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Fixed point results in uniform spaces via simulation functions

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Abstract

In this paper, by using α-admissible mappings embedded in simulation functions, some fixed point results are proved in the setting of a Hausdorff S-complete uniform space. The results obtained generalizes and unifies some known results in the literature.

Keywords: Fixed point, admissible mappings, simulation functions, uniform spaces.
MSC2010: 47H10, 54H25.

1 Introduction

Many generalization of metric spaces abound in literature. The concept of uniform spaces was introduced by Weil [1] while Bourbaki [2] provided the definition of a uniform structure in terms of entourages. Aamri and El Moutawakil [3] provided the definition of A-distance and E-distance and proved some results on common fixed point for some contractive and expansive maps in uniform spaces. Olisama *et al.* [4] introduced the concept of J_{AV}-distance (an analogue of b-metric), ϕ_p-proximal contraction, and ϕ_p-proximal cyclic contraction for non-self-mappings in Hausdorff uniform spaces and proved best proximity point results for these contractive mappings. Recently, Umudu *et al.* [5] generalized the results of Olisama *et al.* [4] by introducing Geraghty p-proximal cyclic quasi-contraction and investigated the existence and uniqueness of best proximity point for the contractions in uniform spaces.
As a generalization of the well known Banach contraction mapping, Khojasteh *et al.* [6] introduced the notion of $\mathcal{Z}$-contraction which is defined by means of a family of functions called simulation functions and proved the existence and uniqueness of fixed point for the class of $\mathcal{Z}$-contraction mappings. Several results have been proved in this direction, see ([6–8]).

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2 Preliminaries

The following definitions are fundamental to our work.

Definition 2.1 [2]. A uniform space (X, Γ) is a nonempty set X equipped with a uniform structure which is a family Γ of subsets of Cartesian product $X \times X$ which satisfy the following conditions:

(i) If $U \in \Gamma$, then U contains the diagonal $\Delta = \{(x, x) : x \in X\}$.

(ii) If $U \in \Gamma$, then $U^{-1} = \{(y, x) : (x, y) \in U\}$ is also in Γ.

(iii) If $U, V \in \Gamma$, then $U \cap V \in \Gamma$.

(iv) If $U \in \Gamma$ and $V \subseteq X \times X$, which contains U, then $V \in \Gamma$.

(v) If $U \in \Gamma$, then there exists $V \in \Gamma$ such that whenever (x, y) and (y, z) are in V, then (x, z) is in U.

Γ is called the uniform structure or uniformity of U and its elements are called entourages.

Definition 2.2 [9]. Let (X, Γ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an

(a) A - distance if, for any $V \in \Gamma$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$;

(b) E - distance if p is an A - distance and $p(x, z) \leq p(x, y) + p(y, z), \forall x, y, z \in X$.

Definition 2.3 [4]. Let (X, Γ) be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be a

(c) J_{AV}-distance if p is an A - distance and $p(x, z) \leq s[p(x, y) + p(y, z)], \forall x, y, z \in X$, $s \geq 1$.

Note that the function p reduces to an E-distance if the constant s is taken as 1.

Example in [4] shows that a uniform space equipped with J_{AV} distance function is a generalisation of a uniform space equipped with an E-distance function.

Definition 2.4 [9]. Let (X, Γ) be a uniform space and p an A-distance on X.

(a) If $V \in \Gamma, (x, y) \in V$, and $(y, x) \in V$, x and y are said to be V-close. A sequence (x_n) is a Cauchy sequence for Γ if, for any $V \in \Gamma$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$. The sequence $(x_n) \in X$ is a p-Cauchy sequence if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

(b) X is said to be S-complete if for any p-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$.

(c) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(d) X is said to be p-bounded if $\delta_p(X) = sup\{p(x, y) : x, y \in X\} < \infty$.

To guarantee the uniqueness of the limit of the Cauchy sequence for Γ, the uniform space (X, Γ) needs to be Hausdorff.

Definition 2.5 [2]. A uniform space (X, Γ) is said to be Hausdorff if and only if the intersection of all the $V \in \Gamma$ reduces to the diagonal Δ of X, $\Delta = \{(x, x), x \in X\}$. In other words, $(x, y) \in V$ for all $V \in \Gamma$ implies $x = y$.

The concept of α-admissible mappings have been used in many works. Popescu [10] defined the

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concept of triangular α-orbital admissible mapping as an improvement of triangular α-admissible mapping ([11, 12]).

Definition 2.6 [10]. Let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}^+$ be a function.

(a) T is called α-orbital admissible if $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.

(b) T is called triangular α-orbital admissible if T is α-orbital admissible and $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.

The class of simulation function was introduced by Khojasteh *et al.* [6] as follows.

Definition 2.7 [6]. Let $\varsigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping, then ς is called a simulation function if it satisfies the following conditions:

(ς_1) $\varsigma(0, 0) = 0$;

(ς_2) $\varsigma(t, s) < s - t$ for all $s > 0$.

(ς_3) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then $\limsup_{n\to\infty} \varsigma(t_n, s_n) < 0$.

The set of all simulation functions are denoted by $\mathcal{Z}$.

Definition 2.8 [6] Let (X, d) be a metric space, $T : X \to X$ be a mapping and $\varsigma \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ς if the following is satisfied:

$$\varsigma(d(Tx, Ty), d(x, y)) \geq 0 \quad for \ \ all \ \ x, y \in X.$$

Examples of the simulation function and $\mathcal{Z}$-contraction are also provided in [6].

In this paper, we consider some fixed point results in uniform spaces for the class of $\mathcal{Z}$-contraction via admissible mappings embedded in simulation function as a generalization of some fixed point results obtained in a metric space.

3 Main Results

We begin with the following definitions.

Definition 3.1. Let (X, Γ) be a uniform space such that p is an E-distance. Let $T : X \to X$ be a self mapping, $\alpha : X \times X \to \mathbb{R}^+$ and $\varsigma \in \mathcal{Z}$. Then T is called an α-$\mathcal{Z}$-contraction with respect to ς if

$$\varsigma\left(\alpha(x, y)p(Tx, Ty), p(x, y)\right) \geq 0 \quad for \ \ all \ \ x, y \in X. \tag{3.1}$$

Remark 3.2.

1. Suppose the uniform space is reduced to a metric space i.e $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ then the self mapping T is a α-$\mathcal{Z}$ contraction with respect to ς [8]
2. Suppose the uniform space is reduced to a metric space i.e $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ and $\alpha(x, y) = 1$, then the self mapping T is a $\mathcal{Z}$ contraction with respect to ς [6].

Definition 3.3. Let (X, Γ) be a uniform space such that p is an E-distance. Let $T : X \to X$ be a self mapping, $\alpha : X \times X \to \mathbb{R}^+$ and $\varsigma \in \mathcal{Z}$. Then T is called a generalized α-$\mathcal{Z}$-contraction with respect to ς if for all $x, y \in X$.

$$\varsigma(\alpha(x, y)p(Tx, Ty), M(x, y)) \geq 0 \tag{3.2}$$

where $M(x,y) = \max\left\{p(x,y), p(x,Tx), p(y,Ty), \dfrac{p(x,Ty)+p(y,Tx)}{2}\right\}$.

The following is the first main result.

Theorem 3.4. Let X be a S-complete Hausdorff uniform space such that p is an E-distance, $T : X \to X$ a generalized α-$\mathcal{Z}$-contraction with respect to ς and the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping.

(ii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

(iii) T is continuous.

Then T has a fixed point $x^* \in X$.

Proof: By hypothesis (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define a sequence $\{x_n\} \in X$ by letting $x_{n+1} = Tx_n$ for all $n \geq 0$. If $x_n = x_{n+1}$, then T has a fixed point. Consequently, henceforth, we shall assume that $x_n \neq x_{n+1}$ for all n. And so $p(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$. Since T is α-orbital admissible, then

$$\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \Rightarrow \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1.$$

Recursively, we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for} \quad \text{all} \quad n \in \mathbb{N} \cup 0. \tag{3.3}$$

Using (3.2) and (3.3), for all $n \in \mathbb{N}$

$$\begin{aligned} 0 &\leq \varsigma\left(\alpha(x_n, x_{n-1})p(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})\right) \\ &= \varsigma\left(\alpha(x_n, x_{n-1})p(x_{n+1}, x_n), M(x_n, x_{n-1})\right) \\ &< M(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})p(x_{n+1}, x_n) \end{aligned}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\left\{p(x_n, x_{n-1}), p(x_n, Tx_n), p(x_{n-1}, Tx_{n-1}), \frac{p(x_n, Tx_{n-1}) + p(x_{n-1}, Tx_n)}{2}\right\} \\ &= \max\left\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_{n+1})}{2}\right\} \\ &\leq \max\left\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \frac{p(x_{n-1}, x_n) + p(x_n, x_{n+1})}{2}\right\} \\ &= \max\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\}. \end{aligned}$$

If $M(x_n, x_{n-1}) = p(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, then

$$0 \leq \varsigma\left(\alpha(x_n, x_{n-1})p(x_{n+1}, x_n), p(x_n, x_{n+1})\right)$$

$$< p(x_n, x_{n+1}) - \alpha(x_n, x_{n-1})p(x_n, x_{n+1}) \leq 0$$

which is a contradiction. Therefore, $M(x_n, x_{n-1}) = p(x_n, x_{n-1})$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} 0 &\leq \varsigma\left(\alpha(x_n, x_{n-1})p(x_{n+1}, x_n), p(x_n, x_{n-1})\right) \\ &< p(x_n, x_{n-1}) - \alpha(x_n, x_{n-1})p(x_n, x_{n+1}). \end{aligned} \tag{3.4}$$

Consequently,

$$p(x_n, x_{n+1}) \leq \alpha(x_n, x_{n-1})p(x_n, x_{n+1}) < p(x_n, x_{n-1}) \tag{3.5}$$

for all $n \in \mathbb{N}$. Thus, the sequence $\{p(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers and so, there exists a non negative number r such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{3.6}$$

To prove that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, suppose on the contrary that $r > 0$. Now considering equations (3.4), (3.5) and condition (ς_3), we have

$$0 \leq \limsup_{n\to\infty} \varsigma\left(\alpha(x_n, x_{n-1})p(x_{n+1}, x_n), M(x_n, x_{n-1})\right) < 0$$

which is a contradiction. Therefore, $r = 0$. To show that the sequence $\{x_n\}$ is p-Cauchy. Assume for contradiction that $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ such that, for all $k > 0$, we can find $n, m \in \mathbb{N}$ with $m(k) > n(k) > k$ with $p(x_{n(k)}, x_{m(k)}) \geq \epsilon$. Let $m(k)$ be the smallest number satisfying the condition above. Thus, $p(x_{n(k)}, x_{m(k)-1}) < \epsilon$.
Therefore, using triangle inequality, we have

$$\begin{aligned} \epsilon &\leq p(x_{n(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + p(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, we have

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \epsilon. \tag{3.7}$$

Since $|p(x_{n(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{m(k)})| \leq p(x_{m(k)}, x_{m(k)-1})$, we have

$$\lim_{k\to\infty} p(x_{n(k)}, x_{m(k)-1}) = \epsilon. \tag{3.8}$$

Likewise,

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \lim_{k\to\infty} p(x_{m(k)}, x_{n(k)-1}) = \epsilon. \tag{3.9}$$

By condition (i), T is triangular orbital admissible and we have

$$\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1, \quad for \ \ all \ \ k \geq 1. \tag{3.10}$$

T is also a generalized α-$\mathcal{Z}$-contraction with respect to ς and using (3.10) gives

$$\begin{aligned} 0 &\leq \varsigma(\alpha(x_{n(k)-1}, x_{m(k)-1})p(Tx_{n(k-1)}, Tx_{m(k)-1}), M(x_{n(k)-1}, x_{m(k)-1})) \\ &= \varsigma(\alpha(x_{n(k)-1}, x_{m(k)-1})p(x_{n(k)}, x_{m(k)}), M(x_{n(k)-1}, x_{m(k)-1})) \\ &< M(x_{n(k)-1}, x_{m(k)-1}) - \alpha(x_{n(k)-1}, x_{m(k)-1})p(x_{n(k)}, x_{m(k)}) \end{aligned}$$

where

$$\begin{aligned} M(x_{m(k)-1}, x_{n(k)-1}) = \max\{&p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, Tx_{m(k)-1}), p(x_{n(k)-1}, Tx_{n(k)-1}), \\ &\frac{p(x_{m(k)-1}, Tx_{n(k)-1}) + p(x_{n(k)-1}, Tx_{m(k)-1})}{2}\} \end{aligned}$$

Using (3.2),(3.7),(3.8) and (3.9)

$$\lim_{k\to\infty} M(x_{n(k)-1}, x_{m(k)-1}) = \lim_{k\to\infty} p(x_{n(k)}, x_{m(k)}) = \epsilon$$

Clearly, we deduce that

$$0 < p(x_{n(k)}, x_{m(k)}) < \alpha(x_{n(k)-1}, x_{m(k)-1})p(x_{n(k)}, x_{m(k)}) < M(x_{n(k)-1}, x_{m(k)-1}),$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 7, No. 2, pp. 56 - 64
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and considering (ς_3),

$$0 \leq \limsup_{k\to\infty} \varsigma\left(\alpha(x_{n(k)-1}, x_{m(k)-1})p(x_{n(k)}, x_{m(k)}), M(x_{n(k)-1}, x_{m(k)-1})\right) < 0$$

is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. Since the uniform space, (X, Γ), is complete, there exists $w \in X$ such that

$$\lim_{n\to\infty} p(x_n, w) = 0 \tag{3.11}$$

Using (3.11) and the hypothesis that T is continuous, we have

$$\lim_{n\to\infty} p(Tw, x_{n+1}) = p(Tw, Tx_n) = 0. \tag{3.12}$$

By the uniqueness of the limit in a Hausdorff uniform space and using (3.12) we obtain that the fixed point of T is w.

The continuity of T can be replaced by another condition.

Theorem 3.5. Let (X, Γ) be a S-complete Hausdorff uniform space such that p is an E-distance and let $T : X \to X$ be a generalized α-$\mathcal{Z}$-contraction with respect to ς. Suppose the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a unique fixed point $x^* \in X$.

Proof. Following the lines in the proof of Theorem 3.4, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 1$ converges to $w \in X$. To show that w is a fixed point of X, suppose that $x_n \neq w$ for all positive integer n and $p(Tw, w) > 0$. By condition (iii), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, w) \geq 1$ for all $k \in \mathbb{N}$. By (3.2), we have

$$\varsigma(\alpha(x_{n_k}, w)p(Tx_{n(k)}, Tw), M(x_{n_k}, w)) = \varsigma(\alpha(x_{n_k}, w)p(x_{n(k)+1}, Tw), M(x_{n_k}, w)) \geq 0$$

where $M(x_{n(k)}, w) = \max\left\{p(x_{n(k)}, w), p(x_{n(k)}, x_{n(k)+1}), p(w, Tw), \frac{p(x_{n(k)}, Tw) + p(w, Tx_{n(k)})}{2}\right\}$.
By (ς_2),

$$\begin{aligned} 0 &\leq \varsigma(\alpha(x_{n(k)}, w)p(x_{n(k)+1}, Tw), M(x_{n(k)}, w)) \\ &\leq M(x_{n(k)}, w) - \alpha(x_{n(k)}, w)p(x_{n(k)+1}, Tw) \end{aligned}$$

This implies $p(x_{n(k)+1}, Tw) < M(x_{n(k)}, w)$.

Taking limits as k tends to infinity,

$$\lim_{k\to\infty} M(x_{n(k)}, w) = p(w, Tw)$$

and

$$\lim_{k\to\infty} p(x_{n(k)+1}, Tw) = p(w, Tw).$$

Therefore, using (ς_3) we obtain

$$0 \leq \limsup_{k\to\infty} \varsigma(\alpha(x_{n_k}, w)p(x_{n(k)+1}, Tw), M(x_{n(k)}, w)) < 0,$$

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VOL. 7, NO. 2, PP. 56 - 64
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which is a contradiction. Therefore, $p(Tw, w) = 0$ and w is fixed point of X.

To prove uniqueness of a fixed point result, consider the hypothesis.

(J) : For any two fixed points, $x, y \in Fix(T)$, then $\alpha(x, y) = 1$, where $Fix(T)$ denotes the set of fixed points of T.

Theorem 3.6. Adding condition (J) to the hypothesis of Theorem 3.4 (resp. Theorem 3.5), we obtain that x^* is the unique fixed point of T.

Proof. We assume by contradiction that there exists $w_1, w_2 \in X$ such that $w_1 = Tw_1$ and $w_2 = Tw_2$ where $w_1 \neq w_2$. Then by hypothesis (J), $\alpha(w_1, w_2) = 1$. Using (3.2) and (ς_2), we have

$$\begin{aligned} 0 &\leq \varsigma(\alpha(w_1, w_2)p(Tw_1, Tw_2), M(w_1, w_2)) \\ &= \varsigma\left(\alpha(w_1, w_2)p(w_1, w_2), \max\left\{p(w_1, w_2), p(w_1, Tw_1)p(w_2, Tw_2)\frac{p(w_1, Tw_2) + p(w_2, Tw_1)}{2}\right\}\right) \\ &= \varsigma(\alpha(w_1, w_2)p(w_1, w_2), p(w_1, w_2)) \\ &< p(w_1, w_2) - \alpha(w_1, w_2)p(w_1, w_2) = 0 \end{aligned}$$

which is a contradiction. Hence, $w_1 = w_2$.

Corollary 3.7. Let (X, Γ) be a S-complete Hausdorff uniform space such that p is an E-distance and let $T : X \to X$ be an α-$\mathcal{Z}$-contraction with respect to ς. Suppose the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all $k \in \mathbb{N}$.

Then T has a fixed point $x^* \in X$.

Proof. The proof follows from Theorem 3.4 and Theorem 3.5. if $M(x, y) = p(x, y)$

We give an example to illustrate Theorem 3.4.

Example 3.8. Let $X = [0, \infty)$ equipped with the usual metric and p be a E-distance defined by

$$p(x, y) = \begin{cases} x, & \text{if} \quad x \neq y, \\ 0, & \text{if} \quad x = y. \end{cases}$$

Then p is a E distance and X is a complete uniform space. Let a mapping $T : X \to X$ be defined by $T(x) = \frac{1}{3}x$ for all $x \in X$, $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if} \quad [0, 2], \\ 0, & \text{if} \quad otherwise. \end{cases}$$

and $\varsigma(t, s) = \dfrac{s}{s+1} - t$ for all $t, s \in [0, \infty)$.

Then for all $x, y \in X$, Condition (iii) of Theorem 3.4 is satisfied with $x_1 = 1$. Condition (iv) of Theorem 3.4 is satisfied with $x_n = T^n x_1 = \frac{1}{3^n}$. Obviously, condition (ii) is satisfied. Let x, y be such that $\alpha(x, y) \geq 1$. Then, $x, y \in [0, 1]$, and so $Tx, Ty \in [0, 1]$. Moreover, $\alpha(y, Ty) = \alpha(x, Tx) = 1$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 7, No. 2, pp. 56 - 64
https://doi.org/10.52968/28308162

and $\alpha(Tx, T^2x) = 1$. Thus, T is triangular α-orbital admissible and hence (ii) is satisfied. Finally, we shall prove that (i) is satisfied. If $0 \leq x, y \leq 1$, then $\alpha(x, y) = 1$, and we have

$$\begin{aligned} \varsigma(p(Tx,Ty),p(x,y)) &= \frac{p(x,y)}{1+p(x,y)} - p(Tx,Ty) \\ &= \frac{x}{1+x} - \left(\frac{x}{3} - \frac{y}{3}\right) \\ &= \frac{x}{1+x} - \left(\frac{x-y}{3}\right) \geq 0. \end{aligned}$$

All conditions of Theorems 3.4 are satisfied, and hence T has a unique fixed point $x^* = 0$.

Set $\Gamma = \{(x, y) \in X^2 : d(x, y) < \epsilon\}$ in Corollary 3.7, then the following result in the literature is obtained.

Corollary 3.9 [8]. Let (X, Γ) be a complete metric space and let $T : X \to X$ be an α-$\mathcal{Z}$-contraction with respect to ς. Suppose the following conditions are satisfied:

(i) T is a triangular α-orbital admissible mapping;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(iii) T is continuous.

Then there exists an element $x^* \in X$ such that $x^* = Tx^*$.

Competing Financial Interests

The authors declare no competing financial interests.

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