doi.org/10.6084/m9.figshare.13829129.

Proof 3.1 *We have*

$$\begin{aligned}\zeta_n(s) &= \sum_{k=1}^{\infty} \frac{\Lambda_k^n}{[k(k+2n+1)]^s} \\ &= \sum_{k=1}^{\infty} \frac{4^s \Lambda_k^n}{\left[(2k+2n+1)^2-(2n+1)^2\right]^s} \\ &= \sum_{k=1}^{\infty} \frac{4^s \Lambda_k^n}{(2k+2n+1)^{2s}} \left[1-\left(\frac{2n+1}{2k+2n+1}\right)^2\right]^{-s}. \end{aligned} \tag{3.0.4}$$

Upon applying the generalised binomial expansion and interchanging the summation, we get

$$\zeta_n(s) = \sum_{m=0}^{\infty} \left[\left(\sum_{k=1}^{\infty} \frac{4^s \Lambda_k^n}{(2k+2n+1)^{2s+2m}}\right) \binom{s+m-1}{m} (2n+1)^{2m}\right]. \tag{3.0.5}$$

Substituting the multiplicity (2.0.5) into (3.0.5) and using the identity ([51][eq. 2.2(26)])

$$\zeta(s, a+1) = \zeta(s, a) - a^{-s}, \tag{3.0.6}$$

we have

$$\zeta_n(s) = \frac{2}{(2n+1)!(2n-1)!} \sum_{m=0}^{\infty} \binom{s+m-1}{m} \left(\frac{2n+1}{2}\right)^{2m} \sum_{\ell=0}^{2n-1} \mathscr{A}_\ell^n \zeta\left(2(s-\ell+m)-1, \frac{2n+3}{2}\right). \tag{3.0.7}$$

Noting that the residue of the Hurwitz zeta function in (3.0.7) is 1/2, and that $\zeta_n(s)$ has simple poles at $s = 2n-k$, we obtain the residue formula (3.0.3) as required.

A different method of finding the residues of $\zeta_n(s)$ is considered in [58].

Corollary 3.2 *The following relation holds:*

$$a_k^n = \frac{\omega_n}{\Gamma(2n)} \sum_{\ell=0}^{k} \frac{\mathscr{A}_{\ell+2n-1-k}^n}{\ell!} \Gamma(2n-k+\ell) \left(\frac{2n+1}{2}\right)^{2\ell}. \tag{3.0.8}$$

4 Determinants of Laplacians on $\mathsf{P}^n(\mathbb{H})$

This section presents the main results of this paper, namely, explicit closed-form formulae for the determinants of the Laplacians on $\mathsf{P}^n(\mathbb{H})$ $(n \geq 1)$. The approach involves the explicit evaluation of the zeta functions $\zeta_n(s)$ using a generalised binomial expansion.

We proceed by first considering the spectral zeta function $\widetilde{\zeta}_n(s)$ associated with the shifted Laplacian $\widetilde{\Delta}_n = \Delta_n + (2n+1)^2/4$ having eigenvalues $\widetilde{\lambda}_k^n = k(k+2n+1) + (2n+1)^2/4 = (k+(2n+1)/2)^2$ and the usual multiplicity Λ_k^n. In this regard, we have the following theorem.

Theorem 4.1 *The following formula holds for the determinant of the Laplacian $\widetilde{\Delta}_n$:*

$$\log\det\left(\widetilde{\Delta}_n\right) = \sum_{j=0}^{2n-1} \widetilde{\mathscr{A}_j^n} \left[\left(2^{-2j-1}\log 2\right)\mathscr{B}_j - \left(2^{-2j-1}-1\right)\zeta'(-2j-1)\right],$$

where $\mathscr{B}_j = B_{2j+2}/(2j+2)$ and $\widetilde{\mathscr{A}_j^n} = \frac{4\mathscr{A}_j^n}{(2n+1)!(2n-1)}$.

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doi.org/10.6084/m9.figshare.13829129.

Proof 4.1 *The spectral zeta function $\widetilde{\zeta}_n$ is defined, in a similar way, by the Dirichlet-type series*

$$\widetilde{\zeta}_n(s) = \sum_{k=0}^{\infty} \frac{4^s \Lambda_k^n}{(2k+2n+1)^{2s}}, \qquad Re\, s > 2n. \tag{4.0.1}$$

Upon inserting the multiplicity (2.0.5) in (4.0.1) we obtain

$$\begin{aligned}\widetilde{\zeta}_n(s) &= \sum_{j=0}^{2n-1} \frac{2\mathscr{A}_j^n}{(2n+1)!(2n-1)!}\zeta\left(2s-2j-1, \frac{2n+1}{2}\right)\\ &= \sum_{j=0}^{2n-1} \frac{2\mathscr{A}_j^n}{(2n+1)!(2n-1)!}\zeta\left(2s-2j-1, \frac{1}{2}\right)\\ &= \sum_{j=0}^{2n-1} \frac{2\mathscr{A}_j^n}{(2n+1)!(2n-1)!}\left(2^{2s-2j-1}-1\right)\zeta_R(2s-2j-1),\end{aligned} \tag{4.0.2}$$

where $\zeta_R(z)$ is the well-known Riemann zeta function defined by (see [51][Sec. 2.3])

$$\zeta_R(z) = \sum_{j=1}^{\infty} \frac{1}{j^z}, \qquad Re\, z > 1. \tag{4.0.3}$$

In arriving at the last equation in (4.0.2) we have used the functional relation

$$\zeta\left(z, \frac{1}{2}\right) = (2^z - 1)\,\zeta_R(z). \tag{4.0.4}$$

In particular, $\zeta(z,1) = \zeta_R(z)$. Differentiating $\widetilde{\zeta}_n(s)$ (given in (4.0.2)) at $s = 0$ and using the relation

$$\zeta_R(-j) = \begin{cases} -\frac{1}{2}, & j = 0,\\ -\mathscr{B}_{\frac{j-1}{2}}, & j = 1, 2, \cdots, \end{cases} \tag{4.0.5}$$

we have

$$\widetilde{\zeta}_n'(0) = \sum_{j=0}^{2n-1} \frac{4\mathscr{A}_j^n}{(2n+1)!(2n-1)!}\left[\left(2^{-2j-1}-1\right)\zeta_R'(-2j-1) - \left(2^{-2j-1}\log 2\right)\mathscr{B}_j\right].$$

A generalisation of (4.0.5) is ([57])

$$\zeta(-j, x) = -\mathscr{B}_{\frac{j-1}{2}}(x) = -\frac{B_{j+1}(x)}{j+1}, \tag{4.0.6}$$

where the Bernoulli polynomial $B_j(x)$ is given by

$$\frac{ze^{xz}}{e^{z-1}} = \sum_{j=0}^{\infty} B_j(x)\frac{z^j}{j!}, \quad |z| < 2\pi.$$

Hence we obtain the formula

$$\det\left(\widetilde{\Delta}_n\right) = \exp\left(-\sum_{j=0}^{2n-1} \frac{4\mathscr{A}_j^n}{(2n+1)!(2n-1)}\left[\left(2^{-2j-1}-1\right)\zeta'(-2j-1) - \left(2^{-2j-1}\log 2\right)\mathscr{B}_j\right]\right).$$

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doi.org/10.6084/m9.figshare.13829129.

We are now set to compute the more complicated $\det \Delta_n$, and this is given in the following theorem.

Theorem 4.2 *The determinant of the Laplacian Δ_n admits the following formula:*

$$\begin{aligned}\log\det\left(\Delta_n\right) = \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\Bigg[& \left(2^{-2m}\log 2\right)\mathscr{B}_m - \left(2^{-2m}-2\right)\zeta'(-2m-1) \\ & - \mathcal{FP}\left(\sum_{\ell=1}^{\infty}\frac{\left(\frac{2n+1}{2}\right)^{2\ell}}{\ell}\zeta\left(2\ell-2m-1,\frac{2n+3}{2}\right)\right)\Bigg] - 2\log\left(\frac{2n+1}{2}\right),\end{aligned} \tag{4.0.7}$$

where $\mathcal{FP}(f)$ denotes the finite part of the meromorphic function f.

Proof 4.2 *Upon inserting the multiplicity (2.0.5) and applying the generalised binomial expansion to the eigenvalue λ_k^n, one gets*

$$\begin{aligned}\Lambda_k^n\left[\left(k+\frac{2n+1}{2}\right)^2-\left(\frac{2n+1}{2}\right)^2\right]^{-s} = & \frac{2}{(2n+1)!(2n-1)!}\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{-s}{\ell} \\ & \times\sum_{m=0}^{2n-1}\mathscr{A}_m^n\left(k+\frac{2n+1}{2}\right)^{2m-2\ell-2s+1}\left(\frac{2n+1}{2}\right)^{2\ell}.\end{aligned} \tag{4.0.8}$$

As a result we have

$$\begin{aligned}\zeta_n(s) &= \sum_{k=1}^{\infty}\frac{\Lambda_k^n}{\left[k\left(k+2n+1\right)\right]^s} \\ &= \frac{2}{(2n+1)!(2n-1)!}\sum_{\ell=0}^{\infty}\frac{\Gamma(s+\ell)}{\ell!\Gamma(s)}\sum_{m=0}^{2n-1}\mathscr{A}_m^n\zeta\left(2s-2m+2\ell-1,\frac{2n+3}{2}\right)\left(\frac{2n+1}{2}\right)^{2\ell},\end{aligned} \tag{4.0.9}$$

where we have used the identity ([51][2.2(26)])

$$\zeta(s,a+1)=\zeta(s,a)-a^{-s} \tag{4.0.10}$$

and the fact that

$$\binom{-s}{\ell} = (-1)^{\ell}\frac{\Gamma(s+\ell)}{\ell!\Gamma(s)}, \qquad \ell\geq 0. \tag{4.0.11}$$

It follows that

$$\zeta_n(s) = \sum_{\ell=0}^{\infty}\frac{\Gamma(s+\ell)}{\ell!\Gamma(s)}\sum_{m=0}^{2n-1}\frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\zeta\left(2s-2m+2\ell-1,\frac{2n+3}{2}\right)\left(\frac{2n+1}{2}\right)^{2\ell} \tag{4.0.12}$$

or

$$\begin{aligned}\zeta_n(s) = & \sum_{m=0}^{2n-1}\frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\zeta\left(2s-2m-1,\frac{2n+3}{2}\right) \\ & + \sum_{\ell=1}^{\infty}\frac{\Gamma(s+\ell)}{\ell!\Gamma(s)}\sum_{m=0}^{2n-1}\frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\zeta\left(2s+2\ell-2m-1,\frac{2n+3}{2}\right)\left(\frac{2n+1}{2}\right)^{2\ell},\end{aligned} \tag{4.0.13}$$

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doi.org/10.6084/m9.figshare.13829129.

which upon using the relation (4.0.10) gives

$$\begin{aligned}\zeta_n(s) =& \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\zeta\left(2s-2m-1, \frac{2n+1}{2}\right) - \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n \left(\frac{2n+1}{2}\right)^{2m-2s+1}}{(2n+1)!(2n-1)!} \\ &+ \sum_{\ell=1}^{\infty} \frac{\Gamma(s+\ell)}{\ell!\Gamma(s)} \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n \left(\frac{2n+1}{2}\right)^{2\ell}}{(2n+1)!(2n-1)!}\zeta\left(2s+2\ell-2m-1, \frac{2n+3}{2}\right) \\ =&\widetilde{\zeta}_n(s) - \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^2 \left(\frac{2n+1}{2}\right)^{2m-2s+1}}{(2n+1)!(2n-1)!} \\ &+ \sum_{\ell=1}^{\infty} \frac{\Gamma(s+\ell)}{\ell!\Gamma(s)} \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n \left(\frac{2n+1}{2}\right)^{2\ell}}{(2n+1)!(2n-1)!}\zeta\left(2s+2\ell-2m-1, \frac{2n+3}{2}\right). \end{aligned} \tag{4.0.14}$$

Upon differentiating (4.0.14) with respect to s and setting $s = 0$ in the resulting equation yields

$$\begin{aligned}\zeta_n'(0) =&\widetilde{\zeta}_n'(0) + 2\log\left(\frac{2n+1}{2}\right) \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n \left(\frac{2n+1}{2}\right)^{2m+1}}{(2n+1)!(2n-1)!} \\ &+ \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!}\mathcal{FP}\left(\sum_{\ell=1}^{\infty} \frac{\left(\frac{2n+1}{2}\right)^{2\ell}}{\ell}\zeta\left(2\ell-2m-1, \frac{2n+3}{2}\right)\right). \end{aligned} \tag{4.0.15}$$

It is interesting to see that

$$\sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n \left(\frac{2n+1}{2}\right)^{2m+1}}{(2n+1)!(2n-1)!} = 1, \qquad n \geq 1, \tag{4.0.16}$$

and consequently we obtain

$$\begin{aligned}\zeta_n'(0) =&\widetilde{\zeta}_n'(0) + 2\log\left(\frac{2n+1}{2}\right) + \sum_{m=0}^{2n-1} \frac{2\mathscr{A}_m^n}{(2n+1)!(2n-1)!} \times \\ &\times \mathcal{FP}\left(\sum_{\ell=1}^{\infty} \frac{\left(\frac{2n+1}{2}\right)^{2\ell}}{\ell}\zeta\left(2\ell-2m-1, \frac{2n+3}{2}\right)\right). \end{aligned}$$

Therefore, we obtain the following closed-form formula for the determinant of the Laplacian Δ_n :

$$\det\left(\Delta_n\right) = \exp\left(-\widetilde{\zeta}_n'(0) - 2\log\left(\frac{2n+1}{2}\right) - \frac{2H_n}{(2n+1)!(2n-1)!}\right), \tag{4.0.17}$$

where

$$H_n = \mathcal{FP}\left(\sum_{\ell=1}^{\infty} \frac{\left(\frac{2n+1}{2}\right)^{2\ell}}{\ell} \sum_{m=0}^{2n-1} \mathscr{A}_m^n \zeta\left(2\ell-2m-1, \frac{2n+3}{2}\right)\right). \tag{4.0.18}$$

Remark 4.3 *A way of regularising the Hurwitz zeta function $\zeta(z, \cdot)$ where it has a pole is to use the limit formula ([51][eq. 2.2(15)])*

$$\mathcal{FP}(\zeta(1, a)) = \lim_{m \nearrow 1}\left(\zeta(m, a) - \frac{1}{m-1}\right) = -\psi(a), \tag{4.0.19}$$

where ψ is the digamma function ([20]) defined explicitly in terms of the gamma function by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

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Example 4.4 *For the special case $n = 1$, we have*

$$\widetilde{\zeta}_1'(0) = \frac{1}{12}\zeta_R'(-1) - \frac{7}{12}\zeta_R'(-3) + \frac{61}{1440}\log 2.$$

Hence, we obtain the explicit formula

$$det\left(\widetilde{\Delta}_1\right) = 2^{-\frac{61}{1440}} \exp\left(-\frac{1}{12}\zeta_R'(-1) + \frac{7}{12}\zeta_R'(-3)\right).$$

It therefore follows that

$$det\left(\Delta_1\right) = \exp\left(-\widetilde{\zeta}_1'(0) - \log\frac{9}{4} - \frac{H_1}{3}\right), \tag{4.0.20}$$

where

$$H_1 = -\frac{1}{4}\mathcal{FP}\left(\sum_{\ell=1}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-1,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell}\right) + \mathcal{FP}\left(\sum_{\ell=1}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-3,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell}\right).$$

By using the limit formula (4.0.19) *and the identity* (4.0.6) *with $B_2(x) = x^2 - x + 1/6$ we obtain*

$$H_1 = -\frac{141}{32} - \frac{63}{32}\psi\left(\frac{5}{2}\right) - \frac{1}{4}\sum_{\ell=2}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-1,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell} + \sum_{\ell=3}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-3,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell}. \tag{4.0.21}$$

To evaluate the first series on the right-hand side of (4.0.21) *we use the identity (see [59][eq. (4.11)])*

$$\begin{aligned}\sum_{j=1}^{\infty}\zeta(2j+1,a)\frac{z^{2j+2}}{j+1} &= \left[\psi(a)-1\right]z^2 + (a-1)[\log\Gamma(a+z) + \log\Gamma(a-z)] \\ &\quad - \log G(a+z) - \log G(a-z) \\ &\quad + 2(1-a)\log\Gamma(a) + 2\log G(a), \quad |z| < |a|,\end{aligned} \tag{4.0.22}$$

where G is the Barnes G-function ([60]) defined in terms of the gamma function by

$$G(z+1) = \Gamma(z)G(z), \ z \in \mathbb{C}, \ \ G(1) = 1. \tag{4.0.23}$$

Thus we obtain

$$\begin{aligned}\sum_{\ell=2}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-1,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell} &= \frac{9}{4}\psi\left(\frac{5}{2}\right) + \frac{55}{12}\log 2 - \frac{3}{2}\log 3 - 3\log A - 2 \\ &= \frac{9}{4}\psi\left(\frac{5}{2}\right) + \frac{55}{12}\log 2 - \frac{3}{2}\log 3 + 3\zeta_R'(-1) - \frac{9}{4},\end{aligned} \tag{4.0.24}$$

where we have used the identity ([51][eq. 1.4(8)])

$$G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}}\pi^{-\frac{1}{4}}e^{\frac{1}{8}}A^{-\frac{3}{2}}. \tag{4.0.25}$$

and the relation ([30])

$$\log A = -\zeta_R'(-1) + \frac{1}{12}. \tag{4.0.26}$$

Here A is the Glaisher-Kinkelin constant given by $A \approxeq 1.282427130$.

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doi.org/10.6084/m9.figshare.13829129.

For the explicit evaluation of the second series on the right-hand side of (4.0.21) *we employ the equality ([51][eq. (3.2.67)], [57])*

$$\begin{aligned}&\sum_{j=1}^{\infty}\frac{\zeta(2j+1,a)}{j+l+1}z^{2j+2l+2}\\&=\sum_{j=0}^{2l+1}\frac{\Gamma(2l+2)}{j!\Gamma(2l-j+2)}\left[\zeta'(-j,a-z)-(-1)^j\zeta'(-j,a+z)\right]z^{2l-j+1}\\&-\sum_{m=1}^{l}\frac{\zeta(1-2m,a)}{l-m+1}z^{2l-2m+2}-\frac{z^{2l+2}}{l+1}[\psi(2l+2)-\psi(a)+C]\\&-2\zeta'(-2l-1,a),\qquad l\geq 0,|z|<|a|,\end{aligned}$$

with $l=1$*,* $a=5/2$ *and* $z=3/2$ *to see that*

$$\begin{aligned}\sum_{\ell=3}^{\infty}\frac{1}{\ell}\zeta\left(2\ell-3,\frac{5}{2}\right)\left(\frac{3}{2}\right)^{2\ell}=&\frac{27}{2}\zeta_R'(-1)+2\zeta_R'(-3)-2\zeta'\left(-3,\frac{5}{2}\right)+\frac{1}{8}\log 2+\frac{27}{8}\log 3\\&-\frac{15}{64}+\frac{81}{32}\psi\left(\frac{5}{2}\right).\end{aligned}$$

Hence we have

$$H_1=\frac{51}{4}\zeta_R'(-1)+2\zeta_R'(-3)-2\zeta'\left(-3,\frac{5}{2}\right)-\frac{49}{48}\log 2+\frac{15}{4}\log 3-\frac{441}{64}.$$

It therefore follows that

$$det\ (\Delta_1)=2^{\frac{3309}{1440}}3^{-\frac{13}{4}}\exp\left(-\frac{13}{3}\zeta_R'(-1)-\frac{1}{12}\zeta_R'(-3)+\frac{2}{3}\zeta'\left(-3,\frac{5}{2}\right)+\frac{147}{64}\right).$$

We can evaluate the higher cases $n\geq 2$ similarly. See [51, 57, 59, 61] for the evaluation of series involving zeta functions.

Acknowledgements

The author is grateful to the reviewers for their valuable suggestions and contributions on improving the paper.

Competing financial interests

The author declares no competing financial interests".

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