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The Computational Solution of First Order Delay Differential Equations Using Second Derivative Block Backward Differentiation Formulae

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Abstract

In this paper, we implemented second derivative block backward differentiation formulae methods in solving first order delay differential equations without the application of interpolation methods in investigating the delay argument. The delay argument was evaluated using a suitable idea of sequence which we incorporated into some first order delay differential equations before its numerical evaluations. The construction of the continuous expressions of these of block methods was executed through the use of second derivative backward differentiation formulae method on the bases of linear multistep collocation approach using matrix inversion method to derive the discrete schemes. After the numerical experiments, the new proposed method was observed to be convergent, stable and less time consuming. From the numerical solutions obtained, the scheme for step number $k = 4$ performed better in terms of accuracy than that of the schemes for step numbers $k = 3$ and 2 when compared with other existing methods.

Keywords: First order delay differential equations, Second derivative backward differentiation formulae, Block method.
MSC2010: 34K28; 34A08; 74H15.

1 Introduction

Research has revealed that most real life situations are more realistic when they are modeled using delay differential equations (DDEs).This is because the unknown function of the delay differential equations (DDEs) does not only depends on the current value but also depends on the past value which is called a delay term. Delay differential equation (DDE) is one of the mathematical models that commonly possess the result in differential equations with time delay. In literature, various

types of numerical methods have been developed and implemented in treating the problems of the delay differential equations (DDEs). Most scholars adopted the use of interpolation techniques in the evaluation of the delay term of the delay differential equations in different field of life. These interpolation techniques such as Hermite, Nordsieck, Newton divided difference and Neville's interpolation were applied by [1–15] in solving delay differential equations numerically have some limitations which affected the accuracy of their method. One of the limitations encountered by these researchers in the use of interpolation techniques to evaluate the delay term of DDEs was studied by [16] that the computational method use in solving DDEs should be at least the same with the order of the interpolating polynomials which is very hard to achieve; otherwise, the accuracy of the method will not be preserved. Therefore, it is required that in the evaluation of the delay term, using an accurate and efficient formula should be considered.
In order to overcome the limitation posed by using interpolation techniques in evaluating the delay term, we applied the valid expression of the sequence formulated by [17] and incorporate it into the first order delay differential equations before its numerical evaluations. This approach has been successfully applied by [18–21] in finding the numerical solution of first order delay differential equations without the application of the interpolation techniques in evaluating the delay term.
In this paper, we formulated and applied second derivative block backward differentiation formulae method in solving some first order delay differential equations (DDEs) of this form
$y'(t) = f(t, y(t), y(t-\tau))$, for $t > t_0, \tau > 0$

$$y(t) = e(t), \quad \text{for} \quad t \leq t_0 \tag{1.1}$$

where $e(t)$ is the initial function, τ is called the delay, $(t-\tau)$ is called the delay argument and $y(t-\tau)$ is the solution of the delay term. The results obtained after the application of the proposed method shall be compared to other existing methods studied by [17, 21] to prove its advantage.

2 Construction Techniques

2.1 Construction of Second Derivative Backward Differentiation Formulae Method

In [22] the $k-$step Backward Differentiation Formulae Methods was derived as

$$\sum_{b=0}^{z} \alpha_b(x)\, y_{a+b} = w\, \beta_b(x) f(x_b, y(x_b)) \tag{2.1}$$

And its second derivative [22] was expressed as

$$\sum_{b=0}^{z} \alpha_b(x)\, y_{a+b} = w\, \beta_b(x) f(x_b, y(x_b) + w^2\, \gamma_b(x) g(x_b, y(x_b) \tag{2.2}$$

where $\alpha_b(x)$, $\beta_b(x)$ and $\gamma_b(x)$ are continuous coefficients of the method defined as

$$\alpha_b(x) = \sum_{e=0}^{u+v-1} \alpha_{b,e+1} x^e \quad \text{for} \quad b = \{0, 1, ..., u-1\} \tag{2.3}$$

$$w\beta_b(x) = \sum_{e=0}^{u+v-1} w\beta_{b,e+1} x^e \quad \text{for} \quad b = \{0, 1, ..., v-1\} \tag{2.4}$$

$$w^2\, \gamma_b(x) = \sum_{e=1}^{z} w^2\, \gamma_{b,e+1}\, x^e \quad \text{for} \quad b = \{0, 1, ..., v-1\} \tag{2.5}$$

where $x_0, \ldots, x_{v-1}$ are the v collocation points, x_{a+b} , $b = 0, 1, 2, ..., u-1$ are the u arbitrarily chosen interpolation points and w is the constant step size.
To get $\alpha_b(x)$, $\beta_b(x)$ and $\gamma_b(x)$, [23] developed a matrix equation of the form

$$RQ = \mathrm{I} \tag{2.6}$$

where I represents the unit matrix of dimension $(u+v)\times(u+v)$ and R and Q are matrices presented as

$$R = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & . & \alpha_{u-1,1} & w\,\beta_{0,1} & . & w\,\beta_{v-1,1} & w^2\,\gamma_{0,1} & . & w^2\,\gamma_{v-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & . & \alpha_{u-1,2} & w\,\beta_{0,2} & . & w\,\beta_{v-1,2} & w^2\,\gamma_{0,2} & . & w^2\,\gamma_{v-1,2} \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ \alpha_{0,b+v} & \alpha_{1,b+v} & . & \alpha_{u-1,b+v} & w\,\beta_{0,b+v} & . & w\,\beta_{v-1,b+v} & w^2\,\gamma_{0,b+v} & . & w^2\,\gamma_{v-1,b+v} \end{pmatrix} \tag{2.7}$$

$$Q = \begin{pmatrix} 1 & x_a & x_a^2 & x_a^3 & x_a^4 & . & . & x_a^{u+v-1} \\ 1 & x_{a+1} & x_{a+1}^2 & x_{a+1}^3 & x_{a+1}^4 & . & . & x_{a+1}^{u+v-1} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 1 & x_{a+u-1} & x_{a+u-1}^2 & x_{a+u-1}^3 & x_{a+u-1}^4 & . & . & x_{a+u-1}^{u+v-1} \\ 0 & 1 & 2\,x_a & 3x_a^2 & 4\,x_a^3 & . & . & (u+v-1)x_a^{u+v-2} \\ 0 & 1 & 2\,x_{a+1} & 3x_{a+1}^2 & 4x_{a+1}^3 & . & . & (u+v-1)x_{a+1}^{u+v-2} \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 2\,x_{a+u-1} & 3x_{a+u-1}^2 & 4x_{a+u-1}^3 & . & . & (u+v-1)x_{a+u-1}^{u+v-2} \\ 0 & 0 & 2 & 6\,x_a & 12x_a^2 & . & . & (u+v-1)(u+v-2)x_a^{u+v-3} \\ 0 & 0 & 2 & 6\,x_{a+1} & 12x_{a+1}^2 & . & . & (u+v-1)(u+v-2)x_{a+1}^{u+v-3} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 2 & 6\,x_{a+u-1} & 12x_{a+u-1}^2 & . & . & (u+v-1)(u+v-2)x_{a+u-1}^{u+v-3} \end{pmatrix} \tag{2.8}$$

From (2.6) the columns of $R = Q^{-1}$ give the continuous coefficients of the continuous scheme (2.2).

2.2 Construction of Second Derivative Block Backward Differentiation Formulae Method for k=2

Here, the number of interpolation points, $u = 2$ and the number of collocation points $v = 2$. Therefore, (2.2) becomes

$$y(x) = \alpha_0(x)\,y_a + \alpha_1(x)\,y_{a+1} + \alpha_2(x)\,y_{a+2} + \alpha_3(x)\,y_{a+3} + w\beta_3(x)\,f_{a+3} + w^2\,\gamma_3(x)\,g_{a+3} \tag{2.9}$$

The matrix Q in (2.6) becomes

$$Q = \begin{pmatrix} 1 & x_a & x_a^2 & x_a^3 \\ 1 & x_a + w & (x_a + w)^2 & (x_a + w)^3 \\ 0 & 1 & 2x_{a+4w} & 3\,(x_a + 2w)^2 \\ 0 & 0 & 2 & 6x_a + 12w \end{pmatrix} \tag{2.10}$$

The inverse of the matrix $R = Q^{-1}$ is computed using Maple 18 to obtain the continuous scheme is obtained using (2.6) and evaluating it at $x = x_{a+2}$ and its derivative at $x = x_{a+1}$, the following discrete schemes are obtained

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$$y_{a+1} = \frac{5}{6}\,w^2\,g_{a+2} + \frac{7}{3}w\,f_{a+1} - \frac{4}{3}w\,f_{a+2} + y_a$$

$$y_{a+2} = -\frac{1}{7}\,y_a + \frac{8}{7}\,y_{a+1} + \frac{6}{7}w\,f_{a+2} - \frac{2}{7}\,w^2\,g_{n+2} \tag{2.11}$$

2.3 Construction of Second Derivative Block Backward Differentiation Formulae Method for k=3

Here, also the number of interpolation points, $u = 3$ and the number of collocation points, $v = 2$. Therefore, (2.2) becomes

$$y(x) = \alpha_0(x)\,y_a + \alpha_1(x)\,y_{a+1} + \alpha_2(x)\,y_{a+2} + w\,\beta_3(x)\,f_{a+3} + w^2\,\gamma_3(x)\,g_{a+3} \tag{2.12}$$

The matrix Q in (2.6) becomes

$$Q = \begin{pmatrix} 1 & x_a & x_a^2 & x_a^3 & x_a^4 \\ 1 & x_{a+w} & (x_{a+w})^2 & (x_{a+w})^3 & (x_{a+w})^4 \\ 1 & x_{a+2w} & (x_{a+2w})^2 & (x_{a+2w})^3 & (x_{a+2w})^4 \\ 0 & 1 & 2x_{a+6w} & 3\,(x_{a+3w})^2 & 4\,(x_{a+3w})^3 \\ 0 & 0 & 2 & 6x_{a+18w} & 12\,(x_{a+3w})^2 \end{pmatrix} \tag{2.13}$$

The inverse of the matrix $R = Q^{-1}$ is computed using Maple 18 to obtain the continuous scheme is obtained using (2.6) and evaluating it at $x = x_{a+3}$ and its derivative at $x = x_{a+1}$ and $x = x_{a+2}$, the following discrete schemes are obtained

$$y_{a+1} = \frac{7}{32}\,w^2\,g_{a+3} - \frac{85}{64}w\,f_{a+1} - \frac{23}{64}w\,f_{a+3} - \frac{11}{32}\,y_a + \frac{43}{32}\,y_{a+2}$$
$$y_{a+2} = \frac{2}{7}\,w^2\,g_{a+3} + \frac{10}{7}w\,f_{a+2} - \frac{4}{7}w\,f_{a+3} - \frac{1}{7}\,y_a + \frac{8}{7}\,y_{a+1}$$
$$y_{a+3} = \frac{4}{85}\,y_a - \frac{27}{85}\,y_{a+1} + \frac{108}{85}\,y_{a+2} + \frac{66}{85}\,wf_{a+3} - \frac{18}{85}\,w^2\,g_{a+3} \tag{2.14}$$

2.4 Construction of Second Derivative Block Backward Differentiation Formulae Method for k=4

Here, also the number of interpolation points, $u = 4$ and the number of collocation points, $v = 2$. Therefore, (2.2) becomes

$$y(x) = \alpha_0(x)\,y_a + \alpha_1(x)\,y_{a+1} + \alpha_2(x)\,y_{a+2} + \alpha_3(x)\,y_{a+3} + w\beta_4(x)\,f_{a+4} + w^2\,\gamma_4(x)\,g_{a+4} \tag{2.15}$$

Also the matrix Q in (2.6) becomes

$$Q = \begin{pmatrix} 1 & x_a & x_a^2 & x_a^3 & x_a^4 & x_a^5 \\ 1 & x_a+w & (x_a+w)^2 & (x_a+w)^3 & (x_a+w)^4 & (x_a+w)^5 \\ 1 & x_a+2w & (x_a+2w)^2 & (x_a+2w)^3 & (x_a+2w)^4 & (x_a+2w)^5 \\ 1 & x_a+3w & (x_a+3w)^2 & (x_a+3w)^3 & (x_a+3w)^4 & (x_a+3w)^5 \\ 0 & 1 & 2x_a+8w & 3\,(x_a+4w)^2 & 4\,(x_a+4w)^3 & 5\,(x_a+4w)^4 \\ 0 & 0 & 2 & 6x_a+24w & 12\,(x_a+4w)^2 & 20\,(x_a+4w)^3 \end{pmatrix} \tag{2.16}$$

The inverse of the matrix $R = Q^{-1}$ is computed using Maple 18 to obtain the continuous scheme is also obtained using (2.6) and evaluating it at $x = x_{a+4}$ and its derivative at $x = x_{a+1}, x = x_{a+2}$ and $x = x_{a+3}$, the following discrete schemes are obtained

$$y_{a+1}=-\frac{29}{297}w^2g_{a+4}-\frac{830}{891}wf_{a+1}+\frac{155}{891}wf_{a+4}-\frac{37}{198}y_a+\frac{119}{66}y_{a+2}-\frac{61}{99}y_{a+3}$$
$$y_{a+2}=\frac{62}{229}w^2g_{a+4}-\frac{830}{229}wf_{a+2}-\frac{120}{229}wf_{a+4}+\frac{127}{687}y_a-\frac{424}{229}y_{a+1}+\frac{8}{3}y_{a+3}$$
$$y_{a+3}=\frac{333}{2249}w^2g_{a+4}+\frac{2490}{2249}wf_{a+3}-\frac{765}{2249}wf_{a+4}-\frac{187}{4498}y_a-\frac{711}{2249}y_{a+1}+\frac{441}{346}y_{a+2}$$
$$y_{a+4}=-\frac{9}{415}y_a+\frac{64}{415}y_{a+1}-\frac{216}{415}y_{a+2}+\frac{576}{415}y_{a+3}+\frac{60}{83}wf_{a+4}-\frac{72}{415}w^2g_{a+4} \tag{2.17}$$

3 Convergence Analysis

Here, the investigations of order, error constant, consistency, zero stability and region of the absolute stability of (2.11), (2.14) and (2.17) will be carried out.

3.1 Order and Error Constant

The SDBBDFM (2.2) is said to be of order Ω if $C_0=C_1=\ldots C_\Omega=0$ and the first non-zero coefficient $C_{\Omega+1}\neq 0$ is the error constant as developed by [25]. The order and error constant for (2.11) are obtained as follows
$C_0=C_1=\begin{pmatrix}0 & 0\end{pmatrix}^T$ but $C_2=\begin{pmatrix}\frac{5}{6} & -\frac{2}{7}\end{pmatrix}^T$. Therefore, (2.11) has order $\Omega=1$ and error constants,

$$C_2=\begin{pmatrix}\frac{5}{6} & -\frac{2}{7}\end{pmatrix}^T.$$

Following the same approach, (2.14) can be presented as
$C_0=C_1=\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$ but $C_2=\begin{pmatrix}\frac{7}{32} & \frac{2}{7} & -\frac{18}{85}\end{pmatrix}^T$. Therefore, (2.14) has order $\Omega=1$ and error constants, $C_2=\begin{pmatrix}\frac{7}{32} & \frac{2}{7} & -\frac{18}{85}\end{pmatrix}^T$.
Applying the same approach, (2.17) can be obtained as
$C_0=C_1=\begin{pmatrix}0 & 0 & 0 & 0\end{pmatrix}^T$ but $C_2=\begin{pmatrix}-\frac{29}{297} & \frac{62}{229} & \frac{333}{2249} & -\frac{72}{415}\end{pmatrix}^T$. Therefore, (2.17) has order $\Omega=1$ and error constants, $C_2=\begin{pmatrix}-\frac{29}{297} & \frac{62}{229} & \frac{333}{2249} & -\frac{72}{415}\end{pmatrix}^T$.

3.2 Consistency

Since $\Omega=1$ in (2.11), (2.14) and (2.17) satisfying the condition for consistency of order $\Omega\geq 1$ as stated by [25], then the discrete schemes are said to be consistent.

3.3 Zero Stability

The discrete schemes (2.11), (2.14) and (2.17) are said to be zero stable if the no root of the first characteristic polynomial is greater than 1.
The zero stability for (2.11) is examined as follows

$$\begin{aligned}\begin{pmatrix}1 & 0\\ -\frac{8}{7} & 1\end{pmatrix}\begin{pmatrix}y_{a+1}\\ y_{a+2}\end{pmatrix} &= \begin{pmatrix}0 & -1\\ 0 & \frac{1}{7}\end{pmatrix}\begin{pmatrix}y_{a-1}\\ y_a\end{pmatrix}+w\begin{pmatrix}\frac{7}{3} & -\frac{4}{3}\\ 0 & \frac{6}{7}\end{pmatrix}\begin{pmatrix}f_{a+1}\\ f_{a+2}\end{pmatrix}\\ &+w\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}f_{a-1}\\ f_a\end{pmatrix}+w^2\begin{pmatrix}0 & \frac{5}{6}\\ 0 & -\frac{2}{7}\end{pmatrix}\begin{pmatrix}g_{a+1}\\ g_{a+2}\end{pmatrix}+w^2\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}g_{a-1}\\ g_a\end{pmatrix}\end{aligned}$$

where
$L_2^{(1)}=\begin{pmatrix}1 & 0\\ -\frac{8}{7} & 1\end{pmatrix}$, $L_1^{(1)}=\begin{pmatrix}0 & -1\\ 0 & \frac{1}{7}\end{pmatrix}$, $M_2^{(1)}=\begin{pmatrix}\frac{7}{3} & -\frac{4}{3}\\ 0 & \frac{6}{7}\end{pmatrix}$ and $N_2^{(1)}=\begin{pmatrix}0 & \frac{5}{6}\\ 0 & -\frac{2}{7}\end{pmatrix}$

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$$\begin{aligned}\mu(\delta) &= \det(\delta L_2^{(1)} - L_1^{(1)}) \\ &= \left|\delta\, L_2^{(1)} - L_1^{(1)}\right| = 0\end{aligned} \tag{3.1}$$

Now we have,

$$\mu\left(\delta\right) = \left|\delta \begin{pmatrix} 1 & 0 \\ -\frac{8}{7} & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{7} \end{pmatrix}\right| = \begin{pmatrix} \delta & 0 \\ -\frac{8}{7}\delta & \delta \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & \frac{1}{7} \end{pmatrix}$$

$$\Rightarrow \mu\left(\delta\right) = \begin{pmatrix} \delta & 1 \\ -\frac{8}{7}\delta & \delta - \frac{1}{7} \end{pmatrix}$$

Using Maple (2.17) software, we obtain

$$\mu(\delta) = \delta^2 + \delta$$

$$\Rightarrow \delta^2 + \delta = 0$$

$\Rightarrow \delta_1 = -1, \delta_2 = 0$. Since $|\delta_i| < 1,\ i = 1, 2$, the discrete schemes in (2.11) is zero stable. Applying the same technique for (2.14) and presented as follows

$$\begin{aligned}\begin{pmatrix} 1 & -\frac{43}{32} & 0 \\ -\frac{8}{7} & 1 & 0 \\ \frac{27}{85} & -\frac{108}{85} & 1 \end{pmatrix}\begin{pmatrix} y_{a+1} \\ y_{a+2} \\ y_{a+3} \end{pmatrix} &= \begin{pmatrix} 0 & 0 & \frac{11}{32} \\ 0 & 0 & \frac{1}{7} \\ 0 & 0 & -\frac{4}{85} \end{pmatrix}\begin{pmatrix} y_{a-2} \\ y_{a-1} \\ y_a \end{pmatrix} + w\begin{pmatrix} -\frac{85}{64} & 0 & -\frac{23}{64} \\ 0 & \frac{10}{7} & -\frac{4}{7} \\ 0 & 0 & \frac{66}{85} \end{pmatrix}\begin{pmatrix} f_{a+1} \\ f_{a+2} \\ f_{a+3} \end{pmatrix} \\ &+ w\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} f_{a-2} \\ f_{a-1} \\ f_a \end{pmatrix} + w^2\begin{pmatrix} 0 & 0 & \frac{7}{32} \\ 0 & 0 & \frac{2}{7} \\ 0 & 0 & -\frac{18}{85} \end{pmatrix}\begin{pmatrix} g_{a+1} \\ g_{a+2} \\ g_{a+3} \end{pmatrix} \\ &+ w^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_{a-2} \\ g_{a-1} \\ g_a \end{pmatrix}\end{aligned}$$

where
$L_2^{(2)} = \begin{pmatrix} 1 & -\frac{43}{32} & 0 \\ -\frac{8}{7} & 1 & 0 \\ \frac{27}{85} & -\frac{108}{85} & 1 \end{pmatrix}$, $L_1^{(2)} = \begin{pmatrix} 0 & 0 & \frac{11}{32} \\ 0 & 0 & \frac{1}{7} \\ 0 & 0 & -\frac{4}{85} \end{pmatrix}$, $M_2^{(2)} = \begin{pmatrix} -\frac{85}{64} & 0 & -\frac{23}{64} \\ 0 & \frac{10}{7} & -\frac{4}{7} \\ 0 & 0 & \frac{66}{85} \end{pmatrix}$ and
$N_2^{(2)} = \begin{pmatrix} 0 & 0 & \frac{7}{32} \\ 0 & 0 & \frac{2}{7} \\ 0 & 0 & -\frac{18}{85} \end{pmatrix}$

$$\begin{aligned}\mu(\delta) &= \det\left(\delta L_2^{(2)} - L_1^{(2)}\right) \\ &= \left|\delta\, L_2^{(2)} - L_1^{(2)}\right| = 0.\end{aligned} \tag{3.2}$$

Now we have,

$$\mu\left(\delta\right) = \left|\delta\begin{pmatrix} 1 & -\frac{43}{32} & 0 \\ -\frac{8}{7} & 1 & 0 \\ \frac{27}{85} & -\frac{108}{85} & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{11}{32} \\ 0 & 0 & \frac{1}{7} \\ 0 & 0 & -\frac{4}{85} \end{pmatrix}\right| = \left|\begin{pmatrix} \delta & -\frac{43}{32}\delta & 0 \\ -\frac{8}{7}\delta & \delta & 0 \\ \frac{27}{85}\delta & -\frac{108}{85}\delta & \delta \end{pmatrix} - \begin{pmatrix} 0 & 0 & \frac{11}{32} \\ 0 & 0 & \frac{1}{7} \\ 0 & 0 & -\frac{4}{85} \end{pmatrix}\right|$$

$$\Rightarrow \mu\left(\delta\right) = \begin{pmatrix} \delta & -\frac{43}{32}\delta & -\frac{11}{32} \\ -\frac{8}{7}\delta & \delta & -\frac{1}{7} \\ \frac{27}{85}\delta & -\frac{108}{85}\delta & \delta + \frac{4}{85} \end{pmatrix}$$

The following are obtained using Maple (2.17) software,

$$\mu(\delta) = -\frac{15}{28}\delta^3 - \frac{15}{28}\delta^2$$

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$$\Rightarrow -\frac{15}{28}\,\delta^3 - \frac{15}{28}\,\delta^2 = 0$$

$\Rightarrow \delta_1 = -1, \delta_2 = 0, \delta_3 = 0$. Since $|\delta_i| < 1,\ i = 1, 2, 3$, the discrete schemes in (2.14) is zero stable. By the same technique (2.17) can be presented as follows

$$\begin{pmatrix} 1 & -\frac{119}{66} & \frac{61}{99} & 0 \\ -\frac{424}{229} & 1 & -\frac{8}{3} & 0 \\ \frac{711}{2249} & -\frac{441}{346} & 1 & 0 \\ -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 \end{pmatrix} \begin{pmatrix} y_{a+1} \\ y_{a+2} \\ y_{a+3} \\ y_{a+4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{37}{198} \\ 0 & 0 & 0 & -\frac{127}{687} \\ 0 & 0 & 0 & -\frac{187}{4498} \\ 0 & 0 & 0 & \frac{9}{415} \end{pmatrix} \begin{pmatrix} y_{a-3} \\ y_{a-2} \\ y_{a-1} \\ y_a \end{pmatrix}$$
$$+w \begin{pmatrix} -\frac{830}{891} & 0 & 0 & \frac{155}{891} \\ 0 & -\frac{830}{229} & 0 & -\frac{120}{229} \\ 0 & 0 & \frac{2490}{2249} & -\frac{765}{2249} \\ 0 & 0 & 0 & \frac{60}{83} \end{pmatrix} \begin{pmatrix} f_{a+1} \\ f_{a+2} \\ f_{a+3} \\ f_{a+4} \end{pmatrix}$$
$$+w \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_{a-3} \\ f_{a-2} \\ f_{a-1} \\ f_a \end{pmatrix} + w^2 \begin{pmatrix} 0 & 0 & 0 & -\frac{29}{297} \\ 0 & 0 & 0 & \frac{62}{229} \\ 0 & 0 & 0 & \frac{333}{2249} \\ 0 & 0 & 2 & -\frac{72}{415} \end{pmatrix} \begin{pmatrix} g_{a+1} \\ g_{a+2} \\ g_{a+3} \\ g_{a+4} \end{pmatrix}$$
$$+w^2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_{a-3} \\ g_{a-2} \\ g_{a-1} \\ g_a \end{pmatrix}$$

where

$$L_2^{(3)} = \begin{pmatrix} 1 & -\frac{119}{66} & \frac{61}{99} & 0 \\ -\frac{424}{229} & 1 & -\frac{8}{3} & 0 \\ \frac{711}{2249} & -\frac{441}{346} & 1 & 0 \\ -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 \end{pmatrix}, L_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & \frac{37}{198} \\ 0 & 0 & 0 & -\frac{127}{687} \\ 0 & 0 & 0 & -\frac{187}{4498} \\ 0 & 0 & 0 & \frac{9}{415} \end{pmatrix}, M_2^{(3)} = \begin{pmatrix} -\frac{830}{891} & 0 & 0 & \frac{155}{891} \\ 0 & -\frac{830}{229} & 0 & -\frac{120}{229} \\ 0 & 0 & \frac{2490}{2249} & -\frac{765}{2249} \\ 0 & 0 & 0 & \frac{60}{83} \end{pmatrix}$$

and

$$N_2^{(3)} = \begin{pmatrix} 0 & 0 & 0 & -\frac{29}{297} \\ 0 & 0 & 0 & \frac{62}{229} \\ 0 & 0 & 0 & \frac{333}{2249} \\ 0 & 0 & 2 & -\frac{72}{415} \end{pmatrix}$$

$$\begin{aligned} \mu(\delta) &= \det\left(\delta L_2^{(3)} - L_1^{(3)}\right) \\ &= \left|\delta\, L_2^{(3)} - L_1^{(3)}\right| = 0 \end{aligned} \tag{3.3}$$

Now we have,

$$\begin{aligned} \mu(\delta) &= \left|\delta \begin{pmatrix} 1 & -\frac{119}{66} & \frac{61}{99} & 0 \\ -\frac{424}{229} & 1 & -\frac{8}{3} & 0 \\ \frac{711}{2249} & -\frac{441}{346} & 1 & 0 \\ -\frac{64}{415} & \frac{216}{415} & -\frac{576}{415} & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \frac{37}{198} \\ 0 & 0 & 0 & -\frac{127}{687} \\ 0 & 0 & 0 & -\frac{187}{4498} \\ 0 & 0 & 0 & \frac{9}{415} \end{pmatrix}\right| \\ &= \left|\begin{pmatrix} \delta & -\frac{119}{66}\delta & \frac{61}{99}\delta & 0 \\ -\frac{424}{229}\delta & \delta & -\frac{8}{3}\delta & 0 \\ \frac{711}{2249}\delta & -\frac{441}{346}\delta & \delta & 0 \\ -\frac{64}{415}\delta & \frac{216}{415}\delta & -\frac{576}{415}\delta & \delta \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \frac{37}{198} \\ 0 & 0 & 0 & -\frac{127}{687} \\ 0 & 0 & 0 & -\frac{187}{4498} \\ 0 & 0 & 0 & \frac{9}{415} \end{pmatrix}\right| \\ \Rightarrow \mu(\delta) &= \begin{pmatrix} \delta & -\frac{119}{66}\delta & \frac{61}{99}\delta & -\frac{37}{198} \\ -\frac{424}{229}\delta & \delta & -\frac{8}{3}\delta & \frac{127}{687} \\ \frac{711}{2249}\delta & -\frac{441}{346}\delta & \delta & \frac{187}{4498} \\ -\frac{64}{415}\delta & \frac{216}{415}\delta & -\frac{576}{415}\delta & \delta - \frac{9}{415} \end{pmatrix} \end{aligned}$$

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The following are obtained using Maple (2.16) software,

$$\mu(\delta) = \frac{13778000}{16995693}\,\delta^4 + \frac{13778000}{16995693}\,\delta^3$$

$$\Rightarrow \frac{13778000}{16995693}\,\delta^4 + \frac{13778000}{16995693}\,\delta^3 = 0$$

$\Rightarrow \delta_1 = -1, \delta_2 = 0, \delta_3 = 0, \delta_4 = 0$. Since $|\delta_i| < 1,\ i = 1, 2, 3, 4$, the discrete schemes in (2.17) is zero stable.

3.4 Convergence

Since (2.11), (2.14) and (2.17) are both consistent and zero stable, therefore they are convergent.

3.5 Region of Absolute Stability

The regions of absolute stability of the numerical methods for DDEs are considered. We considered finding the $C-$ and $D-$stability by applying (2.11), (2.14) and (2.17) to the DDEs of this form:

$$\begin{array}{ll} y'(t) = \phi y(\mathrm{t}) + \omega y(t-\tau), & t \geq t_0 \\ y(t) = \varphi(t), & t \leq t_0 \end{array} \tag{3.4}$$

where $\varphi(t)$ is the initial function, ϕ, ω are complex coefficients, $\tau = mw, m \in \mathbb{Z}^+$, w is the step size and $w = \frac{\tau}{m}, m$ is a positive integer. Let $R_1 = w\phi$ and $R_2 = w\omega$, then from (2.11), then the $C-$ and $D-$ stability of (2.11), (2.14) and (2.17) are investigated, plotted and presented in figure 1 to 6 below with use of Maple 18 and MATLAB.

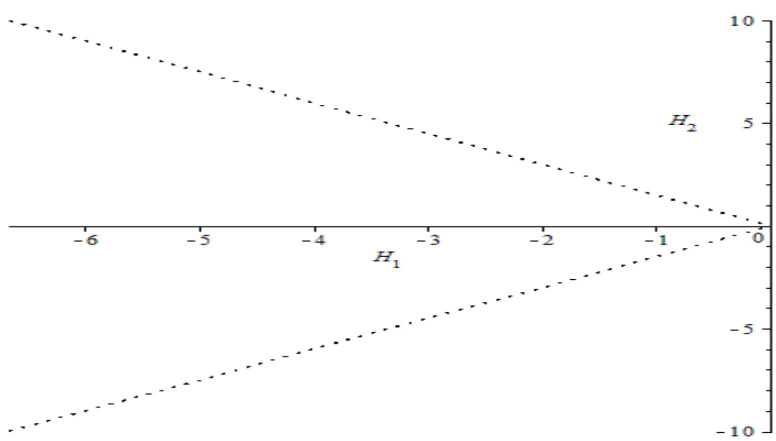

Figure 1: Region of $C-$stability (SDBBDFM)in (2.11)

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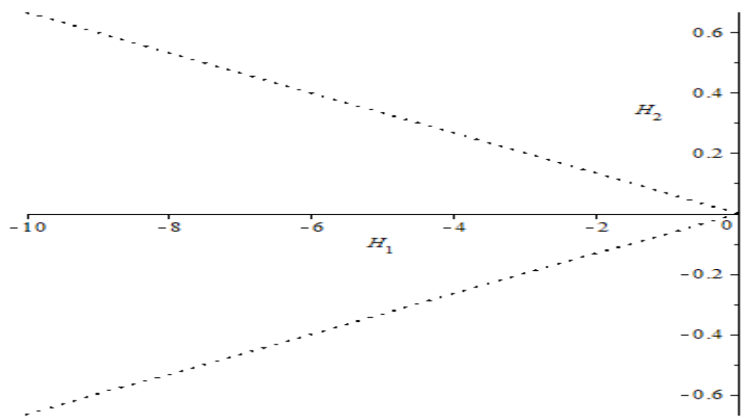

Figure 2: Region of $C-$stability (SDBBDFM)in (2.14)

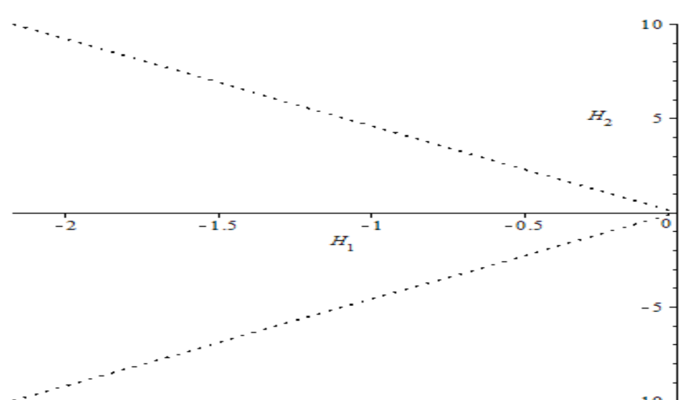

Figure 3: Region of $C-$stability (SDBBDFM)in (2.17)

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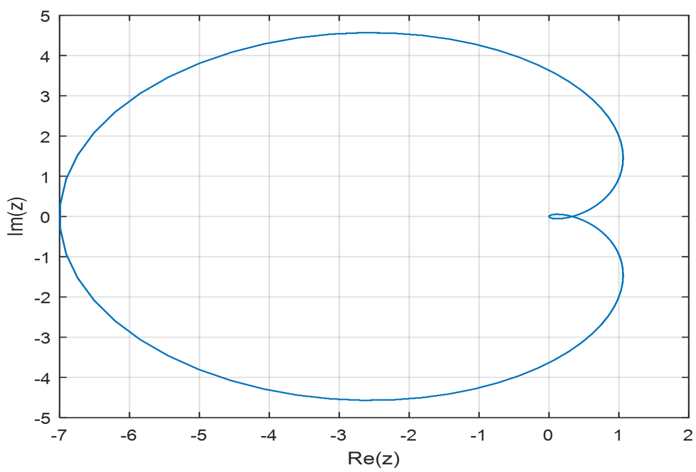

Figure 4: Region of $D-$stability (SDBBDFM)in (2.11)

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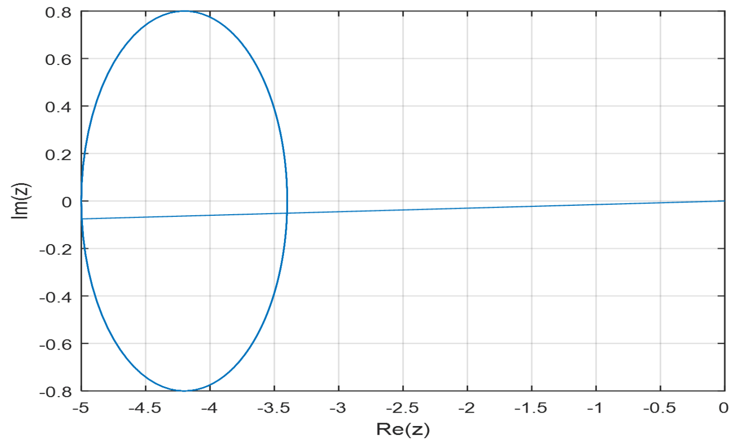

Figure 5: Region of $D-$stability (SDBBDFM) in (2.14)

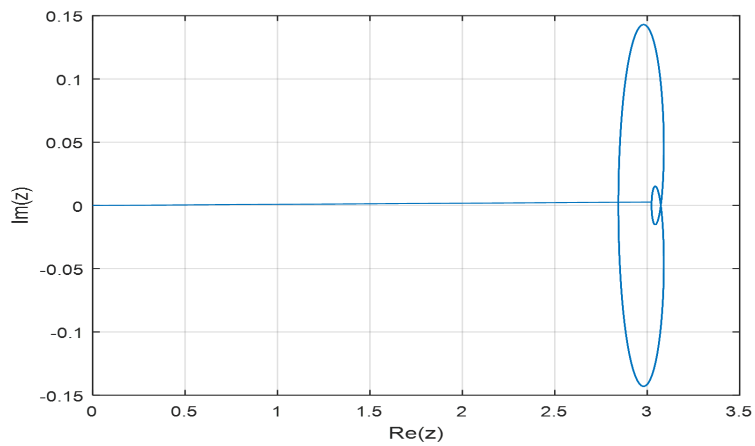

Figure 6: Region of $D-$stability (SDBBDFM) in (2.17)

The $C-$stability regions in Figs 1 to 3 lie inside the open-ended region while the $D-$ stability regions in Figs 3 to 6 lie inside the enclosed region.

4 Numerical Computations

In this section, some first-order delay differential equations shall be solved using (2.11), (2.14) and (2.17) of the discrete schemes been established. The delay argument shall be implemented using the idea of sequence derived by [17].

4.1 Applications of SDBBDFM to Solve Some First Order DDEs

Problem 1

$$y^{'}(t) = -1000y(t) + y(t - (\ln(1000 - 1))), 0 \leq t \leq 3$$
$$y(t) = e^{-t}, t \leq 0$$

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Exact solution $y(t) = e^{-t}$

Problem 2

$$y^{'}(t) = -1000y(t) + 997e^{-3}y(t-1) + (1000 - 997e^{-3}), 0 \leq t \leq 3$$

$$y(t) = 1 + e^{-3t}, t \leq 0$$

Exact solution $y(t) = 1 + e^{-3t}$

These problems were solved using the discrete schemes in (12), (15) and (18) and the numerical solutions are presented below

t	Exact Solution	$k=2$ **Numerical Solution**	$k=3$ **Numerical Solution**	$k=4$ **Numerical Solution**
0.1	0.990049834	0.990046071	0.990045639	0.990004001
0.2	0.980198673	0.980204092	0.980204329	0.980162693
0.3	0.970445534	0.970442037	0.97026926	0.970419306
0.4	0.960789439	0.960794693	0.960784392	0.959542
0.5	0.951229425	0.951225995	0.951233939	0.951219778
0.6	0.941764534	0.941769684	0.941534196	0.941764353
0.7	0.93239382	0.932390458	0.932388597	0.932403009
0.8	0.923116346	0.923121395	0.923120399	0.92323405
0.9	0.913931185	0.913927889	0.913687709	0.913885631
1.0	0.904837418	0.904842366	0.904832237	0.904800959
1.1	0.895834135	0.895830906	0.895837958	0.895806679
1.2	0.886920437	0.886925287	0.886677455	0.88564472
1.3	0.878095431	0.878092265	0.878090366	0.878089949
1.4	0.869358235	0.86936299	0.869361908	0.869361492
1.5	0.860707976	0.860704873	0.860469921	0.860719883
1.6	0.852143789	0.852148449	0.852138862	0.85238348
1.7	0.843664817	0.843661775	0.843668368	0.843619153
1.8	0.835270211	0.835274779	0.835038436	0.835232943
1.9	0.826959134	0.826956152	0.826954348	0.826930176
2.0	0.818730753	0.81873523	0.818734195	0.81741484
2.1	0.810584246	0.810581323	0.810359058	0.810582997
2.2	0.802518798	0.802523187	0.802514152	0.802525615
2.3	0.794533603	0.794530738	0.794536942	0.794548405
2.4	0.786627861	0.786632163	0.786409241	0.78699501
2.5	0.778800783	0.778797975	0.778796274	0.778754609
2.6	0.771051586	0.771055802	0.771054826	0.771013161
2.7	0.763379494	0.763376742	0.763167305	0.763348742

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2.8	0.755783741	0.755787874	0.755779366	0.75441508
2.9	0.748263568	0.748260869	0.748266712	0.748266658
3.0	0.740818221	0.740822272	0.740612304	0.740828758

Numerical Solutions of Problem 1 for SDBBDFM $k = 2, 3$ and 4

t	Exact Solution	$k = 2$ **Numerical Solution**	$k = 3$ **Numerical Solution**	$k = 4$ **Numerical Solution**
0.1	1.970445534	1.970423847	1.970402449	1.970013705
0.2	1.941764534	1.941795768	1.941807493	1.941418749
0.3	1.913931185	1.913911862	1.913164307	1.9136689
0.4	1.886920437	1.886949516	1.886876817	1.87607662
0.5	1.860707976	1.860689767	1.860742997	1.860623911
0.6	1.835270211	1.835297601	1.834311469	1.83526246
0.7	1.810584246	1.810567098	1.810542954	1.810650552
0.8	1.786627861	1.786653657	1.786658441	1.78845191
0.9	1.763379494	1.763363344	1.762416543	1.762989522
1.0	1.740818221	1.740842513	1.740780001	1.740495934
1.1	1.7189237733	1.718908523	1.718951201	1.718667129
1.2	1.697676326	1.697699203	1.696767095	1.68722167
1.3	1.677056874	1.67704255	1.677021786	1.677043804
1.4	1.65704682	1.657068365	1.657071763	1.65709378
1.5	1.637628152	1.637614662	1.636787358	1.637733367
1.6	1.618783392	1.618803683	1.618751268	1.622249
1.7	1.600495579	1.600482875	1.600518321	1.600132834
1.8	1.582748252	1.582767362	1.581976539	1.58243875
1.9	1.565525439	1.565513474	1.565496062	1.565267604
2.0	1.548811636	1.548829633	1.548832403	1.53844499
2.1	1.532591801	1.532580532	1.531885394	1.532640588
2.2	1.516851334	1.516868282	1.516824479	1.516947343
2.3	1.501576069	1.501565457	1.501595042	1.501717903
2.4	1.486752256	1.486768217	1.486106283	1.49173923
2.5	1.472366553	1.472356559	1.472342006	1.472018882
2.6	1.458406011	1.458421043	1.45842335	1.458100223
2.7	1.444858066	1.444848655	1.444267583	1.444592922
2.8	1.431710523	1.431724681	1.431688091	1.42117034

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2.9	1.418951549	1.418942685	1.418967394	1.419055689
3.0	1.40656966	1.406582991	1.406029946	1.406710946

Numerical Solutions of Problem 2 for SDBBBDFM $k = 2, 3$ and 4

5 Analysis and Discussion of Results

Here, the performances of the schemes derived in (2.11), (2.14) and (2.17), shall be implemented in solving the two problems above by computing their absolute errors. The analysis of results is obtained by evaluating absolute difference of exact solutions and numerical solutions. The results are summarized in the tables 5 to 5,

t	$k = 2$ **Error**	$k = 3$ **Error**	$k = 4$ **Error**
0.1	3.76245E-04	4.19475E-06	4.58323E-08
0.2	5.41909E-04	5.65589E-06	3.59802E-08
0.3	3.49705E-05	1.76274E-06	2.62275E-08
0.4	5.25355E-05	5.04695E-06	1.247439E-09
0.5	3.4298E-05	4.514E-06	9.647E-09
0.6	5.15002E-05	2.30338E-06	1.80484E-09
0.7	3.36171E-05	5.22331E-06	9.18939E-09
0.8	5.04821E-05	4.05271E-06	1.7704E-09
0.9	3.29587E-05	2.43476E-06	4.55541E-09
1.0	4.94806E-05	5.18114E-06	3.64592E-09
1.1	3.2297E-05	3.8225E-07	2.74561E-09
1.2	4.85018E-05	2.42982E-07	1.275717E-10
1.3	3.16632E-05	5.06482E-07	5.48162E-10
1.4	4.7541E-05	3.6725E-07	3.2567E-10
1.5	3.10313E-05	2.38055E-07	1.19066E-10
1.6	4.66003E-05	4.92727E-07	2.39691E-10
1.7	3.0417E-05	3.5515E-07	4.56641E-10
1.8	4.56779E-05	2.31775E-07	3.72685E-10
1.9	2.98164E-05	4.78614E-07	2.89577E-10
2.0	4.47732E-05	3.44232E-07	1.315913E-10
2.1	2.92257E-05	2.25188E-07	1.24887E-11
2.2	4.38854E-05	4.64636E-07	6.81744E-11
2.3	2.8647E-05	3.3392E-07	1.48025E-11
2.4	4.30163E-05	2.1862E-07	3.67149E-11

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2.5	2.80787E-05	4.50897E-07	4.61744E-11
2.6	4.2165E-05	3.24E-07	3.84248E-11
2.7	2.75234E-05	2.12189E-08	3.07526E-12
2.8	4.13294E-05	4.37576E-08	1.368661E-12
2.9	2.69818E-05	3.14402E-08	3.09082E-12
3.0	4.05122E-05	2.05917E-08	1.05372E-12

Absolute Errors for SDBBBDFM $k = 2, 3$ and 4 using problem 1

t	$k = 2$ **Error**	$k = 3$ **Error**	$k = 4$ **Error**
0.1	2.16865E-05	4.30845E-06	4.31829E-07
0.2	3.12344E-05	4.29594E-06	3.45785E-07
0.3	1.93233E-05	7.66878E-06	2.62285E-08
0.4	2.90793E-05	4.36197E-06	1.0843817E-08
0.5	1.82094E-05	3.50206E-06	8.40654E-08
0.6	2.73896E-05	9.58742E-06	7.75141E-08
0.7	1.7148E-05	4.1292E-06	6.6306E-08
0.8	2.57959E-05	3.05799E-06	1.824049E-09
0.9	1.61503E-05	9.62951E-06	3.89972E-09
1.0	2.42923E-05	3.82197E-06	3.22287E-09
1.1	1.52104E-05	2.74676E-06	2.56604E-09
1.2	2.28769E-05	9.09231E-06	1.0454656E-09
1.3	1.43245E-05	3.50885E-06	1.30705E-09
1.4	2.15452E-05	2.49432E-06	4.69602E-09
1.5	1.34896E-05	8.40794E-06	1.05215E-09
1.6	2.02912E-05	3.21238E-06	3.465608E-09
1.7	1.27038E-05	2.27422E-06	3.62745E-09
1.8	1.91096E-05	7.71713E-06	3.09502E-09
1.9	1.19647E-05	2.93767E-07	2.57835E-10
2.0	1.79969E-05	2.07669E-07	1.0366646E-10
2.1	1.1269E-05	7.06407E-07	4.8787E-10
2.2	1.69475E-05	2.68555E-07	9.60085E-10
2.3	1.06121E-05	1.89729E-07	1.41834E-10
2.4	1.5961E-05	6.45973E-07	4.986974E-10
2.5	9.99374E-06	2.45467E-07	3.47671E-11

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2.6	1.50317E-05	1.73387E-07	3.05788E-11
2.7	9.41122E-06	5.90483E-08	2.65144E-12
2.8	1.41576E-05	2.24324E-08	1.0540183E-13
2.9	8.86425E-06	1.58448E-09	1.0414E-13
3.0	1.33313E-05	5.39714E-09	1.41286E-13

Absolute Errors of SDBBDFM $k = 2, 3$ and 4 using problem 2

5.1 Comparison of Results

The results obtained from tables 5 to 5 shall be compared to other existing methods studied by [17, 21] to prove its advantage. The notations used in the tables 1 to 2 below are stated as

SDBBDFM = Second Derivative Block Backward Differentiation Formulae Methods for step numbers $k = 2, 3$ and 4.
CBBDFM = Conventional Block Backward Differentiation Formulae Method for step numbers $k = 2$ and 3 in [17].
HEBBDFM = Hybrid Extended Block Backward Differentiation Formulae Methods for step numbers $k = 3$ and 4 in [21].
MAXE = Maximum Error.

Numerical Method	Compared MAXEs with [17, 21]
SDBBDFM MAXE for k = 2	5.25E-05
SDBBDFM MAXE for k = 3	4.38E-08
SDBBDFM MAXE for k = 4	3.09E-12
CBBDFM MAXE for k = 2	1.66E-05
CBBDFM MAXE for k = 3	2.22E-07
HEBBDFM MAXE for k = 2	2.97E-06
HEBBDFM MAXE for k = 3	5.74E-08
HEBBDFM MAXE for k = 4	3.90E-09

Table 1: Comparison between the Maximum Absolute Errors of SDBBDFM $k = 2, 3$ and 4 with [17, 21] for constant step size $w = 0.01$ Using Problem 1.

CPU time of SDBBDFM for $k = 2$ is $0.395s$, $k = 3$ is $0.244s$ and $k = 4$ is $0.1.98s$

Numerical Method	Compared MAXEs with [17, 21]
SDBBDFM MAXE for k = 2	9.99E-06
SDBBDFM MAXE for k = 3	5.40E-09
SDBBDFM MAXE for k = 4	1.41E-13
CBBDFM MAXE for k = 2	1.66E-05
CBBDFM MAXE for k = 3	2.22E-07
HEBBDFM MAXE for k = 2	3.33E-06
HEBBDFM MAXE for k = 3	9.85E-08
HEBBDFM MAXE for k = 4	8.32E-09

Table 2: Comparison between the Maximum Absolute Errors of SDBBDFM $k = 2, 3$ and 4 with [17, 21] for constant step size $w = 0.01$ Using Problem 2.

CPU time of SDBBDFM for $k = 2$ is $0.420s$, $k = 3$ is $0.370s$ and $k = 4$ is $0.201s$

5.2 Conclusions

In this study, we have proved that second derivative block backward differentiation formulae methods (SDBBDFM) are suitable in solving first order delay differential equations without the application of interpolation methods in investigating the delay argument. After the implementation, we observed that proposed method satisfied the necessary and sufficient conditions of convergence and stability. Also, we observed in tables 5 to 2 that the SDBBDFM for $k = 4$ scheme performed better when compared with other existing methods than the SDBBDFM schemes for step numbers $k = 3$ and $k = 2$ respectively. Therefore, it is recommended that SDBBDFM schemes for step numbers $k = 2, 3,$ and 4 are suitable for solving DDEs. It is also recommended that the SDBBDFM schemes of higher step numbers perform better than the SDBBDFM schemes of lower step numbers. Further study should be carried out for step number $k = 5, 6, 7...$ on the construction of discrete schemes of SDBBDFM for solving DDEs without the application interpolation techniques in investigating the delay term.

Conflicts of Interest

No conflict of interest was declared by the authors.

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