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Refund Clause of Contributions with Predetermined Interest under CEV Model

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Abstract

This work studies the optimal control strategy for a pension plan with refund clause of contributions with predetermined interest under constant elasticity of variance (CEV) in a defined contribution (DC) pension plan. A model which mandates fund managers to refund dead members' accumulations with predetermined interest to their next of kin during the accumulation phase is considered. Also considered herein are investments in a bank security and stock where the stock market price is driven by the CEV model and the remaining accumulations are equally distributed between the remaining members. Furthermore, the game theoretic approach is use in establishing an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation which is a non-linear partial differential equation (PDE). Using mean variance utility function and method of variable separation, explicit solutions of the optimal control strategy and the efficient frontier are obtained. Finally, Numerical simulations and theoretical analysis are used to study the effect of the elasticity parameter β and some other parameters on the optimal control strategy with observations that the elasticity parameter affects the investment strategy of the fund manager significantly. Also, we observed that the optimal control strategy employed by the fund manager is inversely proportional to the risk aversion coefficient, initial fund size, instantaneous volatility and predetermined interest rate but directly proportional to time.

Keywords and Phrases: DC Pension plan, extended HJB equation, optimal control strategies, refund clause of contribution, game theoretic method, mean variance utility, constant elasticity of variance.
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1 Introduction

The study of optimal control laws is key in portfolio optimization and risk management and has attracted much attention over the years. The optimal control law is necessary due to the volatility

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nature of the risky assets available in the financial markets. For a financial institution to make right choices in investment process involving risky assets, it is necessary to consider stochastic volatility models which are not constant in order to understand the unstable nature of the risky assets. Some of such volatility models include the CEV model, the Heston's volatility model, the Ornstein-Uhlenbeck (O-U) model etc.

The CEV model is a direct extension of geometric Brownian motion (GBM) and was first developed by [1]. In [2], the optimal investment problem with taxes, dividend and transaction cost using CEV model and logarithm utility function was studied; they pointed out that the CEV model has the ability of capturing the implied volatility skew. Several authors such as [3,4] studied utility maximization under CEV model in a DC pension plan. In [3], the CEV model was applied to derive a dual solution of a constant relative risk aversion (CRRA) utility function via Legendre transform while [4] extended the work of [3] by solving for the investment strategies with both CRRA and constant absolute risk aversion (CARA) utility function. The optimal reinsurance and investment problem of utility maximization under CEV model was studied in [5]. The optimal portfolio strategy with multiple contributors in a DC pension fund using Legendre transformation method was studied by [6]. In [7], the impact of additional voluntary contribution on the portfolio strategies under CEV model was investigated; they solved the problem using power transformation and change of variable approach. In [8], stochastic salary income of a pension beneficiary was considered and solve for the investment strategy that maximizes the expected power utility of the relationship between the terminal fund and the final salary. In [9], asset allocation problem under a loss-averse preference was investigated. In [10], the optimal strategies for CRRA and CARA using power transformation method by modeling a DC pension fund system with multiple contributors was obtained. [11], investigated a utility optimization problem for a DC pension plan with a stochastic salary income and a stochastic contribution process in a regime-switching economy.

Recently, the study of optimal control laws with refund of contributions clause have been studied by different researchers which include; [12] where they studied optimal control plan for a DC pension with refund of contributions under mean-variance utility. In [13], the same problem in [12] was studied for both accumulation and distribution phases where the risky asset was modelled by Heston volatility model. The strategic optimal portfolio management for a DC plan with refund clause was studied in [14]; in their work, they extended the work of [12] by considering investment in a risk free and two risky assets. In [15], investment strategies with refund clause was studied under inflation and volatility risk; they considered investment in one risk free asset, stock and inflation index bond where the stock market price was modelled by Heston's volatility model. In [16], the DC pension plan with refund clause was studied under affine interest structure. [17] studied optimal investment for the DC plan with refund of premium under jump diffusion process. In their work, the risky asset was modeled by the jump diffusion process. The authors in [18] studied equilibrium investment strategy for DC pension plan with default risk and return of premiums clauses under (CEV) model; they considered investments in treasury, stock and bond. From the above literatures, the authors assumed that the refund contributions were without any kind of interest.

Most recently, the optimal control laws with refund of contributions with predetermined interest have been studied by some authors; in their work, they assumed the refund contributions to the death members' families are with predetermined interest from the risk free asset. These include [19], where the optimal control plan in a DC plan with a risk free and one risky asset were studied when the refund contributions were with predetermined interest and the price of the risky asset was modelled by GBM. In [20], the optimal asset allocation strategy for a DC pension system with refund of contributions with predetermined interest under Heston's volatility model was studied. Since the price process of the risk free asset is deterministic and the interest rate is predetermined, it is possible to determine the interest paid to each death member's family at each point in time during the accumulation phase. This form the foundation of this work, where we intend to extend the work of [12] by developing an optimal control strategy when the risky asset is modelled by CEV model and the refund is with predetermined interest.

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2 Mathematical model of the Financial Market

We assume that the market is made up of risk-free asset (cash) and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $W_t(t)$ is a standard Brownian motion, F is the filtration and denotes the information generated by the Brownian motion $W_t(t)$.

Let $B_t(t)$ and $S_t(t)$ denote the prices of the risk-free asset and risky asset respectively and they are modelled as

$$\frac{dB_t\left(t\right)}{B_t\left(t\right)} = rdt, \tag{2.0.1}$$

$$\frac{dS_t\left(t\right)}{S_t\left(t\right)} = (r+v_1)dt + \sigma {S_t}^{\beta} dW_t. \tag{2.0.2}$$

Where $r > 0$ is the predetermined interest rate of the risk free asset, $(r + v_1)$ is an expected instantaneous rate of return of the risky asset and satisfies the general condition $v_1 > 0$. $\sigma {S_t}^{\beta}$ is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta < 0$.

Let d_1 represent the proportion of the wealth to be invested in risky assets and $d_2 = 1 - d_1$, the proportion to be invested in the risk free asset and p be the contributions received at a given time, which is predetermined, ω_0 represent the initial age of accumulation phase, T is the time frame of the accumulation phase such that $\omega_0 + T$ is the end age. The actuarial symbol δ_{i,ω_0+t} is the mortality rate from time t to $t + i$, tp is the premium accumulated at time t, $tp\delta_{i,\omega_0+t}$ is the premium returned to the death members and $\mathcal{X}\left(t\right)\left(1-d_1\right)\frac{B_{t+\frac{1}{i}}}{B_t}\delta_{i,\omega_0+t}$ is the predetermined interest paid to the death members during the accumulation phase.

Considering the time interval $[t, t+i]$, the differential form associated with the fund size is given as:

$$\mathcal{X}\left(t+i\right) = \left[\begin{array}{c} \mathcal{X}\left(t\right)\left(d_1\frac{S_{t+i}}{S_t} + d_2\frac{B_{t+i}}{B_t}\right) + pi \\ -tp\delta_{i\omega_0+t} - \mathcal{X}\left(t\right)d_2\frac{B_{t+\frac{1}{i}}}{B_t}\delta_{i,\omega_0+t} \end{array}\right]\left(\frac{1}{1-\delta_{i,\omega_0+t}}\right) \tag{2.0.3}$$

$$\mathcal{X}\left(t+i\right) = \left[\mathcal{X}\left(t\right)\left(\begin{array}{c} d_1\left(\frac{S_{t+i}}{S_t} - \frac{S_t}{S_t} + \frac{S_t}{S_t}\right) + \left(1-d_1\right)\left(\frac{B_{t+i}}{B_t} - \frac{B_t}{B_t} + \frac{B_t}{B_t}\right) \\ -\left(1-d_1\right)\delta_{i,\omega_0+t}\left(\frac{B_{t+i}}{B_t} - \frac{B_t}{B_t} + \frac{B_t}{B_t}\right) \\ +pi - tp\delta_{i,\omega_0+t} \end{array}\right)\right]\left(1 + \frac{\delta_{i,\omega_0+t}}{1-\delta_{i,\omega_0+t}}\right) \tag{2.0.4}$$

$$\mathcal{X}\left(t+i\right) = \left[\mathcal{X}\left(t\right)\left(\begin{array}{c} d_1 + 1 - d_1 - \left(1-d_1\right)\delta_{i,\omega_0+t} + d_1\left(\frac{S_{t+\frac{1}{i}}}{S_t} - \frac{S_t}{S_t}\right) \\ +\left(1-d_1\right)\left(\frac{B_{t+i}}{B_t} - \frac{B_t}{B_t}\right) - \left(1-d_1\right)\delta_{i,\omega_0+t}\left(\frac{B_{t+i}}{B_t} - \frac{B_t}{B_t}\right) \\ +pi - tp\delta_{i,\omega_0+t} \end{array}\right)\right]\left(1 + \frac{\delta_{i,\omega_0+t}}{1-\delta_{i,\omega_0+t}}\right) \tag{2.0.5}$$

$$\mathcal{X}\left(t+i\right) - \mathcal{X}\left(t\right) = \left[X\left(t\right)\left(\begin{array}{c} d_1\left(\frac{S_{t+i}-S_t}{S_t}\right) + \left(1-d_1\right)\left(\frac{B_{t+i}-B_t}{B_t}\right)\left(1-\delta_{i\omega_0+t}\right) \\ -\left(1-d_1\right)\delta_{i\omega_0+t} \\ +pi - tp\delta_{i\omega_0+t} \end{array}\right)\right]\left(1 + \frac{\delta_{i,\omega_0+t}}{1-\delta_{i,\omega_0+t}}\right) \tag{2.0.6}$$

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$$\left\{\begin{array}{c}\delta_{i,\omega_0+t}=1-\exp\{-\int_0^i \mu\left(\omega_0+t+s\right)ds\}=\mu\left(\omega_0+t\right)i+O(i),\ \frac{\delta_{i,\omega_0+t}}{1-\delta_{i,\omega_0+t}}=\mu\left(\omega_0+t\right)i+O(i)\\ i\to\infty,\ \delta_{i,\omega_0+t}=\mu\left(\omega_0+t\right)dt,\ \frac{\delta_{i,\omega_0+t}}{1-\delta_{i,\omega_0+t}}=\mu\left(\omega_0+t\right)dt, pi\to pdt,\ \frac{S_{t+i-S_t}}{S_t}\to\frac{dS_t(t)}{S_t(t)},\ \frac{B_{t+i-B_t}}{B_t}\to\frac{dB_t(t)}{B_t(t)}\end{array}\right. \tag{2.0.7}$$

Substituting (2.0.7) into (2.0.6) we have

$$d\mathcal{X}\left(t\right)=\left[\begin{array}{c}\mathcal{X}\left(t\right)\left(\begin{array}{c}d_1\frac{dS_t(t)}{S_t(t)}+\left(1-d_1\right)\frac{dB_t(t)}{B_t(t)}\left(1-\mu\left(\omega_0+t\right)dt\right)\\ -\left(1-d_1\right)\mu\left(\omega_0+t\right)dt\\ +pdt-tp\mu\left(\omega_0+t\right)dt\end{array}\right)\end{array}\right]\left(1+\mu\left(\omega_0+t\right)dt\right) \tag{2.0.8}$$

Substituting (2.0.1) and (2.0.2) into (2.0.8), we have

$$d\mathcal{X}\left(t\right)=\left[\begin{array}{c}\mathcal{X}\left(t\right)\left(\begin{array}{c}d_1\left(\left(r+v_1\right)dt+\sigma {S_t}^{\beta}dW_t\right)+\left(1-d_1\right)rdt\left(1-\mu\left(\omega_0+t\right)dt\right)\\ -\left(1-d_1\right)\mu\left(\omega_0+t\right)dt+\mu\left(\omega_0+t\right)dt\\ +p\left(1-t\mu\left(\omega_0+t\right)\right)dt\end{array}\right)\end{array}\right] \tag{2.0.9}$$

Since $\mu\left(t\right)$ is the force function and ω is the maximal age of the life table. From [12] The force function is given as

$$\mu\left(t\right)=\frac{1}{\omega-t}\quad 0\leq t<\omega. \tag{2.0.10}$$

This implies that

$$\mu\left(\omega_0+t\right)=\frac{1}{\omega-\omega_0-t} \tag{2.0.11}$$

Substituting (2.0.11) into (2.0.9) and simplifying it, we have

$$d\mathcal{X}\left(t\right)=\left\{\mathcal{X}\left(t\right)\left(r+d_1\left(v_1+\frac{1}{\omega-\omega_0-t}\right)\right)+p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right\}dt+d_1\mathcal{X}\left(t\right)\sigma {S_t}^{\beta}dW_t\mathcal{X}\left(0\right)=x_0 \tag{2.0.12}$$

3 Methodology

In this section, we consider a fund manager whose interest is to maximize his surviving member's fund size and minimize the volatility of the wealth accumulated. Hence, there is need to develop an optimal portfolio problem using mean-variance utility as follows:

$$\mathcal{J}\left(t,x,s\right)=\sup_{d_1}\left\{E_{t,x,s}\mathcal{X}^{d_1}\left(T\right)-Var_{t,x,s}\mathcal{X}^{d_1}\left(T\right)\right\} \tag{3.0.1}$$

Next, we follow the approach in [12, 13], by using the variational inequality technique. The control problem in (3.0.1) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{J}(t,x,s)$

$$\left\{\begin{array}{c}\mathcal{N}\left(t,x,s,d_1\right)=E_{t,x,s}[\mathcal{X}^{d_1}\left(T\right)]-\frac{\gamma}{2}Var_{t,x,s}[\mathcal{X}^{d_1}\left(T\right)]\\ =\left(E_{t,x,s}[\mathcal{X}^{d_1}\left(T\right)]-\frac{\gamma}{2}\left(E_{t,x,s}[\mathcal{X}^{d_1}\left(T\right)^2\right]-\left(E_{t,x,s}[\mathcal{X}^{d_1}\left(T\right)]\right)^2\right)\right)\\ \mathcal{J}\left(t,x,s\right)\ =\sup_{d_1}\mathcal{N}\left(t,x,s,d_1\right)\end{array}\right. \tag{3.0.2}$$

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Following [12] the optimal control law ${d_1}^*$ satisfies:

$$\mathcal{J}(t,x,s) = \sup_{d_1} \mathcal{N}(t,x,s,{d_1}^*) \tag{3.0.3}$$

where γ represent the risk-averse coefficient of the members

Let $a^{d_1}(t,x,s) = E_{t,x,s}[\mathcal{X}^{d_1}(T)]$,$b^{d_1}(t,x,s) = E_{t,x,s}[\mathcal{X}^{d_1}(T)^2]$ then
$\mathcal{J}(t,x,s) \quad = \sup_{d_1} k\left(t,x,a^{d_1}(t,x,s), b^{d_1}(t,x,s)\right)$ where

$$k(t,x,s,a,b) = a - \frac{\gamma}{2}(b - a^2) \tag{3.0.4}$$

Theorem 3.1 *(Verification theorem). If there exists three real functions $\mathcal{K},\mathcal{L},\mathcal{M}$:$[0,T]\times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:*

$$\left\{ \begin{array}{c} \sup_{\varphi} \left\{ \begin{array}{c} \mathcal{K}_t - k_t + \left[x\left(r + d_1\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\right) + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right](\mathcal{K}_x - k_x) \\ +(r+v_1)s(\mathcal{K}_s - k_s) + \frac{1}{2}{d_1}^2x^2\sigma^2s^{2\beta}(\mathcal{K}_{xx} - \mathcal{V}_{xx}) \\ +\frac{1}{2}\sigma^2s^{2\beta+2}(\mathcal{K}_{ss} - \mathcal{V}_{ss}) + xd_1\sigma^2s^{2\beta+1}(\mathcal{K}_{xs} - \mathcal{V}_{xs}) \end{array} \right\} = 0 \\ \mathcal{K}(T,x,s) = \ k\left(T,x,s,x,x^2\right) \end{array} \right. \tag{3.0.5}$$

where:

$$\mathcal{V}_{xx} = \gamma\mathcal{L}_x^2, \mathcal{V}_{xs} = \gamma\mathcal{L}_x\mathcal{L}_s, \mathcal{V}_{ss} = \gamma\mathcal{L}_s^2 \tag{3.0.6}$$

$$\left\{ \begin{array}{c} \mathcal{L}_t + \left[x\left(r + d_1\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\right) + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]\mathcal{L}_x \\ +(r+v_1)s\mathcal{L}_s + \frac{1}{2}{d_1}^2x^2\sigma^2s^{2\beta}\mathcal{L}_{xx} + \frac{1}{2}\sigma^2s^{2\beta+2}\mathcal{L}_{ss} + xd_1\sigma^2s^{2\beta+1}\mathcal{L}_{xs} \\ \mathcal{L}(T,x,s) = x \end{array} \right\} = 0 \tag{3.0.7}$$

$$\left\{ \begin{array}{c} \mathcal{M}_t + \left[x\left(r + d_1\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\right) + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)\right]\mathcal{M}_x \\ +(r+v_1)s\mathcal{M}_s + \frac{1}{2}{d_1}^2x^2\sigma^2s^{2\beta}\mathcal{M}_{xx} + \frac{1}{2}\sigma^2s^{2\beta+2}\mathcal{M}_{ss} + xd_1\sigma^2s^{2\beta+1}\mathcal{M}_{xs} \\ \mathcal{M}(T,x,s) = \ x^2 \end{array} \right\} = 0 \tag{3.0.8}$$

Then $\mathcal{J}(t,x,s) = \mathcal{K}(t,x,s)$, $a^{{d_1}^*} = \mathcal{L}(t,x,s)$, $b^{{d_1}^*} = \mathcal{M}(t,x,s)$ for the optimal investment strategy φ^*.

Proof 3.1 *The details of the proof can be found in [21] - [23]*

4 The Optimal Control Law and Efficient Frontier

Our focus now is to obtain the optimal control law by solving (3.0.5), (3.0.7), (3.0.8).

Lemma 4.1 *The optimal control law for the risky asset is given as*

$${d_1}^* = \frac{e^{r(t-T)}}{\gamma x\sigma^2s^{2\beta}}\left[\left(v_1 + \frac{1}{\omega-\omega_0-t}\right) + 2\beta(\omega-\omega_0-t)^{2\beta}e^{2r\beta(t-T)}\int_t^T\left[\frac{v_1 + \frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right]^2e^{2r\beta(T-\tau)}d\tau\right] \tag{4.0.1}$$

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Proof 4.1 *Recall that from (3.0.4),*

$$k_t = k_x = k_{xx} = k_{xa} = k_{xb} = k_{ab} = k_{bb} = 0, k_a = 1 + \gamma a, k_{aa} = \gamma, k_b = -\frac{\gamma}{2} \tag{4.0.2}$$

Substituting (3.0.6), (4.0.2) into (3.0.5) and differentiating it with respect to d_1, we have

$$x\left(v_1 + \frac{1}{\omega - \omega_0 - t}\right)\mathcal{K}_x + d_1 x^2\sigma^2 s^{2\beta}\left(\mathcal{K}_{xx} - \gamma\mathcal{L}_x^2\right) + x\sigma^2 s^{2\beta+1}\left(\mathcal{K}_{xs} - \gamma\mathcal{L}_x\mathcal{L}_s\right) = 0 \tag{4.0.3}$$

Solving equation (4.0.3) for d_1, we have

$$d_1{}^* = -\left[\frac{\mathcal{K}_x\left(v_1 + \frac{1}{\omega-\omega_0-t}\right) + \left(\mathcal{K}_{xs} - \gamma\mathcal{L}_x\mathcal{L}_s\right)\sigma^2 s^{2\beta+1}}{x\sigma^2 s^{2\beta}\left(\mathcal{K}_{xx} - \gamma\mathcal{L}_x^2\right)}\right], \tag{4.0.4}$$

where $d_1{}^*$ is the optimal control law.
From equation (4.0.4), (3.0.5) becomes,

$$\left\{\begin{array}{c} \mathcal{K}_t + \left[rx + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right) - s\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\frac{\mathcal{K}_{xs}-\gamma\mathcal{L}_x\mathcal{L}_s}{\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2}\right]\mathcal{K}_x + (r+v_1)\,s\mathcal{K}_s \\ -\frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2\mathcal{K}_x^2}{2(\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2)\sigma^2 s^{2\beta}} + \frac{1}{2}(\mathcal{K}_{ss} - \gamma\mathcal{L}_s^2)\sigma^2 s^{2\beta+2} - \frac{1}{2}\left(\frac{(\mathcal{K}_{xs}-\gamma\mathcal{L}_x\mathcal{L}_s)^2}{\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2}\right)\sigma^2 s^{2\beta+2} = \end{array}\right\} = 0 \tag{4.0.5}$$

To solve equation (4.0.5), we conjecture a solution for $\mathcal{K}(t,x,s)$ as follows:

$$\left\{\begin{array}{c} \mathcal{K}(t,x,s) = xf(t) + \frac{s^{-2\beta}}{\gamma}g(t) + \frac{1}{\gamma}h(t), \quad f(T) = 1, \ g(T) = 0, \ h(T) = 0 \\ \mathcal{K}_t = f_t x + \frac{g_t s^{-2\beta}}{\gamma} + \frac{h_t(t)}{\gamma}, \ \mathcal{K}_x = f, \ \mathcal{K}_{xx} = \mathcal{K}_{xs} = 0 = 0, \\ \mathcal{K}_s = \frac{-2\beta g(t) s^{-2\beta-1}}{\gamma}, \ \mathcal{K}_{ss} = \frac{2\beta(2\beta+1)g(t)s^{-2\beta-2}}{\gamma} \end{array}\right. \tag{4.0.6}$$

Substituting (4.0.6) into (4.0.5), we have:

$$\left\{\begin{array}{c} f_t x + \frac{g_t s^{-2\beta}}{\gamma} + \frac{h_t}{\gamma} + \left[rfx + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)f + \frac{2\beta f v s^{-2\beta}}{u\gamma}\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\right] \\ -\frac{2\beta g s^{-2\beta}}{\gamma}(r+v_1) + \frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2 f^2}{2\gamma u^2\sigma^2 s^{2\beta}} + \frac{\beta(2\beta+1)\sigma^2 g}{\gamma} \end{array}\right\} = 0 \tag{4.0.7}$$

Simplifying (4.0.7), we have

$$\left\{\begin{array}{c} [f_t + rf]x + \frac{?.^{-2\beta}}{\gamma}\left[g_t - 2\beta g(r+v_1) + \frac{2\beta f v}{u}\left(v_1 + \frac{1}{\omega-\omega_0-t}\right) + \frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2 f^2}{2u^2\sigma^2}\right] \\ +\frac{1}{\gamma}\left[h_t + p\gamma\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)f + \beta(2\beta+1)g\sigma^2\right] \end{array}\right\} = 0 \tag{4.0.8}$$

Since $x \neq 0$, $\frac{s^{-2\beta}}{\gamma} \neq 0$, $\frac{1}{\gamma} \neq 0$, then

$$f_t + rf = 0 \tag{4.0.9}$$

$$g_t - 2\beta g(r+v_1) + \frac{2\beta f v}{u}\left(v_1 + \frac{1}{\omega - \omega_0 - t}\right) + \frac{\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)^2 f^2}{2u^2\sigma^2} = 0 \tag{4.0.10}$$

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$$h_t + p\gamma\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)f + \beta\left(2\beta+1\right)g\sigma^2 = 0 \tag{4.0.11}$$

Similarly, if we substitute equation (4.0.4) *into* (3.0.7), *we have*

$$\left\{\begin{array}{c} \mathcal{L}_t + \left[rx + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right) - s\left(v_1 + \frac{1}{\omega-\omega_0-??}\right)\frac{\mathcal{K}_{xs}-\gamma\mathcal{L}_x\mathcal{L}_s}{\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2}\right]\mathcal{L}_x + \left(r+v_1\right)s\mathcal{L}_s \\ -\frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2\mathcal{K}_x\mathcal{L}_x}{(\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2)\sigma^2 s^{2\beta}} + \frac{1}{2}\sigma^2 s^{2\beta+2}\mathcal{L}_{ss} + \frac{\sigma^2 s^{2\beta}\mathcal{L}_{xx}}{2}\left(\frac{\mathcal{K}_x\left(v_1+\frac{1}{\omega-\omega_0-t}\right)+(\mathcal{K}_{xs}-\gamma\mathcal{L}_x\mathcal{L}_s)\sigma^2 s^{2\beta+1}}{\sigma^2 s^{2\beta}(\mathcal{K}_{xx}-\gamma\mathcal{L}_x^2)}\right)^2 \end{array}\right\} = 0 \tag{4.0.12}$$

Next, we conjecture a solution for $\mathcal{L}(t,x,s)$ *as follows:*

$$\left\{\begin{array}{c} \mathcal{L}(t,x,s) = xu(t) + \frac{s^{-2\beta}}{\gamma}v(t) + \frac{1}{\gamma}w(t), \quad u(T)=1,\ v(T)=0,\ w(T)=0 \\ \mathcal{L}_t = u_t x + \frac{v_t s^{-2\beta}}{\gamma} + \frac{w_t(t)}{\gamma},\ \mathcal{L}_x = u,\ \mathcal{L}_{xx} = \mathcal{L}_{xx} = 0, \\ \mathcal{L}_s = \frac{-2\beta v(t)s^{-2\beta-1}}{\gamma},\quad \mathcal{L}_{ss} = \frac{2\beta(2\beta+1)v(t)s^{-2\beta-2}}{\gamma} \end{array}\right. \tag{4.0.13}$$

Substituting (4.0.13) *into* (4.0.12), *we have*

$$\left\{\begin{array}{c} u_t x + \frac{v_t s^{-2\beta}}{\gamma} + \frac{w_t}{\gamma} + \left[rux + p\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)u + \frac{2\beta v(t)s^{-2\beta}}{\gamma}\left(v_1 + \frac{1}{\omega-\omega_0-t}\right)\right] \\ -\frac{2\beta v(t)s^{-2\beta}}{\gamma}\left(r+v_1\right) + \frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2 f}{\gamma u\sigma^2 s^{2\beta}} + \frac{\beta(2\beta+1)\sigma^2 v(t)}{\gamma} \end{array}\right\} = 0 \tag{4.0.14}$$

Simplifying (4.0.14), *we have*

$$\left\{\begin{array}{c} [u_t + ru]x + \frac{s^{-2\beta}}{\gamma}\left[v_t + 2\beta\left(\frac{1}{\omega-\omega_0-t} - r\right)v + \frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2 f}{?o\sigma^2}\right] \\ +\frac{1}{\gamma}\left[w_t + p\gamma\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)u + \beta\left(2\beta+1\right)\sigma^2 v\ \right] \end{array}\right\} = 0 \tag{4.0.15}$$

Since $x \neq 0$, $\frac{s^{-2\beta}}{\gamma} \neq 0$, $\frac{1}{\gamma} \neq 0$, *then*

$$u_t(t) + ru(t) = 0 \tag{4.0.16}$$

$$v_t + 2\beta\left(\frac{1}{\omega-\omega_0-t} - r\right)v + \frac{\left(v_1+\frac{1}{\omega-\omega_0-t}\right)^2 f}{u\sigma^2} = 0 \tag{4.0.17}$$

$$w_t + p\gamma\left(\frac{\omega-\omega_0-2t}{\omega-\omega_0-t}\right)u + \beta\left(2\beta+1\right)\sigma^2 v = 0 \tag{4.0.18}$$

Solving (4.0.9) – (4.0.11) *and* (4.0.16) – (4.0.18), *we obtain:*

$$f(t) = e^{r(T-t)} \tag{4.0.19}$$

$$g(t) = -e^{2\beta(r+v_1)(t-T)}\left(\int_t^T\left(2\beta\left(v_1 + \frac{1}{\omega-\omega_0-\tau}\right) + \frac{\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)^2}{2\sigma^2}\right)e^{2\beta(r+v_1)(T-\tau)}d\tau\right) \tag{4.0.20}$$

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$$h\left(t\right)=\gamma\left(\frac{p}{r}\left\{e^{r(T-t)}-1\right\}+p\int_t^T\frac{\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right)-\beta\left(2\beta+1\right)\sigma^2\int_t^T g(\tau)d\tau \tag{4.0.21}$$

$$u\left(t\right)=e^{r(T-t)} \tag{4.0.22}$$

$$v\left(t\right)=\frac{\left(\omega-\omega_0-t\right)^{2\beta}}{\sigma^2}e^{2r\beta(t-T)}\int_t^T\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{\left(\omega-\omega_0-\tau\right)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}d\tau \tag{4.0.23}$$

$$w\left(t\right)=\gamma\left(\frac{p}{r}\left\{e^{r(T-t)}-1\right\}+p\int_t^T\frac{\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right)-\beta\left(2\beta+1\right)\sigma^2\int_t^T v(\tau)d\tau \tag{4.0.24}$$

Substituting (4.0.19) (4.0.20), (4.0.21) into (4.0.6)and (4.0.22), (4.0.23), (4.0.24) into (4.0.13) we have:

$$\mathcal{K}\left(t,x,s\right)=\left(\begin{array}{c}\left(xe^{r(T-t)}+\frac{p}{r}\left\{e^{r(T-t)}-1\right\}+p\int_t^T\frac{\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right)\\ -\frac{1}{\gamma}\left(\begin{array}{c}s^{-2\beta}e^{2\beta(r+v_1)(t-T)}\int_t^T\left(\begin{array}{c}2\beta\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)\\+\frac{\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)^2}{2\sigma^2}\end{array}\right)e^{2\beta(r+v_1)(T-\tau)}d\tau\\ +\beta\left(2\beta+1\right)\sigma^2\int_t^T g(\tau)d\tau\end{array}\right)\end{array}\right) \tag{4.0.25}$$

$$\mathcal{L}\left(t,x,s\right)=\left(\begin{array}{c}\left(xe^{r(T-t)}+\frac{p}{r}\left\{e^{r(T-t)}-1\right\}+p\int_t^T\frac{\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\right)\\ +\frac{1}{\gamma}\left(\begin{array}{c}?o^{-2\beta}\frac{(\omega-\omega_0-t)^{2\beta}}{\sigma^2}e^{2r\beta(t-T)}\int_t^T\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}d\tau\\ -\beta\left(2\beta+1\right)\sigma^2\int_t^T v(\tau)d\tau\end{array}\right)\end{array}\right) \tag{4.0.26}$$

Substituting $\mathcal{K}_x$, $\mathcal{K}_{xs}$,$\mathcal{K}_{xx}\mathcal{L}_x$, $\mathcal{L}_s$ into (4.0.4) we obtain (4.0.1) which complete the proof.

Lemma 4.2 *The efficient frontier of the pension fund is given as follows*

$$E_{t,x,s}[X^{{d_1}^*}(T)]=\left(\begin{array}{c}xe^{r(T-t)}+\frac{p}{r}\left\{e^{r(T-t)}-1\right\}+p\int_t^T\frac{\tau}{\omega-\omega_0-\tau}e^{r(T-\tau)}d\tau\\ +\left(\frac{\left(\begin{array}{c}s^{-2\beta}\frac{(\omega-\omega_0-t)^{2\beta}}{\sigma^2}e^{2r\beta(t-T)}\int_t^T\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}d\tau\\ -\beta\left(2\beta+1\right)\sigma^2\int_t^T v\left(\tau\right)d\tau\end{array}\right)\sqrt{Var_{t,x,s}[X^{{d_1}^*}(T)]}}{\frac{2e^{2r\beta(t-T)}}{s^{2\beta}}\left(\begin{array}{c}\frac{(\omega-\omega_0-t)^{2\beta}}{\sigma^2}\\+e^{2v_1\beta(T-\tau)}\end{array}\right)\int_t^T\left(\begin{array}{c}\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}\\ +\left(\begin{array}{c}2\beta\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)\\+\frac{\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)^2}{2\sigma^2}\end{array}\right)e^{2\beta(r+v_1)(T-\tau)}\end{array}\right)d\tau}\right)\end{array}\right) \tag{4.0.27}$$

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Proof 4.2 *Recall that*

$$Var_{t,x}[X^{{d_1}^*}(T)] = E_{t,x}[X^{{d_1}^*}(T)^2] - (E_{t,x}[X^{{d_1}^*}(T)])^2$$

$$Var_{t,x,s}[X^{{d_1}^*}(T)] = \frac{2}{\gamma}(\mathcal{L}(t,x,s) - \mathcal{K}(t,x,s))$$

Substituting (4.0.25) *and* (4.0.26) *for* $\mathcal{K}(t,x,s)$ *and* $\mathcal{L}(t,x,s)$ *in the above equation, we have*

$$Var_{t,x,s}[X^{{d_1}^*}(T)] = \frac{1}{\gamma^2}\left(\frac{2e^{2r\beta(t-T)}}{s^{2\beta}}\left(\begin{array}{c}\frac{(\omega-\omega_0-t)^{2\beta}}{\sigma^2}\\ +e^{2v_1\beta(T-\tau)}\end{array}\right)\int_t^T\left(\begin{array}{c}\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}\\ +\left(\begin{array}{c}2\beta\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)\\ +\frac{\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)^2}{2\sigma^2}\end{array}\right)e^{2\beta(r+v_1)(T-\tau)}\end{array}\right)d\tau\right) \tag{4.0.28}$$

$$\frac{1}{\gamma} = \sqrt{\frac{Var_{t,x,s}[X^{{d_1}^*}(T)]}{\left(\frac{2e^{2r\beta(t-T)}}{s^{2\beta}}\left(\begin{array}{c}\frac{(\omega-\omega_0-t)^{2\beta}}{\sigma^2}\\ +e^{2v_1\beta(T-?o)}\end{array}\right)\int_t^T\left(\begin{array}{c}\left(\frac{v_1+\frac{1}{\omega-\omega_0-\tau}}{(\omega-\omega_0-\tau)^{\beta}}\right)^2 e^{2r\beta(T-\tau)}\\ +\left(\begin{array}{c}2\beta\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)\\ +\frac{\left(v_1+\frac{1}{\omega-\omega_0-\tau}\right)^2}{2\sigma^2}\end{array}\right)e^{2\beta(r+v_1)(T-\tau)}\end{array}\right)d\tau\right)}} \tag{4.0.29}$$

Recall from theorem 3.1, *the expectation is given as*

$$E_{t,x,s}[X^{{d_1}^*}(T)] = \mathcal{L}(t,x,s) \tag{4.0.30}$$

Substituting equation (4.0.26) *and* (4.0.29) *into* (4.0.30), *we obtain* (4.0.27) *which complete the proof.*

Remark 4.3 *In a case where the price of risky asset is mobelled by geometric Brownian motion i.e when* $\beta = 0$, *the optimal control law for the risky asset in* (4.0.1) *reduces to*

$$d_3^* = \frac{e^{r(t-T)}}{\gamma x \sigma^2}\left[v_1 + \frac{1}{\omega - \omega_0 - t}\right] \tag{4.0.31}$$

Lemma 4.4 *Suppose* $v_1 > 0,\ x > 0,\ 0 < r < 1,\ \ \sigma > 0,\ \gamma > 0,\ \ \omega - \omega_0 - t > 0,\ T > t > 0$ *then*

(a) $\frac{dd_3^*}{dr} < 0$

(b) $\frac{dd_3^*}{d\gamma} < 0$

(c) $\frac{dd_3^*}{d\sigma} < 0$

(d) $\frac{dd_3^*}{dx} < 0$

(e) $\frac{dd_3^*}{dt} > 0$

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Proof 4.3 *(a) From* (4.0.31), $d_3^* = \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right]$, *then*

$$\frac{dd_3^*}{dr} = (t-T)\frac{e^{r(t-T)}}{\gamma^2 x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right],$$

Since $(t-T) < 0,\quad \frac{e^{r(t-T)}}{\gamma^2 x\sigma^2} > 0$ *and* $\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] > 0$, *then*

$$\frac{dd_3^*}{dr} = (t-T)\frac{e^{r(t-T)}}{\gamma^2 x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] < 0$$

$$\therefore\ \frac{dd_3^*}{dr} < 0$$

(b) From (4.0.31), $d_3^* = \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right]$, *then*

$$\frac{dd_3^*}{d\gamma} = -\frac{e^{r(t-T)}}{\gamma^2 x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right],$$

since $\frac{e^{r(t-T)}}{\gamma^2 x\sigma^2} > 0$ *and* $\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] > 0$, *then*

$$\frac{dd_3^*}{d\gamma} = -\frac{e^{r(t-T)}}{\gamma^2 x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] < 0$$

$$\therefore\ \frac{dd_3^*}{d\gamma} < 0$$

(c) From (4.0.31), $d_3^* = \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right]$, *then*

$$\frac{dd_3^*}{d\sigma} = -\frac{2e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right],$$

since $\frac{2e^{r(t-T)}}{\gamma x\sigma^2} > 0$ *and* $\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] > 0$, *then*

$$\frac{dd_3^*}{d\sigma} = -\frac{2e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] < 0$$

$$\therefore\ \frac{dd_3^*}{d\sigma} < 0$$

(d) From (4.0.31), $d_3^* = \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right]$, *then*

$$\frac{dd_3^*}{dx} = -\frac{e^{r(t-T)}}{x^2\gamma\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right],$$

since $\frac{e^{r(t-T)}}{x^2\gamma\sigma^2} > 0$ *and* $\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] > 0$, *then*

$$\frac{dd_3^*}{dx} = -\frac{e^{r(t-T)}}{x^2\gamma\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] < 0$$

$$\therefore\ \frac{dd_3^*}{dx} < 0$$

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(e) From (4.0.31), $d_3^* = \frac{e^{r(t-T)}}{\gamma x \sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right]$, *then*

$$\frac{dd_3^*}{dt} = \frac{re^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] + \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[\frac{1}{\omega-\omega_0-t}\right]^2$$

since $\frac{e^{r(t-T)}}{\gamma x\sigma^2} > 0, \quad v_1 > 0, \ r > 0$ *and* $\left[\frac{1}{\omega-\omega_0-t}\right] > 0$, *then*

$$\frac{dd_3^*}{dt} = \frac{re^{r(t-T)}}{\gamma x\sigma^2}\left[v_1 + \frac{1}{\omega-\omega_0-t}\right] + \frac{e^{r(t-T)}}{\gamma x\sigma^2}\left[\frac{1}{\omega-\omega_0-t}\right]^2 > 0$$

$$\therefore \ \frac{dd_3^*}{dt} > 0$$

5 Numerical Simulation

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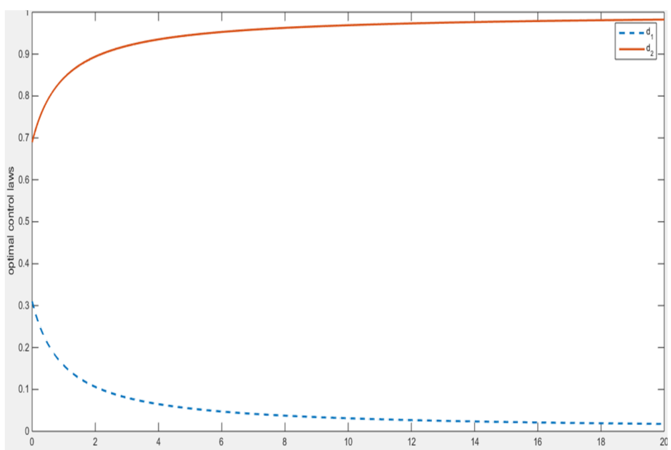

Figure 1: The optimal control laws with time when $\beta = 0$

6 Discussion

From Lemma 4.2, we observed that the expectation is directly dependent on the variance; this means that when members take more risk, their expectation from such investment at the expiration date will be higher compared to members who invest less in stock (risky asset). Also, lemma 4.4 shows that the optimal control strategy for stock is inversely proportional to the following parameters; risk aversion coefficient, initial fund size, instantaneous volatility, predetermined interest rate, and directly proportional to time. Consequently, we observe that the optimal control strategy for the bank security is directly proportional to the risk aversion coefficient, initial fund size, instantaneous volatility and the predetermined interest rate and inversely proportional to time. The consequences of the above lemmas are that members with low risk aversion coefficient will invest more in stock while members with high risk aversion coefficient will invest more in bank security to increase their expectations which is in accordance with lemma 4.2. Secondly, if the initial fund size at the

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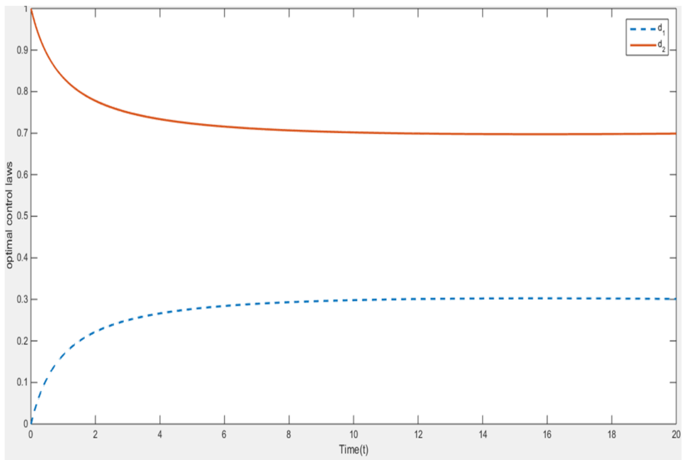

Figure 2: The optimal control laws with time when $\beta = -0.5$

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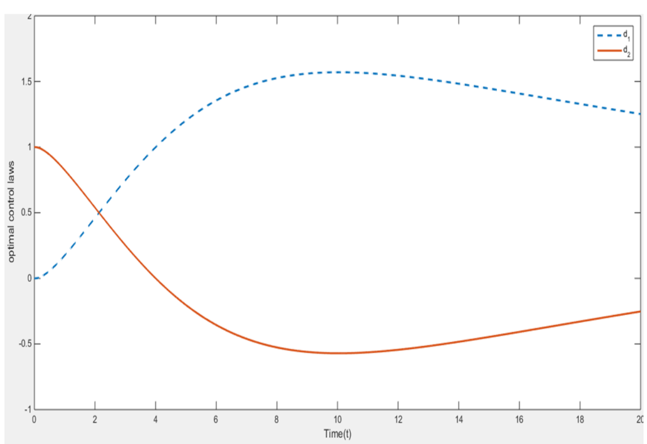

Figure 3: The optimal control laws with time when $\beta = -1$

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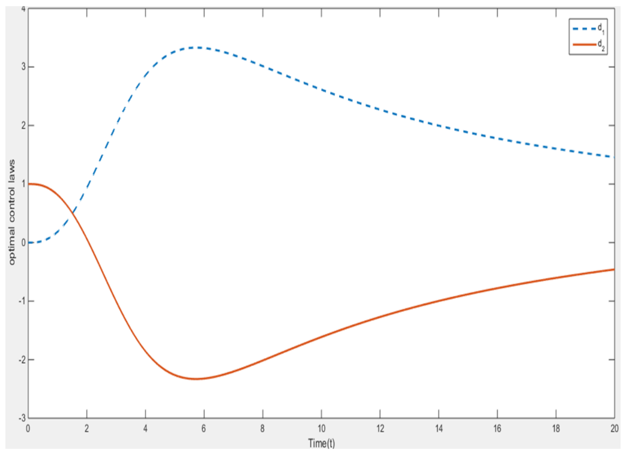

Figure 4: The optimal control laws with time when $\beta = -1.5$

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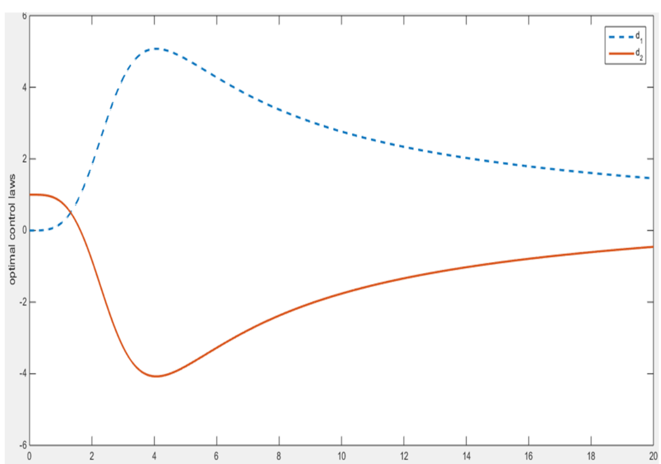

Figure 5: The optimal control laws with time when $\beta = -2$

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time of investment is high, members may decide to reduce the amount of risk to be taken, thereby reducing the proportion of their funds to be invested in stock. Thirdly, in the case where investment offers high interest rate, members may be attracted to such investment hence we observe that high interest rate in bank security may implies more investment in bank security; this is due to fact that many investors naturally do not like taking much risk but desires good returns. Fourthly, since the instantaneous volatility represent the risk coefficient of the stock market price, risk averse members with high instantaneous volatility will investment less in stock and vice versa. Finally, we observed that as retirement age of the remaining members draw closer, there will be anxiety among the remaining members on what they will have at the end of the accumulation period taking into consideration the impact of the refund clause which has taken some funds out of the pension system, these members will like to increase their investments in risky assets in order to boost their investment returns since their benefit depend on the investment returns.
In figure 1, a simulation of the optimal control laws with respect to time is presented when the elasticity parameter $\beta = 0$; it is observed that the members will invest more in the risk free asset (bank security) compared to the risky asset (stock) and continue to increase their investment in bank security as retirement age get closer. Figure 2 shows that when $\beta = -0.5$, the members will invest almost everything in bank security and as time goes on he gradually reduce the proportion of his wealth in bank security and increase investment in stock such that he does not over invest in stock as retirement age approaches. In figure 3, 4 and 5, we observed that when the elasticity parameter $\beta = -1, \ -1.5 \ and -2$, the optimal control laws fluctuate with time showing how volatile the investment in stock are. We observed that at the beginning of investment, members invest almost everything in the risk- free asset and a little while adjust the investment strategies such that they invest equal proportion in both assets and as retirement age approaches, they invest more in risky asset to increase their expected returns; this is because most members may likely die at a age closer to their retirement age and their accumulated funds are being given to their next of kin and fund manager will definitely want to increase the accumulations of the remaining members by investing more in the risky asset. It was observed that the degree of volatility of any investment depend on the elasticity parameter, hence the choice of elasticity parameter is very vital in the determination of optimal investment strategy.

7 Conclusion

In this research, we investigated the optimal control strategy with refund clause of contributions with predetermined interest under constant elasticity of variance (CEV) for a pension fund system. We modelled our problem based on the clause which permits members' next of kin to recover the accumulations of their family members with predetermined interest during the accumulation period in the case of mortality. Investment in a bank security and stock were considered such that the stock market price follows the CEV model. The game theoretic approach is use in establishing an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation. Also, we used the mean variance utility function as our objective function and variable separation technique to obtain explicit solutions of the optimal control strategy and the efficient frontier. Furthermore, Theoretical analysis and numerical simulations were used to analyze the impact of the elasticity parameter β and some other parameters on the optimal control strategy. In conclusion, based on the behavior of the parameters of the investment strategy, this work will guide pension fund managers on how to manage the contributions of the remaining members of the pension scheme after refunds have been made to the death members' next of kin or the families for optimal profit with minimal risks when considering investments in risky assets with stochastic volatilities.

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Competing financial interests

The author declares no competing financial interests.

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