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On The Closed Form Strategies of an Investor under the CEV and CIR Processes

Edikan E. Akpanibah [1,2], Bright O. Osu [1,5*], Everestus O. Eze [1],

Chidi U. Okonkwo [1,4], Ben I. Oruh [1], Udeme O. Ini [3]

1. Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Nigeria.
2. Department of Mathematics and Statistics, Federal University Otuoke, Bayelsa, Nigeria.
3. Department of Mathematics and Computer Science, Niger Delta University, Bayelsa, Nigeria.
4. Department of Mathematics and Statistics, Caritas University, Amorji Nike. Enugu, Nigeria.
5. Department of Mathematics, Abia State University, Uturu.

*Corresponding author: osu.bright@mouau.edu.ng

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Abstract

In this paper, the explicit solutions of the optimal investment plans of an investor with exponential utility function exhibiting constant absolute risk aversion (CARA) under constant elasticity of variance (CEV) and stochastic interest rate is studied. A portfolio comprising of a risk-free asset modelled by the Cox-Ingersoll-Ross (CIR) process and two risky assets modelled by the CEV process is considered, where the instantaneous volatilities of the two risky assets form a 2×2 matrix $n = \{n_{p,q}\}_{2\times 2}$ such that nn^T is positive definite. Using the power transformation and change of variable approach with asymptotic expansion technique, explicit solutions of the optimal investment plans are found. Moreover, numerical simulations are used to study the effects of the interest rate, elasticity parameter, correlation coefficient and the risk averse coefficient on the optimal investment plans.

Keywords: Asymptotic technique, CEV process, Cox-Ingersoll-Ross process, Exponential utility, Optimal investment plan, Power transformation.
MSC2010:97M30, 98B26

1 Introduction

The optimal investment plan of utility maximization is a basic problem in the study of mathematical finance and has attracted attentions from a good number of authors which has led to numerous researches in this area. [1] used the optimal control method to study the optimal investment plan for the first time. [2] - [4] studied the problem of utility maximization using stochastic optimal control theory. Other authors such as [5] - [7], used the Martingale method to solve optimization problems related to optimal investment plan. [3, 8] studied the problem of utility maximization for an incomplete market. The optimal investment plan with stochastic interest rate under geometric

Brownian motion (GBM) has been studied by some authors; these include [9], who studied the investment plan under stochastic interest rate for a case of protected defined contribution (DC) fund. [10] - [11] modelled the risk free interest rate using CIR process to obtain optimal investment plan for a DC plan. In [12]- [13], the risk free interest rate followed the Vasicek model. [14] - [16] studied the optimal investment plan when the interest rate is of affine type. However, all the authors above used the GBM to model the risky assets but [17] showed that the GBM process is not practical in real life since the volatilities of the stock market prices is assumed to be constant

The importance of stochastic volatilities cannot be undermined as it plays a crucial role in the behaviour of the market prices of the risky assets due to its fluctuating nature resulting from various information available in the market especially now that financial institutions in most countries and even the financial markets are currently in serious crisis due to the disgusting effect of the novel corona virus (Covid-19) pandemic. To make a relatively near right decision during investment in assets such as stock, the stochastic volatility models become necessary to understand the fluctuating nature of the stock market price. In this research, the optimal investment plan is investigated for a case where the risky assets follow the CEV model and the risk free asset follows the CIR process. The CEV model is one of the stochastic volatility models used to describe the stock market price behaviours. It was first used in [18] and has the ability to capture the implied volatility skew.

A good number of researches have been done on optimal investment plan under the CEV model. [19] studied the optimal investment plan with dividend, taxes and transaction cost under the CEV model with different utility functions. [20] studied the optimal investment plan and reinsurance problem under the CEV process. [21] - [22] solved the optimal investment problem for a defined contribution (DC) pension plan with return of premiums clauses under different assumptions and assumed that the stock market price follows the CEV process. In all the literature above under CEV processes, the interest rates were assumed to be constant but however, there are some works under the CEV process that their interest rate are stochastic.

In [23], an investor's exponential utility was maximized for a case where the interest rate and stock market price was modelled by CIR and CEV process respectively. They used the Legendre transformation and asymptotic expansion method to determine an explicit solution of the optimal investment plan. They outlined the complexity involved in solving optimization problems that combined both CEV process and stochastic interest rate. Also, they pointed out that in real life applications, interest rates are usually not constant but fluctuating in nature and the volatility of the interest rate generate some market risks; that is to say, when these risks are not considered, we are under estimating the effect of this risk emanating from this interest rate which is critical in influencing the prices of different assets available in financial market. [24] studied the optimal investment plan with stochastic interest rate under the CEV model using logarithm utility; they considered investment in one risk free asset and a risky asset and assumed that the interest rate follows the Cox- Ingersoll-Ross (CIR) process. The power transformation, change of variable and asymptotic approach was used to determine the asymptotic solution of the optimal investment plan. [25] modelled the risky asset with modified CEV process and the interest rate with O-U process and determined the optimal investment plan for an investor with exponential utility. Also, an investor's investment plan with stochastic interest rate under the CEV model and the O-U Process was studied by [26]. In their work, they used two risky assets modelled by the CEV model and a risk free asset modelled by O-U process and observed that the optimal investment plans exhibit a fluctuating effect.

In this paper, the expected exponential utility of an investor's terminal wealth is being maximized by studying the optimal investment plans of an investor exhibiting the CARA. Here, the two risky assets follow the CEV process while risk free interest rate follows the CIR process. More so, we use the power transformation, variable change and asymptotic method to determine asymptotic solutions of the optimal investment plan. The main difference between our work and that of [23] is that we consider investment in two risky assets modelled by the CEV where the instantaneous volatilities of the two risky assets form a 2×2 matrix $n = \{n_{p,q}\}_{2\times 2}$ such that nn^T is positive definite instead of one risky asset. We used the power transformation, variable change instead of Legendre transformation method and dual theory.

2 Materials and Method

2.1 Financial Market Model

Consider a portfolio comprising of one risk free asset and two risky assets in a financial market which is open continuously for an interval $t \in [0,\ T]$, T the expiration date of the investment. Let $\{\mathcal{Z}_0(t),\ \mathcal{Z}_1(t), \mathcal{Z}_2(t) : t \geq\ 0\}$ be standard Brownian motion defined on a complete probability space (Ω, F, P) where Ω is a real space and P is a probability measure and F is the filtration which represents the information generated by the three Brownian motions.
Let $\mathcal{S}_t(t)$ denote the price of the risk free asset at time t and the model is given as follows

$$\begin{cases} \frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} = \mathcal{R}(t)\,dt \\ \mathcal{S}_0(0) = s_0 > 0 \end{cases} \tag{2.1}$$

where $\mathcal{R}(t)$ is the interest rate which follows the CIR process and is given by the stochastic differential equation below

$$\begin{cases} d\mathcal{R}(t) = (a - b\mathcal{R}(t))\,dt - \delta\sqrt{\mathcal{R}(t)}d\mathcal{Z}_0(t) \\ \mathcal{R}(0) = \mathcal{R}_0 > 0 \end{cases}, \tag{2.2}$$

where a, b, and δ are positive real numbers such that the following condition holds $\delta^2 < 2a$ called the Feller's condition [23].
Let $\mathcal{S}_1(t)$ and $\mathcal{S}_2(t)$ denote the prices of two different stocks which are described by the CEV model and the dynamics of the stock market prices are described by the stochastic differential equations at $t \geq 0$ as follows

$$\frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} = m_1 dt + n_{11}\mathcal{S}_1^{\beta}(t)\,d\mathcal{Z}_1(t) + n_{12}\mathcal{S}_1^{\beta}(t)d\mathcal{Z}_2(t) \tag{2.3}$$

$$\frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)} = m_2 dt + n_{21}\mathcal{S}_2^{\beta}(t)\,d\mathcal{Z}_1(t) + n_{22}\mathcal{S}_1^{\beta}(t)d\mathcal{Z}_2(t) \tag{2.4}$$

where m_1 and m_2 are appreciation rate of the two risky assets, $n_{11}, n_{12}, n_{21}, n_{22}$ are instantaneous volatilities and form a 2×2 matrix $n = \{n_{p,q}\}_{2\times 2}$ such that nn^T is positive definite and $\beta < 0$ represent elasticity parameter, see [26] for details. Note that if $\beta = 0$, the stock market price is modelled by GBM.

3 Optimization Problem

Let φ be the optimal investment plan and we define the utility $\mathcal{K}$ attained by the investor from a given state z at time t as

$$\mathcal{N}_{\varphi}(t, \mathcal{R}, s_1,\ s_2, h) = E_{\varphi}\ \mathcal{K}(\mathcal{H}(T)) \mid \mathcal{R}(t) = \mathcal{R}, \mathcal{S}_1(t) = s_1,\ \mathcal{S}_2(t) = s_2,\ \ \mathcal{H}(t) = h], \tag{3.1}$$

where t is the time, $\mathcal{R}$ is the risk free interest rate and h is the wealth. The objective here is to determine the optimal portfolio strategy and the optimal value function of the investor given as

$$\varphi^* \ \text{ and } \ \mathcal{N}(t, \mathcal{R}, s_1,\ s_2, h) = \sup_{\varphi} \mathcal{N}_{\varphi}(t, \mathcal{R}, s_1,\ s_2, h) \tag{3.2}$$

Respectively such that

$$\mathcal{N}_{\varphi^*}(t, \mathcal{R}, s_1,\ s_2, h) = \mathcal{N}(t, \mathcal{R}, s_1,\ s_2, h) \tag{3.3}$$

Let $\mathcal{H}(t)$ be the insurer's wealth at time t and then the differential form associated with the fund size is given as:

$$\mathrm{d}\mathcal{H}(t) = \mathcal{H}(t)\left(\varphi_0 \frac{d\mathcal{S}_0(t)}{\mathcal{S}_0(t)} + \varphi_1 \frac{d\mathcal{S}_1(t)}{\mathcal{S}_1(t)} + \varphi_2 \frac{d\mathcal{S}_2(t)}{\mathcal{S}_2(t)}\right) \tag{3.4}$$

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substituting (2.1), (2.3) and (2.4) into (3.4), we have

$$\mathrm{d}\mathcal{H}(t)=\mathcal{H}(t)\begin{pmatrix}(\varphi_1(m_1-\mathcal{R})+\varphi_2(m_2-\mathcal{R})+\mathcal{R})\,dt\\ +\left(\varphi_1 n_{11}\mathcal{S}_1^{\beta}(t)+\varphi_2 n_{21}\mathcal{S}_2^{\beta}(t)\right)d\mathcal{Z}_1\\ +\left(\varphi_1 n_{12}\mathcal{S}_1^{\beta}(t)+\varphi_2 n_{22}\mathcal{S}_2^{\beta}(t)\right)d\mathcal{Z}_2\\ \mathcal{H}(0)=h_0\end{pmatrix} \tag{3.5}$$

Where φ_0, φ_1 and φ_2 are the optimal investment plans for the risk-free asset and the two risky assets respectively, such that $\varphi_0=1-\varphi_1-\varphi_2$.
Applying the Ito's lemma and maximum principle in [25], the Hamilton Jacobi Bellman (HJB) equation which is a nonlinear PDE associated with (3.5) is obtained by maximizing $\mathcal{N}_{\varphi^*}(t,\mathcal{R},s_1,\ s_2,h)$ subject to the insurer's wealth as follows

$$\left\{\begin{array}{c}\mathcal{N}_t+m_1s_1\mathcal{N}_{s_1}+m_2s_2\mathcal{N}_{s_2}+\frac{1}{2}\mathcal{P}_1s_1^{2\beta+2}\mathcal{N}_{s_1s_1}+\frac{1}{2}\mathcal{P}_2s_2^{2\beta+2}\mathcal{N}_{s_2s_2}\\ +\mathcal{P}_3s_1^{\beta+1}s_2^{\beta+1}\mathcal{N}_{s_1s_2}+\mathcal{R}h\mathcal{N}_h+(a-b\mathcal{R})\mathcal{N}_{\mathcal{R}}+\frac{1}{2}\mathcal{R}\delta^2\mathcal{N}_{\mathcal{R}\mathcal{R}}\\ +\mathcal{P}_4\delta\rho\sqrt{\mathcal{R}}s_1^{\beta+1}\mathcal{N}_{s_1\mathcal{R}}+\mathcal{P}_5\delta\rho\sqrt{\mathcal{R}}s_2^{\beta+1}\mathcal{N}_{s_2\mathcal{R}}\\ +\sup_{\varphi_1,\varphi_2}\left\{\begin{array}{c}\left(\frac{1}{2}\mathcal{P}_1\varphi_1^2s_1^{2\beta}+\mathcal{P}_3\varphi_1\varphi_2s_1^{\beta}s_2^{\beta}+\frac{1}{2}\mathcal{P}_2\varphi_2^2s_2^{2\beta}\right)h^2\mathcal{N}_{hh}\\ ((m_1-\mathcal{R})\varphi_1+(m_2-\mathcal{R})\varphi_2)h\mathcal{N}_h\\ +\left(\mathcal{P}_4\delta\rho\sqrt{\mathcal{R}}\varphi_1s_1^{\beta}+\mathcal{P}_5\delta\rho\sqrt{\mathcal{R}}\varphi_2s_2^{\beta}\right)h\mathcal{N}_{h\mathcal{R}}\\ \left(\mathcal{P}_1s_1^{2\beta+1}\varphi_1+\mathcal{P}_3\varphi_2s_1^{\beta+1}s_2^{\beta}\right)h\mathcal{N}_{hs_1}\\ +\left(\mathcal{P}_3\varphi_2s_1^{\beta+1}s_2^{\beta}+\mathcal{P}_2\varphi_2s_2^{2\beta+1}\right)h\mathcal{N}_{hs_2}\end{array}\right\}\end{array}\right\}=0 \tag{3.6}$$

where

$$\left\{\begin{array}{c}\mathcal{P}_1=n_{11}^2+n_{12}^2,\quad \mathcal{P}_2=n_{21}^2+n_{22}^2,\mathcal{P}_3=n_{11}n_{21}+n_{12}n_{22},\\ \mathcal{P}_4=n_{11}+n_{12},\mathcal{P}_5=n_{21}+n_{22}\end{array}\right.$$

Differentiating (3.6) with respect to φ_1 and φ_2, we obtain the first order maximizing condition for equation (3.6) as

$$\varphi_1^*=\frac{\left[\mathcal{P}_3s_1^{\beta}(m_2-\mathcal{R})-\mathcal{P}_2s_2^{\beta}(m_1-\mathcal{R})\right]}{h(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_1^{2\beta}s_2^{\beta}}\frac{\mathcal{N}_h}{\mathcal{N}_{hh}}-s_1\frac{\mathcal{N}_{hs_1}}{h\mathcal{N}_{hh}}-\frac{(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5)\sqrt{\mathcal{R}}\delta\rho}{h(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_1^{\beta}}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}} \tag{3.7}$$

$$\varphi_2^*=\frac{\left[\mathcal{P}_3s_2^{\beta}(m_1-\mathcal{R})-\mathcal{P}_1s_1^{\beta}(m_2-\mathcal{R})\right]}{h(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_1^{\beta}s_2^{2\beta}}\frac{\mathcal{N}_h}{\mathcal{N}_{hh}}-s_2\frac{\mathcal{N}_{hs_2}}{h\mathcal{N}_{hh}}-\frac{(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4)\sqrt{\mathcal{R}}\delta\rho}{h(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_2^{\beta}}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we have

$$\left\{\begin{array}{c}\mathcal{N}_t+m_1s_1\mathcal{N}_{s_1}+m_2s_2\mathcal{N}_{s_2}+\mathcal{R}h\mathcal{N}_h+\frac{1}{2}\mathcal{P}_1s_1^{2\beta+2}\mathcal{N}_{s_1s_1}+\frac{1}{2}\mathcal{P}_2s_2^{2\beta+2}\mathcal{N}_{s_2s_2}\\ +\mathcal{P}_3s_1^{\beta+1}s_2^{\beta+1}\mathcal{N}_{s_1s_2}+(a-b\mathcal{R})\mathcal{N}_{\mathcal{R}}+\frac{1}{2}\mathcal{R}\delta^2\mathcal{N}_{\mathcal{R}\mathcal{R}}+\mathcal{P}_4\delta\rho\sqrt{\mathcal{R}}s_1^{\beta+1}\mathcal{N}_{s_1\mathcal{R}}\\ +\mathcal{P}_5\delta\rho\sqrt{\mathcal{R}}s_2^{\beta+1}\mathcal{N}_{s_2\mathcal{R}}+\frac{1}{2}\left(\frac{\mathcal{P}_6}{s_1^{\beta}s_2^{\beta}}-\frac{\mathcal{P}_7}{s_1^{2\beta}}-\frac{\mathcal{P}_8}{s_2^{2\beta}}\right)\frac{\mathcal{N}_{\mathcal{R}}^2}{\mathcal{N}_{hh}}-(m_1-\mathcal{R})s_1\frac{\mathcal{N}_h\mathcal{N}_{hs_1}}{\mathcal{N}_{hh}}\\ -(m_2-\mathcal{R})s_2\frac{\mathcal{N}_h\mathcal{N}_{hs_2}}{\mathcal{N}_{hh}}-\delta\rho\sqrt{\mathcal{R}}\left(\frac{\mathcal{P}_9(m_1-\mathcal{R})}{s_1^{\beta}}+\frac{\mathcal{P}_{10}(m_2-\mathcal{R})}{s_2^{\beta}}\right)\frac{\mathcal{N}_h\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}}-\frac{1}{2}\mathcal{P}_1s_1^{2\beta+2}\frac{\mathcal{N}_{hs_1}^2}{\mathcal{N}_{hh}}\\ -\frac{1}{2}\mathcal{P}_2s_2^{2\beta+2}\frac{\mathcal{N}_{hs_2}^2}{\mathcal{N}_{hh}}-\frac{1}{2}\mathcal{P}_{11}\rho^2\delta^2\mathcal{R}\frac{\mathcal{N}_{h\mathcal{R}}^2}{\mathcal{N}_{hh}}-\mathcal{P}_3s_1^{\beta+1}s_2^{\beta+1}\frac{\mathcal{N}_{hs_1}\mathcal{N}_{hs_2}}{\mathcal{N}_{hh}}\end{array}\right\}=0 \tag{3.9}$$

where

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$$\begin{cases} \mathcal{P}_1 = n_{11}^2 + n_{12}^2, \quad \mathcal{P}_2 = n_{21}^2 + n_{22}^2, \mathcal{P}_3 = n_{11}n_{21} + n_{12}n_{22}, \\ \mathcal{P}_4 = n_{11} + n_{12}, \mathcal{P}_5 = n_{21} + n_{22}, \ \mathcal{P}_6 = \frac{2\mathcal{P}_3(m_1-\mathcal{R})(m_2-\mathcal{R})}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)} \\ \mathcal{P}_7 = \frac{\mathcal{P}_2(m_1-\mathcal{R})^2}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}, \mathcal{P}_8 = \frac{\mathcal{P}_1(m_2-\mathcal{R})^2}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}, \mathcal{P}_9 = \frac{(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5)}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}, \\ \mathcal{P}_{10} = \frac{(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4)}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}, \mathcal{P}_{11} = \frac{(\mathcal{P}_2\mathcal{P}_4^2+\mathcal{P}_1\mathcal{P}_5^2-2\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5)}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)} \end{cases} \tag{3.10}$$

From [27], we assumed that the optimal investment plan for risky assets' prices are known based on the assumption that

$$m_1\varphi_1^* + m_2\varphi_2^* = \alpha \tag{3.11}$$

where α is a constant
Substituting (3.7) and (3.8) into (3.11), we derive an expression for $\frac{1}{s_1^\beta s_2^\beta}$ as

$$\frac{1}{s_1^\beta s_2^\beta} = \frac{\mathcal{P}_1\mathcal{P}_2 - \mathcal{P}_3^2}{\mathcal{P}_3(2m_1m_2 - m_1\mathcal{R} - m_2\mathcal{R})} \left[\begin{array}{c} \alpha h\frac{\mathcal{L}_{xx}}{\mathcal{L}_x} + \frac{\mathcal{P}_2 m_1(m_1-\mathcal{R})}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_1^{2\beta}} + m_1 s_1\frac{\mathcal{N}_{hs_1}}{\mathcal{N}_h} + \frac{(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5)m_1\sqrt{\mathcal{R}}\delta\rho}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_1^\beta}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_h} \\ + \frac{\mathcal{P}_1 m_2(m_2-\mathcal{R})}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_2^{2\beta}} + m_2 s_2\frac{\mathcal{N}_{hs_2}}{\mathcal{N}_h} + \frac{(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4)m_2\sqrt{\mathcal{R}}\delta\rho}{(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)s_2^\beta}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_h} \end{array} \right] \tag{3.12}$$

Substituting (3.12) into (3.9), we have

$$\left\{ \begin{array}{c} \mathcal{N}_t + m_1 s_1\mathcal{N}_{s_1} + m_2 s_2\mathcal{N}_{s_2} + (\mathcal{R} + \omega_1)\, h\mathcal{N}_h + \frac{1}{2}\mathcal{P}_1 s_1^{2\beta+2}\mathcal{N}_{s_1s_1} + \frac{1}{2}\mathcal{P}_2 s_2^{2\beta+2}\mathcal{N}_{s_2s_2} \\ +\mathcal{P}_3 s_1^{\beta+1}s_2^{\beta+1}\mathcal{N}_{s_1s_2} + (a - b\mathcal{R})\,\mathcal{N}_\mathcal{R} + \frac{1}{2}\mathcal{R}\delta^2\mathcal{N}_{\mathcal{R}\mathcal{R}} + \mathcal{P}_4\delta\rho\sqrt{\mathcal{R}}s_1^{\beta+1}\mathcal{N}_{s_1\mathcal{R}} \\ +\mathcal{P}_5\delta\rho\sqrt{\mathcal{R}}s_2^{\beta+1}\mathcal{N}_{s_2\mathcal{R}} + \frac{1}{2}\left(\begin{array}{c} \omega_2 s_1^{-2\beta} \\ +\omega_3 s_2^{-2\beta} \end{array} \right)\frac{\mathcal{N}_\mathcal{R}^2}{\mathcal{N}_{hh}} + (\omega_4 - (m_1 - \mathcal{R})\, s_1)\frac{\mathcal{N}_h\mathcal{N}_{hs_1}}{\mathcal{N}_{hh}} \\ + (\omega_5 - (m_2 - \mathcal{R})\, s_2)\frac{\mathcal{N}_h\mathcal{N}_{hs_2}}{\mathcal{N}_{hh}} + \left(\begin{array}{c} \omega_6 s_1^{-\beta} \\ +\omega_7 s_2^{-\beta} \end{array} \right)\frac{\mathcal{N}_h\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}} - \frac{1}{2}\mathcal{P}_1 s_1^{2\beta+2}\frac{\mathcal{N}_{hs_1}^2}{\mathcal{N}_{hh}} \\ -\frac{1}{2}\mathcal{P}_2 s_2^{2\beta+2}\frac{\mathcal{N}_{hs_2}^2}{\mathcal{N}_{hh}} - \frac{1}{2}\mathcal{P}_{11}\rho^2\delta^2\mathcal{R}\frac{\mathcal{N}_{h\mathcal{R}}^2}{\mathcal{N}_{hh}} - \mathcal{P}_3 s_1^{\beta+1}s_2^{\beta+1}\frac{\mathcal{N}_{hs_1}\mathcal{N}_{hs_2}}{\mathcal{N}_{hh}} \end{array} \right\} = 0 \tag{3.13}$$

where,

$$\begin{cases} \omega_1 = \frac{\mathcal{P}_6((\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)\alpha}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}, \ \omega_2 = \frac{\mathcal{P}_2\mathcal{P}_6 m_1(m_1-\mathcal{R})(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}{\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})} - \mathcal{P}_7 \\ \omega_3 = \frac{\mathcal{P}_1\mathcal{P}_6 m_2(m_2-\mathcal{R})(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}{\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})} - \mathcal{P}_8, \omega_4 = \frac{\mathcal{P}_6 m_1 s_1(m_2-\mathcal{R})(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})} \\ \omega_5 = \frac{\mathcal{P}_6 m_2 s_2(m_2-\mathcal{R})(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2)}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}, \omega_6 = \frac{(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5)\mathcal{P}_6 m_1\delta\rho\sqrt{\mathcal{R}}}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})} - \mathcal{P}_9\delta\rho\sqrt{\mathcal{R}}\,(m_1 - \mathcal{R}) \\ \omega_7 = \frac{(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4)\mathcal{P}_6 m_1\delta\rho\sqrt{\mathcal{R}}}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})} - \mathcal{P}_{10}\delta\rho\sqrt{\mathcal{R}}\,(m_2 - \mathcal{R}) \end{cases}.$$

Where, $\mathcal{N}(t, \mathcal{R}, s_1,\ s_2, h) = \mathcal{K}(h)$ and $\mathcal{K}(h)$ is the marginal utility of the investor. Next, we proceed to solve (3.13) for $\mathcal{N}$ using exponential utility, after which we substitute the solution into (3.7) and (3.8) for the optimal investment plan using power transformation, variable change proposed by [28] and asymptotic expansion method in [23].

4 Results & Conclusion

4.1 Optimal Investment Plan for an Investor with CARA Utility

Consider an investor with exponential utility function which exhibit constant absolute risk aversion (CARA).Here, we choose the exponential utility function similar to the one in [21, 27].
Assume the investor takes an exponential utility given as

$$\mathcal{K}\left(h\right)=-\frac{1}{\theta}e^{-\theta h} \tag{4.1}$$

where $\theta>0$ is the risk aversion coefficient.
From equation (4.1), we construct a solution to (3.13) similar to the one in [21, 27] as follows:

$$\begin{cases} \mathcal{N}\left(t,\mathcal{R},s_1,\ s_2,h\right)=-\frac{1}{\theta}e^{-\left(\theta v\left(t,\mathcal{R},s_1\right)+\theta w\left(t,\mathcal{R},s_2\right)+\theta hg(t)\right)} \\ v\left(T,\mathcal{R},s_1\right)=\ w\left(T,\mathcal{R},s_2\right)=0, g\left(T\right)=1 \end{cases} \tag{4.2}$$

$$\left.\begin{array}{c} \mathcal{N}_t=-\theta\mathcal{N}\left(v_t+w_t+hg_t\right),\ \ \mathcal{N}_h=-\theta\mathcal{N}g\ ,\ \ \mathcal{N}_{hh}=\theta^2\mathcal{N}g^2,\ \ \mathcal{N}_{hs_1}=\theta^2\mathcal{N}gv_{s_1} \\ \mathcal{N}_{hs_2}=\theta^2\mathcal{N}gw_{s_2},\ \mathcal{N}_{h\mathcal{R}}=\theta^2\mathcal{N}g\left(v_\mathcal{R}+w_\mathcal{R}\right),\mathcal{N}_{s_1}=-\theta\mathcal{N}v_{s_1},\mathcal{N}_{s_2}=-\theta\mathcal{N}w_{s_2} \\ \mathcal{N}_{s_1s_1}=\mathcal{N}\left(\theta^2v_{s_1}^2-\theta v_{s_1s_1}\right),\ \mathcal{N}_{s_2s_2}=\mathcal{N}\left(\theta^2w_{s_2}^2-\theta w_{s_2s_2}\right),\mathcal{N}_{s_1s_2}=\theta^2\mathcal{N}gv_{s_1}w_{s_2} \\ \mathcal{N}_\mathcal{R}=-\theta\mathcal{N}\left(v_\mathcal{R}+w_\mathcal{R}\right),\mathcal{N}_{\mathcal{R}\mathcal{R}}=\mathcal{N}\left(\theta^2\left(v_\mathcal{R}+w_\mathcal{R}\right)^2-\theta\left(v_{\mathcal{R}\mathcal{R}}+w_{\mathcal{R}\mathcal{R}}\right)\right) \\ \mathcal{N}_{\mathcal{R}s_1}=\mathcal{N}\left(\theta^2\left(v_\mathcal{R}v_{s_1}+w_\mathcal{R}v_{s_1}\right)-\theta v_{\mathcal{R}s_1}\right),\mathcal{N}_{\mathcal{R}s_2}=\mathcal{N}\left(\theta^2\left(v_\mathcal{R}w_{s_2}+w_\mathcal{R}w_{s_2}\right)-\theta w_{\mathcal{R}s_2}\right) \end{array}\right\} \tag{4.3}$$

Substituting (4.3) into (3.13), we have

$$\left\{\begin{array}{c} h\left[g_t+\left(\mathcal{R}+\omega_1\right)g\right] \\ +\left[\begin{array}{c} v_t+w_t+\left(\mathcal{R}+\omega_4\right)s_1v_{s_1}+\left(\mathcal{R}+\omega_5\right)s_2w_{s_2} \\ -\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(v_\mathcal{R}+w_\mathcal{R}\right)^2+\frac{1}{2}\delta^2\mathcal{R}\left(v_{\mathcal{R}\mathcal{R}}+w_{\mathcal{R}\mathcal{R}}\right) \\ +\frac{\omega_2}{2\theta}s_1^{-2\beta}+\frac{\omega_3}{2\theta}s_2^{-2\beta}+\left(v_\mathcal{R}+w_\mathcal{R}\right)\left[a-b\mathcal{R}-\omega_6s_1^{-\beta}-\omega_7s_2^{-\beta}\right] \\ +\frac{1}{2}\mathcal{P}_1s_1^{2\beta+2}v_{s_1s_1}+\mathcal{P}_4\delta\rho\sqrt{\mathcal{R}}s_1^{\beta+1}\left(v_{\mathcal{R}s_1}-\theta\left(v_\mathcal{R}v_{s_1}+w_\mathcal{R}v_{s_1}\right)\right) \\ +\frac{1}{2}\mathcal{P}_2s_2^{2\beta+2}w_{s_2s_2}+\mathcal{P}_5\delta\rho\sqrt{\mathcal{R}}s_2^{\beta+1}\left(w_{\mathcal{R}s_2}-\theta\left(v_\mathcal{R}w_{s_2}+w_\mathcal{R}w_{s_2}\right)\right) \end{array}\right] \end{array}\right\}=0 \tag{4.4}$$

Splitting (4.4) we have

$$\begin{cases} g_t+(\mathcal{R}+\omega_1)g=0 \\ g(T)=1 \end{cases} \tag{4.5}$$

$$\left\{\begin{array}{c} \left[\begin{array}{c} v_t+w_t+\left(\mathcal{R}+\omega_4\right)s_1v_{s_1}+\left(\mathcal{R}+w_5\right)s_2w_{s_2} \\ -\frac{1}{2}\theta^2\mathcal{R}\left(1-\mathcal{P}_{11}^2\right)\left(v_\mathcal{R}+w_\mathcal{R}\right)^2+\frac{1}{2}^2\mathcal{R}\left(v_{\mathcal{R}\mathcal{R}}+w_{\mathcal{R}\mathcal{R}}\right) \\ +\frac{w_2}{2\theta}s_1^{-2\beta}+\frac{w_3}{2\theta}s_2^{-2\beta}+\left(v_\mathcal{R}+w_\mathcal{R}\right)\left[a-b\mathcal{R}-w_6s_1^{-\beta}-w_7s_2^{-\beta}\right] \\ +\frac{1}{2}\mathcal{P}_1s_1^{2\beta+2}v_{s_1s_1}+\mathcal{P}_4\sqrt{\mathcal{R}}s_1^{\beta+1}\left(v_{\mathcal{R}s_1}-\theta\left(v_\mathcal{R}v_{s_1}+w_\mathcal{R}v_{s_1}\right)\right) \\ +\frac{1}{2}\mathcal{P}_2s_2^{2\beta+2}w_{s_2s_2}+\mathcal{P}_5\sqrt{\mathcal{R}}s_2^{\beta+1}\left(w_{\mathcal{R}s_2}-\theta\left(v_\mathcal{R}w_{s_2}+w_\mathcal{R}w_{s_2}\right)\right) \end{array}\right]=0 \\ v\left(T,\mathcal{R},s_1\right)=\ w\left(T,\mathcal{R},s_2\right)=0 \end{array}\right. \tag{4.6}$$

Solving equation (4.5) for g, we obtain

$$g(t)=e^{(\mathcal{R}+w_1)(T-t)} \tag{4.7}$$

Substituting for $w_1=\frac{\mathcal{P}_6\left(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\right.}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}$ in (4.7), we have

$$g(t)=Exp\left[\mathcal{R}+\frac{\mathcal{P}_6\left(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\right.}{2\mathcal{P}_3\left(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R}\right)}\right](T-t) \tag{4.8}$$

Lemma 4.1. *The solution of equation (4.6) is given as*

$$v\left(t,\mathcal{R},s_1\right)+w\left(t,\mathcal{R},s_2\right)=f\left(t,\mathcal{R},y,z\right)=f^1\left(t,\mathcal{R},y,z\right)+\sqrt{f^2\left(t,\mathcal{R},y,z\right)+f^3\left(t,\mathcal{R},y,z\right)}$$

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where

$$
\left\{
\begin{array}{l}
f^1(t,\mathcal{R},y,z) = \left[\begin{array}{c}
\left[\frac{\mathcal{P}_1(2\beta+1)w_2}{4\theta(\mathcal{R}+w_4)} + \frac{\mathcal{P}_2(2\beta+1)w_3}{4\theta(\mathcal{R}+w_5)}\right](T-t) - \frac{\mathcal{P}_1(2\beta+1)w_2}{4\beta\theta(\mathcal{R}+w_4)^2}\left[1-e^{2\beta(\mathcal{R}+w_4)(t-T)}\right] \\
-\frac{\mathcal{P}_1(2\beta+1)w_2}{4\beta\theta(\mathcal{R}+w_4)^2}\left[1-e^{2\beta(\mathcal{R}+w_4)(t-T)}\right] \\
+\frac{w_2}{4\beta\theta(\mathcal{R}+w_4)}\left[1-e^{2\beta(\mathcal{R}+w_4)(t-T)}\right]y \\
+\frac{w_3}{4\beta\theta(\mathcal{R}+w_4)}\left[1-e^{2\beta(\mathcal{R}+w_5)(t-T)}\right]z
\end{array}\right] \\
f^2(t,\mathcal{R},y,z) = \left[\begin{array}{c}
\left[\begin{array}{c}
\left[\frac{\mathcal{P}_1(2\beta+1)w_2}{4\theta(\mathcal{R}+w_4)} + \frac{\mathcal{P}_2(2\beta+1)w_3}{4\theta(\mathcal{R}+w_5)}\right](T-t) - \frac{\mathcal{P}_1(2\beta+1)w_2}{4\beta\theta(\mathcal{R}+w_4)^2}\left[1-e^{2\beta(\mathcal{R}+w_4)(t-T)}\right] \\
-\frac{\mathcal{P}_1(2\beta+1)w_2}{4\beta\theta(\mathcal{R}+w_4)^2}\left[1-e^{2\beta(\mathcal{R}+w_4)(t-T)}\right]
\end{array}\right] \\
+y^{\frac{1}{2}}e^{\beta(\mathcal{R}+w_4)(t-T)} \left[\begin{array}{c}
2\mathcal{P}_4\beta\sqrt{\mathcal{R}}\int_t^T A^1_{2\mathcal{R}} e^{\beta(\mathcal{R}+w_4)(T-\tau)}d\tau \\
-w_6\int_t^T A^1_{1\mathcal{R}} e^{\beta(\mathcal{R}+w_4)(T-\tau)}d\tau \\
-\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T B^2_6 e^{\beta(\mathcal{R}+w_4)(T-\tau)}d\tau
\end{array}\right] \\
+z^{\frac{1}{2}}e^{\beta(\mathcal{R}+w_5)(t-T)} \left[\begin{array}{c}
2\mathcal{P}_5\beta\sqrt{\mathcal{R}}\int_t^T A^1_{3\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
-\omega_7\int_t^T A^1_{1\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
-\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T B^2_7 e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau
\end{array}\right] \\
+\frac{\omega_2 y}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] + \frac{\omega_3 z}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right] \\
+y^{\frac{3}{2}}\omega_6 e^{3\beta(\mathcal{R}+\omega_4)(t-T)}\int_t^T A^1_{2\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+z^{\frac{3}{2}}\omega_7 e^{3\beta(\mathcal{R}+\omega_5)(t-T)}\int_t^T A^1_{3\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau
\end{array}\right]
\end{array}
\right.
$$

$$
\begin{array}{l}
f^3(t,\mathcal{R},y,z) = \left[\begin{array}{c}
\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{1\mathcal{R}}\right)^2 d\tau + 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{2\mathcal{R}} d\tau\tau \\
+2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{3\mathcal{R}} d - \mathcal{P}_1\beta(2\beta+1)\int_t^T C^3_4 d\tau \\
-\mathcal{P}_2\beta(2\beta+1)\int_t^T C^3_5 d\tau - (a-b\mathcal{R})\int_t^T A^1_{1\mathcal{R}} d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{1\mathcal{R}\mathcal{R}} d\tau
\end{array}\right] \\
+y^{\frac{1}{2}}e^{\beta(\mathcal{R}+\omega_4)(t-T)} \left[\begin{array}{c}
2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{4\mathcal{R}} e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+\omega_6\int_t^T B^2_{1\mathcal{R}} e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
-\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T C^3_6 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau
\end{array}\right] \\
+z^{\frac{1}{2}}e^{\beta(\mathcal{R}+\omega_5)(t-T)} \left[\begin{array}{c}
2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{5\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
+\omega_7\int_t^T B^2_{1\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
-\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T C^2_7 e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau
\end{array}\right] \\
+ye^{2\beta(\mathcal{R}+\omega_4)(t-T)} \left[\begin{array}{c}
3\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{6\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
-\frac{\omega_2}{2\theta}\int_t^T e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau - (a-b\mathcal{R})\int_t^T A^1_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+\omega_6\int_t^T B^2_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{2\mathcal{R}\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
-2\mathcal{P}_1(2\beta^2+\beta(2\beta+1))\int_t^T C^3_8 e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau
\end{array}\right] \\
+ze^{2\beta(\mathcal{R}+\omega_5)(t-T)} \left[\begin{array}{c}
3\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{7\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
-\frac{\omega_3}{2\theta}\int_t^T e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau - (a-b\mathcal{R})\int_t^T A^1_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
+\omega_7\int_t^T B^2_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{3\mathcal{R}\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
-2\mathcal{P}_2(2\beta^2+\beta(2\beta+1))\int_t^T C^3_9 e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
+\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau
\end{array}\right] \\
+y^{\frac{3}{2}}\omega_6 e^{3\beta(\mathcal{R}+\omega_4)(t-T)}\int_t^T B^1_{4\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+z^{\frac{3}{2}}\omega_7 e^{3\beta(\mathcal{R}+\omega_5)(t-T)}\int_t^T B^1_{5\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
+y^2 e^{2\beta(\mathcal{R}+\omega_4)(t-T)} \left[\begin{array}{c}
\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{2\mathcal{R}}\right)^2 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+\omega_6\int_t^T B^2_{6\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau
\end{array}\right] \\
+z^2 e^{2\beta(\mathcal{R}+\omega_5)(t-T)} \left[\begin{array}{c}
\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{3\mathcal{R}}\right)^2 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
+\omega_7\int_t^T B^2_{7\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau
\end{array}\right]
\end{array}
\right\}
$$

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Proof. Assume

$$\left\{\begin{array}{c} v(t,\mathcal{R},s_1)+w(t,\mathcal{R},s_2)=f(t,\mathcal{R},y,z),\\ y=s_1^{-2\beta},\quad z=s_2^{-2\beta},\ f(T,\mathcal{R},y,z)=0\end{array}\right. \tag{4.9}$$

Then

$$\left.\begin{array}{c} v_t+w_t=f_t,\quad v_{s_1}=-2\beta s_1^{-2\beta-1}f_y\ ,\ v_{s_1s_1}=2\beta(2\beta+1)s_1^{-2\beta-2}f_y+4\beta^2s_1^{-4\beta-2}f_{yy},\\ v_{\mathcal{R}}+w_{\mathcal{R}}=f_{\mathcal{R}},\quad v_{\mathcal{R}\mathcal{R}}+w_{\mathcal{R}\mathcal{R}}=f_{\mathcal{R}\mathcal{R}},\quad v_{\mathcal{R}s_1}=-2\beta s_1^{-2\beta-1}f_{\mathcal{R}y}\\ w_{s_2}=-2\beta s_2^{-2\beta-1}q_z\ ,\ w_{s_2s_2}=2\beta(2\beta+1)s_2^{-2\beta-2}f_z+4\beta^2s_2^{-4\beta-2}f_{zz},\\ w_{\mathcal{R}s_2}=-2\beta s_2^{-2\beta-1}f_{\mathcal{R}z}\end{array}\right\} \tag{4.10}$$

Substituting (4.10) into (4.6), we have

$$\left\{\begin{array}{c}\left[\begin{array}{c} f_t-2\beta y(\mathcal{R}+\omega_4)f_y-2\beta z(\mathcal{R}+\omega_5)f_z\\ -\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)f_{\mathcal{R}}^2+\frac{1}{2}\delta^2\mathcal{R}f_{\mathcal{R}\mathcal{R}}\\ +\frac{\omega_2}{2\theta}y+\frac{\omega_3}{2\theta}z+f_{\mathcal{R}}\left[a-b\mathcal{R}-\omega_6\sqrt{y}-\omega_7\sqrt{z}\right]\\ +\mathcal{P}_1\beta(2\beta+1)f_y+2\mathcal{P}_1\beta^2yf_{yy}-2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\sqrt{y}\left(f_{\mathcal{R}y}-\theta f_{\mathcal{R}}f_y\right)\\ +\mathcal{P}_2\beta(2\beta+1)f_z+2\mathcal{P}_2\beta^2zf_{zz}-2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\sqrt{z}\left(f_{\mathcal{R}z}-\theta f_{\mathcal{R}}f_z\right)\end{array}\right]=0\\ f(T,\mathcal{R},y,z)=0\end{array}\right. \tag{4.11}$$

We can rewrite (4.11) as

$$(E+F+G)f-\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)f_{\mathcal{R}}^2=0 \tag{4.12}$$

Where

$$E=\left[(a-b\mathcal{R})f_{\mathcal{R}}+\frac{1}{2}\delta^2\mathcal{R}f_{\mathcal{R}\mathcal{R}}\right] \tag{4.13}$$

$$F=\left[\begin{array}{c} f_t+\beta\left(\mathcal{P}_1(2\beta+1)-2y(\mathcal{R}+\omega_4)\right)f_y+\frac{\omega_2}{2\theta}y+\frac{\omega_3}{2\theta}z\\ +\beta\left(\mathcal{P}_2(2\beta+1)-2z(\mathcal{R}+\omega_5)\right)f_z+2\beta^2\mathcal{P}_1yf_{yy}+2\beta^2\mathcal{P}_2zf_{zz}\end{array}\right] \tag{4.14}$$

$$G=\left[\begin{array}{c} 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}f_y-f_{\mathcal{R}y}\right)\sqrt{y}+2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}f_z-f_{\mathcal{R}z}\right)\sqrt{z}\\ -\left(\omega_6\sqrt{y}+\omega_7\sqrt{z}\right)f_{\mathcal{R}}\end{array}\right] \tag{4.15}$$

Next we follow the approach in [11] by applying the asymptotic expansion method to solve the problem in (4.12).
Assume that the volatility follows a slow fluctuating process, we attempt to find an asymptotic solution of (4.12) by a following slow-fluctuating process r_α to replace (2.2), in which $0<\varepsilon\ll 1$ is a small positive parameter:

$$d\mathcal{R}_\varepsilon\ (t)\ =\ (a-b\mathcal{R}_\varepsilon(t))dt-\delta\sqrt{\mathcal{R}_\varepsilon(t)}\,d\mathcal{Z}_0(t), \tag{4.16}$$

Substituting (4.16) into (4.12) and also replacing $a-b\mathcal{R}(t)$ by $\varepsilon(a-b\mathcal{R}(t))$ and $\sqrt{\mathcal{R}}$ by $\sqrt{\varepsilon}\sqrt{\mathcal{R}}$, we will have

$$\left(\varepsilon E+F+\sqrt{\varepsilon}G\right)f_\varepsilon=0 \tag{4.17}$$

Next, we conjecture a solution for (4.17) as follows

$$f_\varepsilon(t,\mathcal{R},y,z)=f^1(t,\mathcal{R},y,z)+\sqrt{\varepsilon}f^2(t,\mathcal{R},y,z)+\varepsilon f^3(t,\mathcal{R},y,z) \tag{4.18}$$

Substituting (4.18) into (4.14) and simplifying it, we have

$$\left(\begin{array}{c} Ff^1(t,\mathcal{R},y,z)+\left[Ff^2(t,\mathcal{R},y,z)+Gf^1(t,\mathcal{R},y,z)\right]\sqrt{\varepsilon}\\ +\left[\begin{array}{c} Ef^1(t,\mathcal{R},y,z)+Ff^3(t,\mathcal{R},y,z)\\ +Gf^2(t,\mathcal{R},y,z)-\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(f_{\mathcal{R}}^1\right)^2\end{array}\right]\varepsilon\end{array}\right)=0 \tag{4.19}$$

This implies that

$$\left\{\begin{array}{c} Ff^1(t,\mathcal{R},y,z)=F\left(p^1(t,\mathcal{R},y)+q^1(t,\mathcal{R},z)\right)=0\\ f^1(T,\mathcal{R},y,z)=0\end{array}\right. \tag{4.20}$$

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$$\left\{\begin{array}{c} Ff_2(t,\mathcal{R},y,z)+Gf_1(t,\mathcal{R},y,z)=0 \\ f^1(T,\mathcal{R},y,z)=f^2(T,\mathcal{R},y,z)=0 \end{array}\right. \tag{4.21}$$

$$\left\{\begin{array}{c} Ef^1(t,\mathcal{R},y,z)+Ff^3(t,\mathcal{R},y,z)+Gf^2(t,\mathcal{R},y,z)=0 \\ f^1(T,\mathcal{R},y,z)=f^2(T,\mathcal{R},y,z)=f^3(T,\mathcal{R},y,z)=0 \end{array}\right. \tag{4.22}$$

From (4.13), (4.14) and (4.15), equation (4.20), (4.21) and (4.22) can be expressed as

$$\left\{\begin{array}{c} \left[\begin{array}{c} f_t^1+\beta\left(\mathcal{P}_1(2\beta+1)-2y(\mathcal{R}+\omega_4)\right)f_y^1+\frac{\omega_2}{2\theta}y+\frac{\omega_3}{2\theta}z \\ +\beta\left(\mathcal{P}_2(2\beta+1)-2z(\mathcal{R}+\omega_5)\right)f_z^1+2\beta^2\mathcal{P}_1f_{yy}^1y+2\beta^2\mathcal{P}_2f_{zz}^1z \end{array}\right]=0 \\ f^1(T,\mathcal{R},y,z)=0 \end{array}\right. \tag{4.23}$$

$$\left\{\begin{array}{c} \left[\begin{array}{c} f_t^2+\beta\left(\mathcal{P}_1(2\beta+1)-2y(\mathcal{R}+\omega_4)\right)f_y^2+\frac{\omega_2}{2\theta}y+\frac{\omega_3}{2\theta}z \\ +\beta\left(\mathcal{P}_2(2\beta+1)-2z(\mathcal{R}+\omega_5)\right)f_z^2+2\beta^2\mathcal{P}_1f_{yy}^2y+2\beta^2\mathcal{P}_2f_{zz}^2z \\ +2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}^1f_y^1-f_{y\mathcal{R}}^1\right)\sqrt{y}-\left(\omega_6\sqrt{y}+\omega_7\sqrt{z}\right)f_{\mathcal{R}}^1 \\ +2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}^1f_z^1-f_{\mathcal{R}z}^1\right)\sqrt{z} \end{array}\right]=0 \\ f^1(T,\mathcal{R},y,z)=f^2(T,\mathcal{R},y,z)=0 \end{array}\right. \tag{4.24}$$

$$\left\{\begin{array}{c} \left[\begin{array}{c} f_t^3+\beta\left(\mathcal{P}_1(2\beta+1)-2y(\mathcal{R}+\omega_4)\right)f_y^3+\frac{\omega_2}{2\theta}y+\frac{\omega_3}{2\theta}z \\ +\beta\left(\mathcal{P}_2(2\beta+1)-2z(\mathcal{R}+\omega_5)\right)f_z^3+2\beta^2\mathcal{P}_1f_{yy}^3y+2\beta^2\mathcal{P}_2f_{zz}^3z \\ +2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}^2f_y^2-f_{y\mathcal{R}}^2\right)\sqrt{y}-\left(\omega_6\sqrt{y}+\omega_7\sqrt{z}\right)f_{\mathcal{R}}^2 \\ +2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\left(\theta f_{\mathcal{R}}^2f_z^2-f_{\mathcal{R}z}^2\right)\sqrt{z}-\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(f_{\mathcal{R}}^1\right)^2 \\ (a-b\mathcal{R})f_{\mathcal{R}}^1+\frac{1}{2}\delta^2\mathcal{R}f_{\mathcal{R}\mathcal{R}}^1 \end{array}\right]=0 \\ f^1(T,\mathcal{R},y,z)=f^2(T,\mathcal{R},y,z)=f^3(T,\mathcal{R},y,z)=0 \end{array}\right. \tag{4.25}$$

Next, we move on to solve equation (4.23), (4.24) and (4.25) for f^1, f^2 *and* f^3
From (4.23), we conjecture a solution of the form

$$\left\{\begin{array}{c} f^1(t,\mathcal{R},y,z)=A_1^1(t,\mathcal{R})+yA_2^1(t,\mathcal{R})+zA_3^1(t,\mathcal{R}) \\ A_1^1(T,\mathcal{R})=A_2^1(T,\mathcal{R})=A_3^1(T,\mathcal{R})=0 \end{array}\right. \tag{4.26}$$

and

$$f_t^1=A_{1t}^1+yA_{2t}^1+zA_{3t}^1,\quad f_y^1=A_2^1, f_{yy}^1=0,\ \ f_z^1=A_3^1\ ,\ \ f_{zz}^1=0\ \} \tag{4.27}$$

Substituting (4.27) in (4.23), we have

$$\left\{\begin{array}{c} A_{1t}^1+\mathcal{P}_1\beta(2\beta+1)A_2^1+\mathcal{P}_2\beta(2\beta+1)A_3^1=0 \\ A_1^1(T,\mathcal{R})=A_2^1(T,\mathcal{R})=A_3^1(T,\mathcal{R})=0 \end{array}\right. \tag{4.28}$$

$$\left\{\begin{array}{c} A_{2t}^1-2\beta(\mathcal{R}+\omega_4)A_2^1+\frac{\omega_2}{2\theta}=0 \\ A_2^1(T,\mathcal{R})=0 \end{array}\right. , \tag{4.29}$$

$$\left\{\begin{array}{c} A_{3t}^1-2\beta(\mathcal{R}+\omega_5)A_3^1+\frac{\omega_3}{2\theta}=0 \\ A_3^1(T,\mathcal{R})=0 \end{array}\right. \tag{4.30}$$

Solving (4.28), (4.29) and (4.30), we have

$$A_1^1(t,\mathcal{R})=\left[\begin{array}{c} \left[\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\theta(\mathcal{R}+\omega_4)}+\frac{\mathcal{P}_2(2\beta+1)\omega_3}{4\theta(\mathcal{R}+\omega_5)}\right](T-t)-\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] \\ -\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] \end{array}\right] \tag{4.31}$$

$$A_2^1(t,\mathcal{R})=\frac{\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] \tag{4.32}$$

$$A_3^1(t,\mathcal{R})=\frac{\omega_3}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right] \tag{4.33}$$

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Hence from (4.26)

$$f^1(t,\mathcal{R},y,z)=\begin{bmatrix}\left[\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\theta(\mathcal{R}+\omega_4)}+\frac{\mathcal{P}_2(2\beta+1)\omega_3}{4\theta(\mathcal{R}+\omega_5)}\right](T-t)-\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\\ -\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\\ +\frac{\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]y\\ +\frac{\omega_3}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right]z\end{bmatrix} \tag{4.34}$$

Next, we proceed to solve (4.24), by assuming a solution of the form

$$f^2(t,\mathcal{R},y,z)=\begin{bmatrix}B_1^2(t,\mathcal{R})+y^{\frac{1}{2}}B_2^2(t,\mathcal{R})+z^{\frac{1}{2}}B_3^2(t,\mathcal{R})+yB_4^2(t,\mathcal{R})\\ +zB_5^2(t,\mathcal{R})+y^{\frac{3}{2}}B_6^2(t,\mathcal{R})+z^{\frac{3}{2}}B_7^2(t,\mathcal{R})\\ B_1^2(T,\mathcal{R})=B_2^2(T,\mathcal{R})=B_3^2(T,\mathcal{R})=B_4^2(T,\mathcal{R})\\ =B_5^2(T,\mathcal{R})=B_6^2(T,\mathcal{R})=B_7^2(T,\mathcal{R})=0\end{bmatrix} \tag{4.35}$$

and

$$\left.\begin{array}{l}f_t^2=B_{1t}^2+y^{\frac{1}{2}}B_{2t}^2+z^{\frac{1}{2}}B_{3t}^2+yB_{4t}^2+zB_{5t}^2+y^{\frac{3}{2}}B_{6t}^2+z^{\frac{3}{2}}B_{7t}^2,\\ f_y^2=\frac{1}{2}y^{-\frac{1}{2}}B_2^2+B_4^2+\frac{3}{2}y^{\frac{1}{2}}B_6^2,\ f_{yy}^2=-\frac{1}{4}y^{-\frac{3}{2}}B_2^2+\frac{3}{4}y^{-\frac{1}{2}}B_6^2,\\ f_z^2=\frac{1}{2}z^{-\frac{1}{2}}B_3^2+B_5^2+\frac{3}{2}z^{\frac{1}{2}}B_7^2\ ,\ f_{zz}^2=-\frac{1}{4}z^{-\frac{3}{2}}B_3^2+\frac{3}{4}z^{-\frac{1}{2}}B_7^2\end{array}\right\} \tag{4.36}$$

Substituting (4.36) into (4.24), we have

$$\begin{cases}B_{1t}^2+\mathcal{P}_1\beta(2\beta+1)B_4^2+\mathcal{P}_2\beta(2\beta+1)B_5^2=0\\ B_1^2(T,\mathcal{R})=B_2^2(T,\mathcal{R})=B_3^2(T,\mathcal{R})=0\end{cases} \tag{4.37}$$

$$\begin{cases}B_{2t}^2-\beta(\mathcal{R}+\omega_4)B_2^2+\frac{3}{2}\beta^2\mathcal{P}_1B_6^2-2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\,A_{2\mathcal{R}}^1-\omega_6A_{1\mathcal{R}}^1=0\\ B_2^2(T,\mathcal{R})=0\end{cases} \tag{4.38}$$

$$\begin{cases}B_{3t}^2-\beta(\mathcal{R}+\omega_5)B_3^2+\frac{3}{2}\beta^2\mathcal{P}_2B_7^2-2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\,A_{3\mathcal{R}}^1-\omega_7A_{1\mathcal{R}}^1=0\\ B_3^2(T,\mathcal{R})=0\end{cases} \tag{4.39}$$

$$\begin{cases}B_{4t}^2-2\beta(\mathcal{R}+\omega_4)B_4^2+\frac{\omega_2}{2\theta}=0\\ B_4^2(T,\mathcal{R})=0\end{cases}, \tag{4.40}$$

$$\begin{cases}B_{5t}^2-2\beta(\mathcal{R}+\omega_5)B_5^2+\frac{\omega_3}{2\theta}=0\\ B_5^2(T,\mathcal{R})=0\end{cases} \tag{4.41}$$

$$\begin{cases}B_{6t}^2-3\beta(\mathcal{R}+\omega_4)B_6^2+\omega_6A_{2\mathcal{R}}^1=0\\ B_6^2(T,\mathcal{R})=0\end{cases}, \tag{4.42}$$

$$\begin{cases}B_{7t}^2-3\beta(\mathcal{R}+\omega_5)B_7^2+\omega_7A_{3\mathcal{R}}^1=0\\ B_7^2(T,\mathcal{R})=0\end{cases} \tag{4.43}$$

Solving equation (4.37) - (4.43), we have

$$\left.\begin{array}{l}B_1^2(t,\mathcal{R})=\begin{bmatrix}\left[\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\theta(\mathcal{R}+\omega_4)}+\frac{\mathcal{P}_2(2\beta+1)\omega_3}{4\theta(\mathcal{R}+\omega_5)}\right](T-t)-\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\\ -\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\end{bmatrix}\\ B_2^2(t,\mathcal{R})=e^{\beta(\mathcal{R}+\omega_4)(t-T)}\begin{bmatrix}2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T A_{2\mathcal{R}}^1e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ -\omega_6\int_t^T A_{1\mathcal{R}}^1e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ -\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T B_6^2e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\end{bmatrix}\\ B_3^2(t,\mathcal{R})=e^{\beta(\mathcal{R}+\omega_5)(t-T)}\begin{bmatrix}2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T A_{3\mathcal{R}}^1e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\\ -\omega_7\int_t^T A_{1\mathcal{R}}^1e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\\ -\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T B_7^2e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\end{bmatrix}\\ B_4^2(t,\mathcal{R})=\frac{\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\\ B_5^2(t,\mathcal{R})=\frac{\omega_3}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right]\\ B_6^2(t,\mathcal{R})=\omega_6e^{3\beta(\mathcal{R}+\omega_4)(t-T)}\int_t^T A_{2\mathcal{R}}^1e^{3\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ B_7^2(t,\mathcal{R})=\omega_7e^{3\beta(\mathcal{R}+\omega_5)(t-T)}\int_t^T A_{3\mathcal{R}}^1e^{3\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\end{array}\right\} \tag{4.44}$$

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Substituting (4.44) into (4.35), we have

$$f^2(t,\mathcal{R},y,z)=\left[\begin{array}{c}\left[\begin{array}{c}\left[\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\theta(\mathcal{R}+\omega_4)}+\frac{\mathcal{P}_2(2\beta+1)\omega_3}{4\theta(\mathcal{R}+\omega_5)}\right](T-t)-\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\\ -\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]\end{array}\right]\\ +y^{\frac{1}{2}}e^{\beta(\mathcal{R}+\omega_4)(t-T)}\left[\begin{array}{c}2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T A^1_{2\mathcal{R}}e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ -\omega_6\int_t^T A^1_{1\mathcal{R}}e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ -\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T B^2_6e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\end{array}\right]\\ +z^{\frac{1}{2}}e^{\beta(\mathcal{R}+\omega_5)(t-T)}\left[\begin{array}{c}2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T A^1_{3\mathcal{R}}e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\\ -\omega_7\int_t^T A^1_{1\mathcal{R}}e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\\ -\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T B^2_7e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\end{array}\right]\\ +\frac{\omega_2 y}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]+\frac{\omega_3 z}{4\beta\theta(\mathcal{R}+\omega_4)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right]\\ +y^{\frac{3}{2}}\omega_6e^{3\beta(\mathcal{R}+\omega_4)(t-T)}\int_t^T A^1_{2\mathcal{R}}e^{3\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau\\ +z^{\frac{3}{2}}\omega_7e^{3\beta(\mathcal{R}+\omega_5)(t-T)}\int_t^T A^1_{3\mathcal{R}}e^{3\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau\end{array}\right] \tag{4.45}$$

Next, we attempt to solve (4.25), by assuming a solution of the form

$$f^3(t,\mathcal{R},y,z)=\left[\begin{array}{c}C^3_1(t,\mathcal{R})+y^{\frac{1}{2}}C^3_2(t,\mathcal{R})+z^{\frac{1}{2}}C^3_3(t,\mathcal{R})+yC^3_4(t,\mathcal{R})+zC^3_5(t,\mathcal{R})\\ +y^{\frac{3}{2}}C^3_6(t,\mathcal{R})+z^{\frac{3}{2}}C^3_7(t,\mathcal{R})+y^2C^3_8(t,\mathcal{R})+z^2C^3_9(t,\mathcal{R})\\ C^3_1(T,\mathcal{R})=C^3_2(T,\mathcal{R})=C^3_3(T,\mathcal{R})=C^3_4(T,\mathcal{R})=C^3_5(T,\mathcal{R})\\ =C^3_6(T,\mathcal{R})=C^3_7(T,\mathcal{R})=C^3_8(T,\mathcal{R})=C^3_9(T,\mathcal{R})=0\end{array}\right] \tag{4.46}$$

and

$$\left.\begin{array}{c}f^3_t=C^3_{1t}+y^{\frac{1}{2}}C^3_{2t}+z^{\frac{1}{2}}C^3_{3t}+yC^3_{4t}+zC^2_{5t}+y^{\frac{3}{2}}C^3_{6t}+z^{\frac{3}{2}}C^3_{7t}+y^2C^3_{8t}+z^2C^3_{9t},\\ f^3_y=\frac{1}{2}y^{-\frac{1}{2}}C^3_2+C^3_4+\frac{3}{2}y^{\frac{1}{2}}C^3_6+2yC^3_8,\ f^3_{yy}=-\frac{1}{4}y^{-\frac{3}{2}}C^3_2+\frac{3}{4}y^{-\frac{1}{2}}C^3_6+2C^3_8,\\ f^3_z=\frac{1}{2}z^{-\frac{1}{2}}C^2_3+C^2_5+\frac{3}{2}z^{\frac{1}{2}}C^2_7+2zC^3_9\ ,f^3_{zz}=-\frac{1}{4}z^{-\frac{3}{2}}C^3_3+\frac{3}{4}z^{-\frac{1}{2}}C^3_7+2C^3_9\\ f^1_{\mathcal{R}}=A^1_{1\mathcal{R}}+yA^1_{2\mathcal{R}}+zA^1_{3\mathcal{R}},f^1_{\mathcal{R}\mathcal{R}}=A^1_{1\mathcal{R}\mathcal{R}}+yA^1_{2\mathcal{R}\mathcal{R}}+zA^1_{3\mathcal{R}\mathcal{R}}\\ f^2_{\mathcal{R}}=B^2_{1\mathcal{R}}+y^{\frac{1}{2}}B^2_{2\mathcal{R}}+z^{\frac{1}{2}}B^2_{3\mathcal{R}}+yB^2_{4\mathcal{R}}+zB^2_{5\mathcal{R}}+y^{\frac{3}{2}}B^2_{6\mathcal{R}}+z^{\frac{3}{2}}B^2_{7\mathcal{R}},\\ f^2_{y\mathcal{R}}=\frac{1}{2}y^{-\frac{1}{2}}B^2_{2\mathcal{R}}+B^2_{4\mathcal{R}}+\frac{3}{2}y^{\frac{1}{2}}B^2_{6\mathcal{R}},\ \ f^2_{z\mathcal{R}}=\frac{1}{2}z^{-\frac{1}{2}}B^2_{3\mathcal{R}}+B^2_{5\mathcal{R}}+\frac{3}{2}z^{\frac{1}{2}}B^2_{7\mathcal{R}}\end{array}\right\} \tag{4.47}$$

Substituting (4.47) into (4.25), we have

$$\left\{\begin{array}{c}\left[\begin{array}{c}C^3_{1t}+\mathcal{P}_1\beta(2\beta+1)C^3_4+\mathcal{P}_2\beta(2\beta+1)C^3_5+(a-b\mathcal{R})A^1_{1\mathcal{R}}+\frac{1}{2}\delta^2\mathcal{R}A^1_{1\mathcal{R}\mathcal{R}}\\ -\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(A^1_{1\mathcal{R}}\right)^2-2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}B^2_{2\mathcal{R}}-2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}B^2_{3\mathcal{R}}\end{array}\right]=0\\ C^3_1(T,\mathcal{R})=0\end{array}\right. \tag{4.48}$$

$$\left\{\begin{array}{c}C^3_{2t}-\beta(\mathcal{R}+\omega_4)C^3_2+\frac{3}{2}\beta^2\mathcal{P}_1C^3_6-2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}B^2_{4\mathcal{R}}-\omega_6B^2_{1\mathcal{R}}=0\\ C^3_2(T,\mathcal{R})=0\end{array}\right. \tag{4.49}$$

$$\left\{\begin{array}{c}C^3_{3t}-\beta(\mathcal{R}+\omega_5)C^3_3+\frac{3}{2}\beta^2\mathcal{P}_2B^2_7-2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}B^2_{5\mathcal{R}}-\omega_7B^2_{1\mathcal{R}}=0\\ C^3_3(T,\mathcal{R})=0\end{array}\right. \tag{4.50}$$

$$\left\{\begin{array}{c}C^3_{4t}-2\beta(\mathcal{R}+\omega_4)C^3_4+2\mathcal{P}_1(2\beta^2+\beta(2\beta+1)C^3_8+(a-b\mathcal{R})A^1_{2\mathcal{R}}+\frac{1}{2}\delta^2\mathcal{R}A^1_{2\mathcal{R}\mathcal{R}}\\ -\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)A^1_{1\mathcal{R}}A^1_{2\mathcal{R}}-3\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}B^2_{6\mathcal{R}}-\omega_6B^2_{2\mathcal{R}}+\frac{\omega_2}{2\theta}=0\\ C^3_4(T,\mathcal{R})=0\end{array}\right. \tag{4.51}$$

$$\left\{\begin{array}{c}C^3_{5t}-2\beta(\mathcal{R}+\omega_5)C^3_5+2\mathcal{P}_2(2\beta^2+\beta(2\beta+1)C^3_9+(a-b\mathcal{R})A^1_{3\mathcal{R}}+\frac{1}{2}\delta^2\mathcal{R}A^1_{3\mathcal{R}\mathcal{R}}\\ -\frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)A^1_{1\mathcal{R}}A^1_{3\mathcal{R}}-3\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}B^2_{7\mathcal{R}}-\omega_7B^2_{3\mathcal{R}}+\frac{\omega_3}{2\theta}=0\\ C^3_5(T,\mathcal{R})=0\end{array}\right. \tag{4.52}$$

$$\left\{\begin{array}{c}C^3_{6t}-3\beta(\mathcal{R}+\omega_4)C^3_6+\omega_6B^2_{4\mathcal{R}}=0\\ C^3_6(T,\mathcal{R})=0\end{array}\right., \tag{4.53}$$

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$$\begin{cases} C^3_{7t} - 3\beta\left(\mathcal{R}+\omega_5\right)C^3_7 + \omega_7 B^2_{5\mathcal{R}} = 0 \\ C^2_7\left(T,\mathcal{R}\right) = 0 \end{cases} \tag{4.54}$$

$$\begin{cases} C^3_{8t} - 2\beta\left(\mathcal{R}+\omega_4\right)C^2_8 - \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(A^1_{2\mathcal{R}}\right)^2 - \omega_6 B^2_{6\mathcal{R}} = 0 \\ C^3_4\left(T,\mathcal{R}\right) = 0 \end{cases}, \tag{4.55}$$

$$\begin{cases} C^2_{9t} - 2\beta\left(\mathcal{R}+\omega_5\right)C^2_9 - \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\left(A^1_{3\mathcal{R}}\right)^2 - \omega_7 B^2_{7\mathcal{R}} = 0 \\ C^2_5\left(T,\mathcal{R}\right) = 0 \end{cases} \tag{4.56}$$

Solving (4.48) - (4.56), we obtain

$$\left.\begin{array}{c}
C^3_1\left(t,\mathcal{R}\right) = \left[\begin{array}{c} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T\left(A^1_{1\mathcal{R}}\right)^2 d\tau + 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{2\mathcal{R}}d\tau\tau \\ +2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{3\mathcal{R}}d - \mathcal{P}_1\beta\left(2\beta+1\right)\int_t^T C^3_4 d\tau \\ -\mathcal{P}_2\beta\left(2\beta+1\right)\int_t^T C^3_5 d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{1\mathcal{R}}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{1\mathcal{R}\mathcal{R}}d\tau \end{array}\right] \\
C^3_2\left(t,\mathcal{R}\right) = e^{\beta\left(\mathcal{R}+\omega_4\right)\left(t-T\right)}\left[\begin{array}{c} 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{4\mathcal{R}}e^{\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ +\omega_6\int_t^T B^2_{1\mathcal{R}}e^{\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ -\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T C^3_6 e^{\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \end{array}\right] \\
C^3_3\left(t,\mathcal{R}\right) = e^{\beta\left(\mathcal{R}+\omega_5\right)\left(t-T\right)}\left[\begin{array}{c} 2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{5\mathcal{R}}e^{\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ +\omega_7\int_t^T B^2_{1\mathcal{R}}e^{\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ -\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T C^2_7 e^{\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \end{array}\right] \\
C^3_4\left(t,\mathcal{R}\right) = e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(t-T\right)}\left[\begin{array}{c} 3\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{6\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ -\frac{\omega_2}{2\theta}\int_t^T e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{2\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ +\omega_6\int_t^T B^2_{2\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{2\mathcal{R}\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ -2\mathcal{P}_1\left(2\beta^2+\beta\left(2\beta+1\right)\int_t^T C^3_8 e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ +\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{2\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \end{array}\right] \\
C^3_5\left(t,\mathcal{R}\right) = e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(t-T\right)}\left[\begin{array}{c} 3\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{7\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ -\frac{\omega_3}{2\theta}\int_t^T e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{3\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ +\omega_7\int_t^T B^2_{3\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{3\mathcal{R}\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ -2\mathcal{P}_2\left(2\beta^2+\beta\left(2\beta+1\right)\int_t^T C^3_9 e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\ +\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{3\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \end{array}\right] \\
C^3_6\left(t,\mathcal{R}\right) = \omega_6 e^{3\beta\left(\mathcal{R}+\omega_4\right)\left(t-T\right)}\int_t^T B^1_{4\mathcal{R}}e^{3\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\
C^3_7\left(t,\mathcal{R}\right) = \omega_7 e^{3\beta\left(\mathcal{R}+\omega_5\right)\left(t-T\right)}\int_t^T B^1_{5\mathcal{R}}e^{3\beta\left(\mathcal{R}+\omega_5\right)\left(T-\tau\right)}d\tau \\
C^3_8\left(t,\mathcal{R}\right) = e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(t-T\right)}\left[\begin{array}{c} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T\left(A^1_{2\mathcal{R}}\right)^2 e^{\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ +\omega_6\int_t^T B^2_{6\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \end{array}\right] \\
C^3_9\left(t,\mathcal{R}\right) = e^{2\beta\left(\mathcal{R}+\omega_5\right)\left(t-T\right)}\left[\begin{array}{c} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T\left(A^1_{3\mathcal{R}}\right)^2 e^{\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \\ +\omega_7\int_t^T B^2_{7\mathcal{R}}e^{2\beta\left(\mathcal{R}+\omega_4\right)\left(T-\tau\right)}d\tau \end{array}\right]
\end{array}\right\} \tag{4.57}$$

Substituting (4.57) into (4.46), we obtain

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$$
\begin{aligned}
f^3(t,\mathcal{R},y,z) = \Bigg\{ & \begin{bmatrix} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{1\mathcal{R}}\right)^2 d\tau + 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{2\mathcal{R}}d\tau\tau \\ +2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{3\mathcal{R}}d - \mathcal{P}_1\beta\left(2\beta+1\right)\int_t^T C^3_4 d\tau \\ -\mathcal{P}_2\beta\left(2\beta+1\right)\int_t^T C^3_5 d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{1\mathcal{R}}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{1\mathcal{R}\mathcal{R}}d\tau \end{bmatrix} \\
& + y^{\frac{1}{2}} e^{\beta(\mathcal{R}+\omega_4)(t-T)} \begin{bmatrix} 2\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{4\mathcal{R}} e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ +\omega_6\int_t^T B^2_{1\mathcal{R}} e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ -\frac{3}{2}\beta^2\mathcal{P}_1\int_t^T C^3_6 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \end{bmatrix} \\
& + z^{\frac{1}{2}} e^{\beta(\mathcal{R}+\omega_5)(t-T)} \begin{bmatrix} 2\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{5\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ +\omega_7\int_t^T B^2_{1\mathcal{R}} e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ -\frac{3}{2}\beta^2\mathcal{P}_2\int_t^T C^2_7 e^{\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \end{bmatrix} \\
& + y e^{2\beta(\mathcal{R}+\omega_4)(t-T)} \begin{bmatrix} 3\mathcal{P}_4\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{6\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ -\frac{\omega_2}{2\theta}\int_t^T e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ +\omega_6\int_t^T B^2_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{2\mathcal{R}\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ -2\mathcal{P}_1(2\beta^2+\beta\left(2\beta+1\right)\int_t^T C^3_8 e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ +\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{2\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \end{bmatrix} \\
& + z e^{2\beta(\mathcal{R}+\omega_5)(t-T)} \begin{bmatrix} 3\mathcal{P}_5\delta\rho\beta\sqrt{\mathcal{R}}\int_t^T B^2_{7\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ -\frac{\omega_3}{2\theta}\int_t^T e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau - \left(a-b\mathcal{R}\right)\int_t^T A^1_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ +\omega_7\int_t^T B^2_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau - \frac{1}{2}\delta^2\mathcal{R}\int_t^T A^1_{3\mathcal{R}\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ -2\mathcal{P}_2(2\beta^2+\beta\left(2\beta+1\right)\int_t^T C^3_9 e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\ +\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T A^1_{1\mathcal{R}}A^1_{3\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \end{bmatrix} \\
& + y^{\frac{3}{2}}\omega_6 e^{3\beta(\mathcal{R}+\omega_4)(t-T)}\int_t^T B^1_{4\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\
& + z^{\frac{3}{2}}\omega_7 e^{3\beta(\mathcal{R}+\omega_5)(t-T)}\int_t^T B^1_{5\mathcal{R}} e^{3\beta(\mathcal{R}+\omega_5)(T-\tau)}d\tau \\
& + y^2 e^{2\beta(\mathcal{R}+\omega_4)(t-T)} \begin{bmatrix} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{2\mathcal{R}}\right)^2 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ +\omega_6\int_t^T B^2_{6\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \end{bmatrix} \\
& + z^2 e^{2\beta(\mathcal{R}+\omega_5)(t-T)} \begin{bmatrix} \frac{1}{2}\theta\delta^2\mathcal{R}\left(1-\mathcal{P}_{11}\rho^2\right)\int_t^T \left(A^1_{3\mathcal{R}}\right)^2 e^{\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \\ +\omega_7\int_t^T B^2_{7\mathcal{R}} e^{2\beta(\mathcal{R}+\omega_4)(T-\tau)}d\tau \end{bmatrix} \Bigg\}
\end{aligned}
\tag{4.58}
$$

Therefore, from (4.18), we have

$$f_\varepsilon\left(t,\mathcal{R},y,z\right) = f^1\left(t,\mathcal{R},y,z\right) + \sqrt{\varepsilon} f^2\left(t,\mathcal{R},y,z\right) + \varepsilon f^3\left(t,\mathcal{R},y,z\right)$$

where $f^1\left(t,\mathcal{R},y,z\right), f^2\left(t,\mathcal{R},y,z\right)$ and $f^3\left(t,\mathcal{R},y,z\right)$ are given in equation (4.34), (4.45) and (4.58) respectively. □

Hence, lemma 4.1 is proved.

Lemma 4.2. *The optimal value function is given as*

$$\mathcal{N}\left(t,\mathcal{R},s_1,\ s_2,h\right) = -\frac{1}{\theta}e^{-\theta\left(f_\varepsilon(t,\mathcal{R},y,z)+hg(t)\right)} \tag{4.59}$$

where

$$f_\varepsilon\left(t,\mathcal{R},y,z\right) = f^1\left(t,\mathcal{R},y,z\right) + \sqrt{\varepsilon} f^2\left(t,\mathcal{R},y,z\right) + \varepsilon f^3\left(t,\mathcal{R},y,z\right)$$

$$g(t) = Exp\left[\mathcal{R} + \frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2 - \mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3\left(2m_1m_2 - m_1\mathcal{R} - m_2\mathcal{R}\right)}\right]\left(T-t\right)$$

Proof. Substituting equation (4.8) and lemma 4.1 into (4.2), then lemma 4.2 is proved. □

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Lemma 4.3. *The optimal investment plans are given as*

$$\varphi_1^* = \frac{1}{h}\left[\begin{array}{c}\frac{\left[\mathcal{P}_3 s_1^{\beta}(m_2-\mathcal{R})-\mathcal{P}_2 s_2^{\beta}(m_1-\mathcal{R})\right]}{\theta\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{2\beta}s_2^{\beta}}\\ +\frac{2\beta}{s_1^{2\beta}}f_y+\frac{(\mathcal{P}_3\mathcal{P}_5-\mathcal{P}_2\mathcal{P}_4)\sqrt{\mathcal{R}}\,\delta\rho}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{\beta}}f_{\mathcal{R}}\end{array}\right] Exp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3\left(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R}\right)}\right](t-T) \tag{4.60}$$

$$\varphi_2^* = \frac{1}{h}\left[\begin{array}{c}\frac{\left[\mathcal{P}_3 s_2^{\beta}(m_1-\mathcal{R})-\mathcal{P}_1 s_1^{\beta}(m_2-\mathcal{R})\right]}{\theta\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{2\beta}s_2^{\beta}}\\ +\frac{2\beta}{s_2^{2\beta}}f_z+\frac{(\mathcal{P}_3\mathcal{P}_4-\mathcal{P}_1\mathcal{P}_5)\sqrt{\mathcal{R}}\,\delta\rho}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_2^{\beta}}f_{\mathcal{R}}\end{array}\right] Exp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3\left(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R}\right)}\right](t-T) \tag{4.61}$$

where

$$\begin{cases} f_y = A_2^1+\sqrt{\varepsilon}\left(\frac{1}{2}y^{-\frac{1}{2}}B_2^2+B_4^2+\frac{3}{2}y^{\frac{1}{2}}B_6^2\right)+\varepsilon\left(\frac{1}{2}y^{-\frac{1}{2}}C_2^3+C_4^3+\frac{3}{2}y^{\frac{1}{2}}C_6^3+2yC_8^3\right)\\ f_z = A_3^1+\sqrt{\varepsilon}\left(\frac{1}{2}z^{-\frac{1}{2}}B_2^2+B_4^2+\frac{3}{2}z^{\frac{1}{2}}B_6^2\right)+\varepsilon\left(\frac{1}{2}z^{-\frac{1}{2}}C_2^3+C_4^3+\frac{3}{2}z^{\frac{1}{2}}C_6^3+2zC_8^3\right)\\ f_{\mathcal{R}} = \left[\begin{array}{c} A_{1\mathcal{R}}^1+yA_{2\mathcal{R}}^1+zA_{3\mathcal{R}}^1\\ +\sqrt{\varepsilon}\left(B_{1\mathcal{R}}^2+y^{\frac{1}{2}}B_{2\mathcal{R}}^2+z^{\frac{1}{2}}B_{3\mathcal{R}}^2+yB_{4\mathcal{R}}^2+zB_{5\mathcal{R}}^2+y^{\frac{3}{2}}B_{6\mathcal{R}}^2+z^{\frac{3}{2}}B_{7\mathcal{R}}^2\right)\\ +\varepsilon\left(C_{1\mathcal{R}}^3+y^{\frac{1}{2}}C_{2\mathcal{R}}^3+z^{\frac{1}{2}}C_{3\mathcal{R}}^3+yC_{4\mathcal{R}}^3+zC_{5\mathcal{R}}^2+y^{\frac{3}{2}}C_{6\mathcal{R}}^3+z^{\frac{3}{2}}C_{7\mathcal{R}}^3+y^2C_{8\mathcal{R}}^3+z^2C_{9\mathcal{R}}^3,\right)\end{array}\right]\end{cases}$$

$$\begin{cases} \mathcal{P}_1 = n_{11}^2+n_{12}^2,\quad \mathcal{P}_2 = n_{21}^2+n_{22}^2, \mathcal{P}_3 = n_{11}n_{21}+n_{12}n_{22},\\ \mathcal{P}_4 = n_{11}+n_{12}, \mathcal{P}_5 = n_{21}+n_{22},\ \mathcal{P}_6 = \frac{2\mathcal{P}_3(m_1-\mathcal{R})(m_2-\mathcal{R})}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)}\\ \mathcal{P}_7 = \frac{\mathcal{P}_2(m_1-\mathcal{R})^2}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)}, \mathcal{P}_8 = \frac{\mathcal{P}_1(m_2-\mathcal{R})^2}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)}, \mathcal{P}_9 = \frac{\left(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5\right)}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)},\\ \mathcal{P}_{10} = \frac{\left(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4\right)}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)}, \mathcal{P}_{11} = \frac{\left(\mathcal{P}_2\mathcal{P}_4^2+\mathcal{P}_1\mathcal{P}_5^2-2\mathcal{P}_3\mathcal{P}_4\mathcal{P}_5\right)}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)}\end{cases}$$

Proof. Recall from equation (3.7) and (3.8), we have

$$\varphi_1^* = \frac{\left[\mathcal{P}_3 s_1^{\beta}\left(m_2-\mathcal{R}\right)-\mathcal{P}_2 s_2^{\beta}\left(m_1-\mathcal{R}\right)\right]}{h\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{2\beta}s_2^{\beta}}\frac{\mathcal{N}_h}{\mathcal{N}_{hh}} - s_1\frac{\mathcal{N}_{hs_1}}{h\mathcal{N}_{hh}} - \frac{\left(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5\right)\sqrt{\mathcal{R}}\,\delta\rho}{h\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{\beta}}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}}$$

$$\varphi_2^* = \frac{\left[\mathcal{P}_3 s_2^{\beta}\left(m_1-\mathcal{R}\right)-\mathcal{P}_1 s_1^{\beta}\left(m_2-\mathcal{R}\right)\right]}{h\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{\beta}s_2^{2\beta}}\frac{\mathcal{N}_h}{\mathcal{N}_{hh}} - s_2\frac{\mathcal{N}_{hs_2}}{h\mathcal{N}_{hh}} - \frac{\left(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4\right)\sqrt{\mathcal{R}}\,\delta\rho}{h\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_2^{\beta}}\frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}}$$

From equation (4.3), (4.8) and (4.10), we have

$$\begin{cases} \frac{\mathcal{N}_h}{\mathcal{N}_{hh}} = -\frac{1}{\theta g} = -\frac{1}{\theta}Exp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}\right](t-T)\\ \frac{\mathcal{N}_{hs_1}}{\mathcal{N}_{hh}} = \frac{v_{s_1}}{g} = -2\beta s_1^{-2\beta-1}f_yExp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}\right](t-T)\\ \frac{\mathcal{N}_{hs_2}}{\mathcal{N}_{hh}} = \frac{w_{s_2}}{g} = -2\beta s_2^{-2\beta-1}f_zExp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}\right](t-T)\\ \frac{\mathcal{N}_{h\mathcal{R}}}{\mathcal{N}_{hh}} = \frac{(v_{\mathcal{R}}+w_{\mathcal{R}})}{g} = f_{\mathcal{R}}Exp\left[\mathcal{R}+\frac{\mathcal{P}_6(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha}{2\mathcal{P}_3(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R})}\right](t-T)\end{cases} \tag{4.62}$$

where

$$\begin{cases} f_y = A_2^1+\sqrt{\varepsilon}\left(\frac{1}{2}y^{-\frac{1}{2}}B_2^2+B_4^2+\frac{3}{2}y^{\frac{1}{2}}B_6^2\right)+\varepsilon\left(\frac{1}{2}y^{-\frac{1}{2}}C_2^3+C_4^3+\frac{3}{2}y^{\frac{1}{2}}C_6^3+2yC_8^3\right)\\ f_z = A_3^1+\sqrt{\varepsilon}\left(\frac{1}{2}z^{-\frac{1}{2}}B_2^2+B_4^2+\frac{3}{2}z^{\frac{1}{2}}B_6^2\right)+\varepsilon\left(\frac{1}{2}z^{-\frac{1}{2}}C_2^3+C_4^3+\frac{3}{2}z^{\frac{1}{2}}C_6^3+2zC_8^3\right)\\ f_{\mathcal{R}} = \left[\begin{array}{c} A_{1\mathcal{R}}^1+yA_{2\mathcal{R}}^1+zA_{3\mathcal{R}}^1\\ +\sqrt{\varepsilon}\left(B_{1\mathcal{R}}^2+y^{\frac{1}{2}}B_{2\mathcal{R}}^2+z^{\frac{1}{2}}B_{3\mathcal{R}}^2+yB_{4\mathcal{R}}^2+zB_{5\mathcal{R}}^2+y^{\frac{3}{2}}B_{6\mathcal{R}}^2+z^{\frac{3}{2}}B_{7\mathcal{R}}^2\right)\\ +\varepsilon\left(C_{1\mathcal{R}}^3+y^{\frac{1}{2}}C_{2\mathcal{R}}^3+z^{\frac{1}{2}}C_{3\mathcal{R}}^3+yC_{4\mathcal{R}}^3+zC_{5\mathcal{R}}^2+y^{\frac{3}{2}}C_{6\mathcal{R}}^3+z^{\frac{3}{2}}C_{7\mathcal{R}}^3+y^2C_{8\mathcal{R}}^3+z^2C_{9\mathcal{R}}^3,\right)\end{array}\right]\end{cases}$$

Substituting (4.62) into (3.7) and (3.8), this completes the proof. □

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Remark 4.4. *If the risk free interest rate $\mathcal{R}$ is not stochastic, i.e. $\varepsilon = 0$ and $\rho = 0$, then*

$$\varphi_1^* = \frac{1}{h}\left[\begin{array}{c} \frac{\left[\mathcal{P}_3 s_1^{\beta}(m_2-\mathcal{R})-\mathcal{P}_2 s_2^{\beta}(m_1-\mathcal{R})\right]}{\theta\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{2\beta}s_2^{\beta}} + \frac{2\beta}{s_1^{2\beta}}A_2^1 \\ -\frac{(\mathcal{P}_2\mathcal{P}_4-\mathcal{P}_3\mathcal{P}_5)\sqrt{\mathcal{R}}\ \delta\rho}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{\beta}}\left(A_{1\mathcal{R}}^1 + yA_{2\mathcal{R}}^1 + zA_{3\mathcal{R}}^1\right) \end{array}\right] Exp\left[\mathcal{R} + \frac{\mathcal{P}_6\left(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha\right.}{2\mathcal{P}_3\left(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R}\right)}\right](t-T) \tag{4.63}$$

$$\varphi_2^* = \frac{1}{h}\left[\begin{array}{c} \frac{\left[\mathcal{P}_3 s_2^{\beta}(m_1-\mathcal{R})-\mathcal{P}_1 s_1^{\beta}(m_2-\mathcal{R})\right]}{\theta\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_1^{2\beta}s_2^{\beta}} + \frac{2\beta}{s_2^{2\beta}}A_3^1 \\ -\frac{(\mathcal{P}_1\mathcal{P}_5-\mathcal{P}_3\mathcal{P}_4)\sqrt{\mathcal{R}}\ \delta\rho}{\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)s_2^{\beta}}\left(A_{1\mathcal{R}}^1 + yA_{2\mathcal{R}}^1 + zA_{3\mathcal{R}}^1\right) \end{array}\right] Exp\left[\mathcal{R} + \frac{\mathcal{P}_6\left(\left(\mathcal{P}_1\mathcal{P}_2-\mathcal{P}_3^2\right)\alpha\right.}{2\mathcal{P}_3\left(2m_1m_2-m_1\mathcal{R}-m_2\mathcal{R}\right)}\right](t-T) \tag{4.64}$$

where

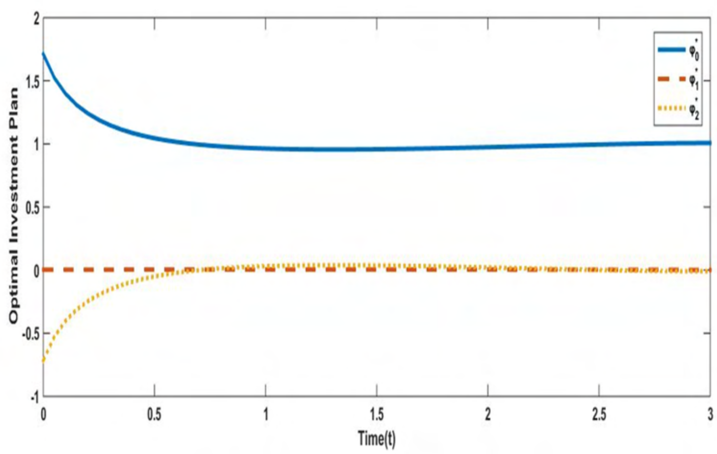
$$A_1^1(t,\mathcal{R}) = \left[\begin{array}{c} \left[\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\theta(\mathcal{R}+\omega_4)} + \frac{\mathcal{P}_2(2\beta+1)\omega_3}{4\theta(\mathcal{R}+\omega_5)}\right](T-t) - \frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] \\ -\frac{\mathcal{P}_1(2\beta+1)\omega_2}{4\beta\theta(\mathcal{R}+\omega_4)^2}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right] \end{array}\right]$$

$$A_2^1(t,\mathcal{R}) = \frac{\omega_2}{4\beta\theta\left(\mathcal{R}+\omega_4\right)}\left[1-e^{2\beta(\mathcal{R}+\omega_4)(t-T)}\right]$$

$$A_3^1(t,\mathcal{R}) = \frac{\omega_3}{4\beta\theta\left(\mathcal{R}+\omega_4\right)}\left[1-e^{2\beta(\mathcal{R}+\omega_5)(t-T)}\right]$$

5 Sensitivity Analysis

Here we present some numerical simulations to study the effects of some parameters on the optimal investment plan under logarithm utility. To achieve this, the following data will be used unless otherwise stated $n_{11} = 1$, $n_{12} = 0.9$, $n_{21} = 0.85$, $n_{11} = 0.8$, $\beta = -1, m_1 = 0.4$, $m_2 = 0.3$, $h = 1$, $\alpha = 1$, $\rho = -0.5$, $\mathcal{R}(0) =$ 0.05, $\mathcal{S}_1(0) = 1.5$, $\mathcal{S}_2(0) = 1.2$, $T = 3$



Evolution of the optimal investment plan φ_0^*, φ_1^*, and φ_2^*

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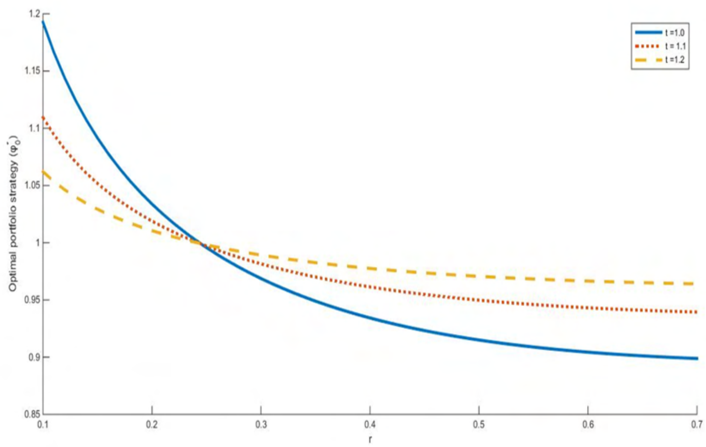
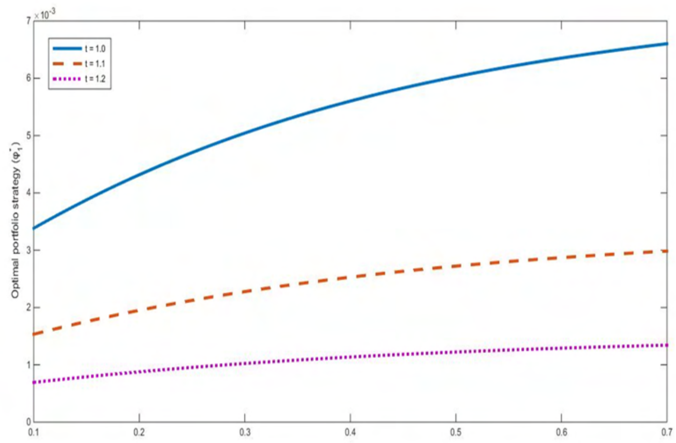
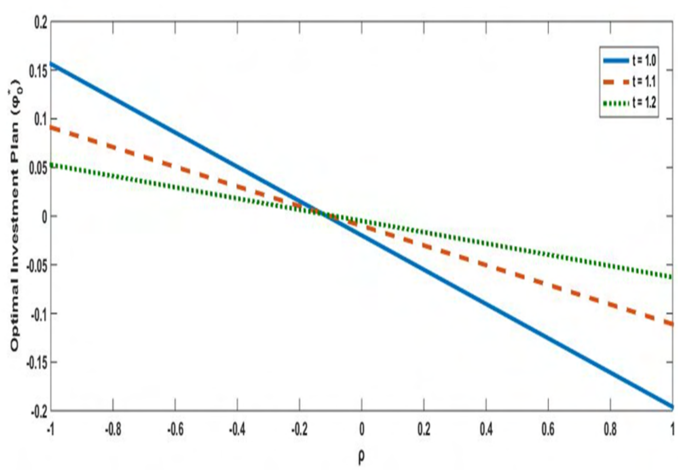


The impact of the risk free interest r on φ_1^*



The impact of the risk free interest r on φ_0^*



The impact of the correlation coefficient ρ on φ_0^*

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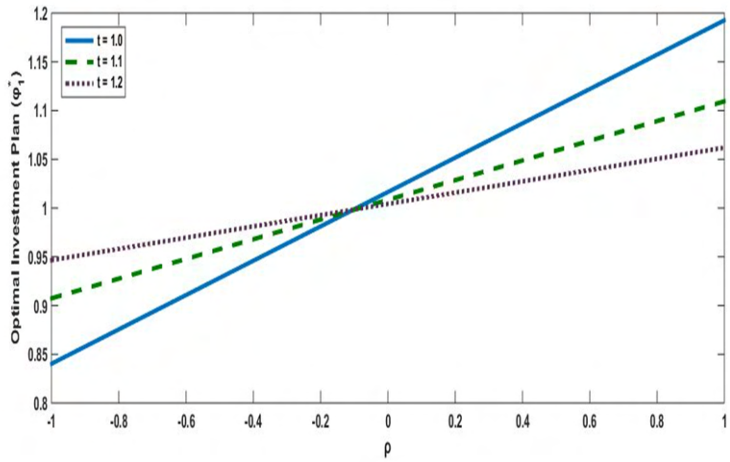
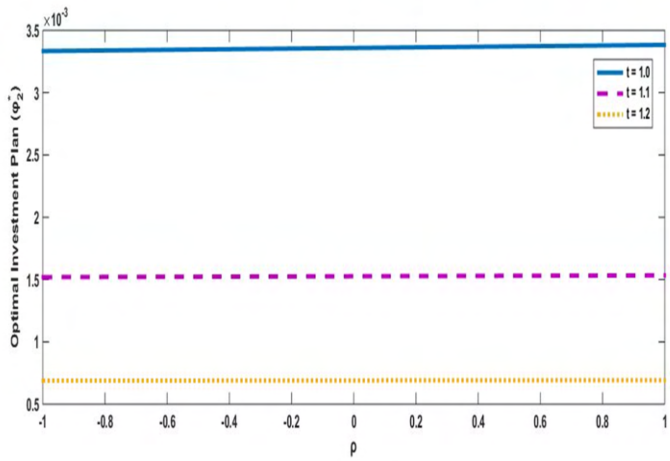
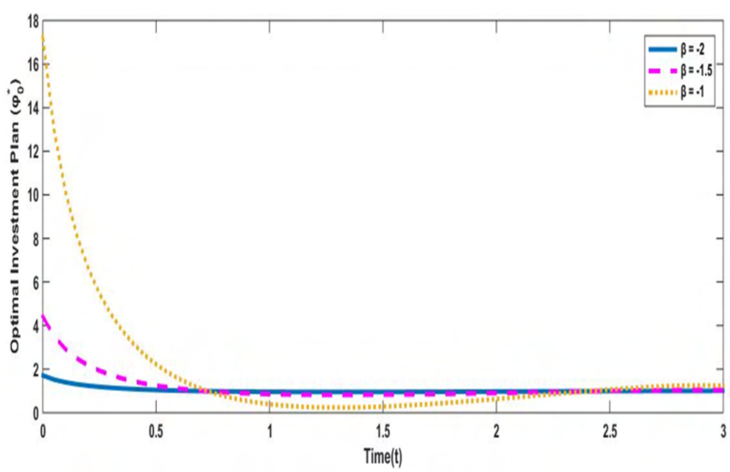


The impact of the correlation coefficient ρ on φ_1^*



The impact of the correlation coefficient ρ on φ_2^*



Evolution of φ_0^* with different elasticity parameter β

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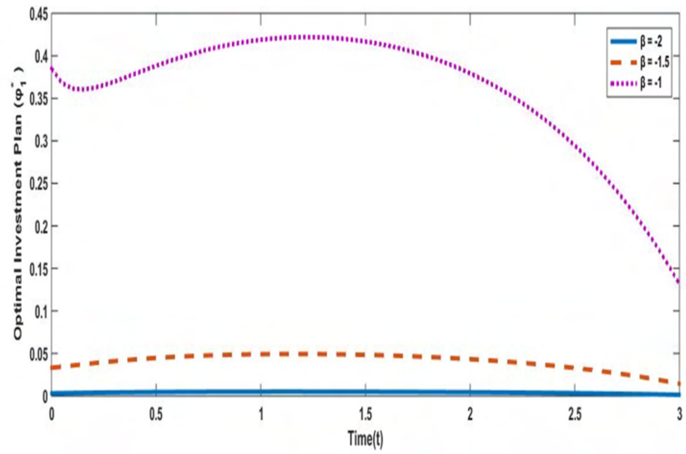
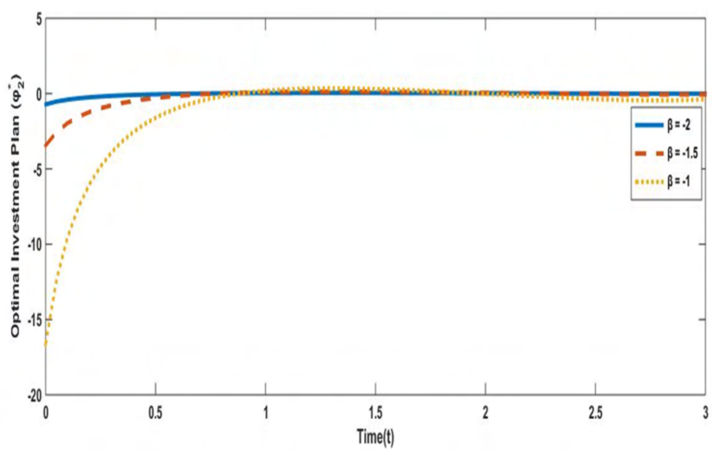


Evolution of φ_1^* with different elasticity parameter β



Evolution of φ_2^* with different elasticity parameter β

6 Discussion

The impact of sensitive parameters on the optimal investment plan is analysed. In Figure 5, the simulation of optimal investment plan of the three assets is given against time; the graph shows that at the initial time, the investor will invest more in the risk free asset and less in the other two risky assets and as expiration date draws closer the investor will begin to invest more in risky and reduce investment in risk free asset. In figure 5 and 5, the investment plan for the risk free asset increases with interest rate increases while that of the risky asset decreases with interest rate.This is because the investors prefer investment in risk free asset than risky asset whenever the interest rate appreciates since it is risk-less. Figure 5, 5, and 5 give the analysis of the impact of the correlation coefficient ρ on the optimal investment plans φ_0^*, φ_1^* and φ_2^*, it is observed that as ρ increases φ_0^* decreases, φ_1^* increases significantly while φ_2^* experience little or no increase. Also, we observed also that as the investment time draws closer to it expiring date, the insurer will invest less in the risky assets and more in the risk free asset. Figure 5, shows that as β reduces, φ_0^* increases which implies that investors with high θ may invest more in risk free asset to prevent more loss especially when the market is very volatile. On the contrary, figures 5 and 5, show that as β reduces, the optimal investment plan for the two risky assets decreases which implies that investors with high θ may be more scared to invest in the risky assets when the market is highly volatile. Furthermore,

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at the initial stage of investment we observed a huge disparity between the investment plan when $\beta = -1$ when compared to $\beta = -1.5$ and -2. This confirm that the choice of $\beta = -1$ could be most suitable in choosing an investment plan.
From equation (4.60) and (4.61), the optimal investment plan decreases with an increase in the initial wealth. Also, from remark 4.4, the optimal investment plan for the two risky assets reduce to the result in [27] when $\mathcal{R}$ is not stochastic.

7 Conclusion

This paper investigated explicit solutions of the optimal investment plans of an investor with exponential utility function exhibiting CARA under CEV model and stochastic interest rate. We considered a portfolio with risk-free asset modelled by Cox- Ingersoll-Ross (CIR) process and two risky assets modelled by the CEV process.The power transformation and change of variable approach with asymptotic expansion technique was used to determine explicit solutions of the optimal investment plans. Furthermore, we present some numerical simulations to study the effect of the interest rate, elasticity parameter, correlation coefficient and the risk averse coefficient on the optimal investment plans.Finally, when the interest rate is constant, our result is similar to the result in [27].

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