IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
VOL. 8, NO. 2, PP. 1 - 14
HTTPS://DOI.ORG/10.6084/M9.FIGSHARE.21505032

K-Step Block Implicit Adams Method for Approximate Solution of Initial Value Problems using Eulerian Polynomial

J. A. Osilagun[1*], R. O. Olagbenro[2], R. I. Abdulganiy[3], M. R. Odekunle[4]

1-4 Department of Mathematics, University of Lagos, Nigeria
* Corresponding author: josilagun@unilag.edu.ng, olagbenrorodiyyah63@gmail.com, rabdulganiy@unilag.edu.ng, odekunle@mautech.edu.ng

Article Info
Received: 14 July 2022 Revised: 02 September 2022
Accepted: 10 September 2022 Available online: 28 November 2022

Abstract

This paper constructs and applied a class of k-step Block Implicit Adams Method (BIAM) for the integration of first order initial value problems in which the Eulerian polynomial is employed as the basis function .The BIAM for the initial value problems is constructed via multistep collocation techniques and applied in block structure as simultaneous numerical integration which makes it self-starting. An analysis of BIAM shows that the proposed method is zero stable, consistent, convergence and A-stable. Application of the BIAM on some standard ordinary differential problems revealed that BIAM is efficient and a good approximating scheme.

Keywords: Block implicit Adams method, Eulerian polynomials, A-Stability, Approximate solution, Equidistant grid point. Region of absolute stability(RAS).
MSC2010: 65L05, 45J05, 44A10, 34A34, 65L10.

1 Introduction

Many real life processes are often formulated by non-trivial differential equations. As such, only a few often seldom be solved analytically to obtain exact or closed form solution especially the non-linear. This propels our quest for approximate solution via appropriate numerical method. This paper, focuses on an efficient and accurate numerical method for the initial value problem

$$y^{'}(x) = f(x,y) \ , \ y(x_0) = y_0 \in \Re^n \ , a \leq x \leq b \ , \ y(x) \in \Re^n \ . \tag{1.1}$$

whose solution is to be given as a set of mesh

$$\Omega = \{x_n : x_n = x_o + nh \ , n \in Z^+\} \tag{1.2}$$

$h$is a fixed step size, $f(x,y)$ is a real value continuous function which satisfies the existence and uniqueness conditions entrenched in Lambert [1]. A k-step block implicit Adams method (K-BIAM)

http://ijmso.unilag.edu.ng/article

with particular emphasis on a new choice of basis functions known as Eulerian polynomials (see Hirzebruch, [2]). The zeal for efficient numerical method for the appropriate approximate solution of (1.1) has been on the increase in recent years. Various techniques for the formulation of continuous approximation scheme for the integral of ordinary differential equations have been carried out by a quite number of researchers. Some of whom have employed either the monomials (x^r), $r = 0\,(1)\,n$ as in Anake et al [3] , Fox and Parker [4] or the canonical polynomials $[Q_k(x)]$, $k = 0\,(1)\,n$.

In the derivation of continuous linear multistep methods(LMMs), several techniques and approaches have appeared in literature with different basis function such as chebyshev polynomial in the work of Adeniyi and Alabi [5] the power series function in Okunuga and Ehigie [7]), Akinfewa *et al* [8] and Hermite polynomials as basis function in Abioyar *et al* [9]. The advantages of using Block method inherent in Ramos and Singh [10], inspired the work of this paper. A block method of step length of four that adopts Newton's polynomials approximation was proposed by Osilagun *et al* [11] to generate the starting value of four steps implicit scheme for solution of second order ordinary differential equations was derived through interpolation and collocation method. For further details on the numerical methods of solving the initial value problems (1.1), see for instance Enright [12], Adeniyi et al [13] Yahaya and Kumleng [14], Abdulganiy *et al* [15] Ngwane and Jator [16], Akinfenwa *et al* [17–19] Sahi *et al* [20], Alabi *et al* [21], Henrici [22], Adeniyi and Alabi [23].

Inspite of the various techniques that have been suggested in the derivation of block methods, this paper presents the use of Eulerian polynomials as a basis function on the collocation which lead to some continuous schemes easily linked to linear multistep methods.

2 Methodology

2.1 Derivation of $K-$Step Block Implicit Adams Method (BIAM) for continuous approximation

The form of general Block Implicit Adams Method is

$$y_{n+k}-y_{n+k-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2.1}$$

where β_j , $j = 0\,(1)\,k$ are unknown parameters uniquely determined via the multistep techniques and h is the step length such that $x_{n+j} - x_j = nh$. The exact solution $y(x)$ is assumed to exist and unique in which is approximated by the interpolating function given by

$$y\,(x) = \sum_{j=0}^{k} a_j \frac{E(j,x)}{j!} \tag{2.2}$$

where $\frac{E(j,x)}{j!}$ is the coefficient of the Euler polynomial whose generating function is

$$\frac{2e^{xt}}{e^{x+1}} = \sum_{j=0}^{k} \frac{E(j,x)}{j!} \tag{2.3}$$

The following conditions are imposed (2.2)

$$y\,(x_{n+j}) = y_{n+j} \quad j = k-1 \tag{2.4}$$

$$y\prime\,(x_{n+j}) = f_{n+j} \quad j = 0(1)k \tag{2.5}$$

Using the Euler polynomial basis function defined by Equation (2.1) and the imposed conditions equation (2.4) and (2.5) result to a linear system of $(k+2)$ equations with $(k+2)$ unknown. The resulting system of equations is solved by Gaussian elimination method or Crammer's rule to obtain the coefficients a_j. The values of such $a_j^\prime s$ are substituted into equation (2.1) to obtain a continuous scheme in the form

$$y\,(x) = y_{n+k-1}+h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2.6}$$

The proposed method is obtained by evaluating (2.6) at $x = x_{n+k}$ which is of the form

$$y_{n+k}-y_{n+k-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} \tag{2.7}$$

while the $(k-1)$ complementary methods are obtained by collocating equation (8) at $x = x_{n+j}$, $j = 0\,(1)\,(k-2)$, which take the form given by equation (2.7)

$$y_{n+1}-y_{n+k-1} = h\sum_{j=0}^{k}\beta_j f_{n+j} \ , \ r = 0\,(1)\,(k-2) \tag{2.8}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 1 - 14
https://doi.org/10.6084/m9.figshare.21505032

The two discrete scheme, That is (2.7) and (2.8) constitute the proposed Block Implicit Adams Method (BIAM). A specific scheme will emerge for a fixed value of k, ($k = 2, 3, 4$).

2.1.1 2-Step BIAM

For a fixed value of $k = 2$, using equation (2.1) for $k = 2$ and imposing the conditions equation (2.4) and (2.5) respectively, give a system of four equation in four unknowns.

$$\begin{pmatrix} 1 & x_{n+1}-\frac{1}{2} & \frac{1}{2}x_{n+1}^2-\frac{1}{2}x_{n+1} & \frac{1}{6}x_{n+1}^3-\frac{1}{4}x_{n+1}^2+\frac{1}{24} \\ 0 & 1 & x_n-\frac{1}{2} & \frac{1}{2}x_n^2-\frac{1}{2}x_n \\ 0 & 1 & x_{n+1}-\frac{1}{2} & \frac{1}{2}x_{n+1}^2-\frac{1}{2}x_{n+1} \\ 0 & 1 & x_{n+2}-\frac{1}{2} & \frac{1}{2}x_{n+2}^2-\frac{1}{2}x_{n+2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ f_n \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \tag{2.9}$$

Solving for a_j , $j = 0\,(1)\,3$ and substituting the values of a_j into equation (2.2) to obtain the continuous scheme of the form

$$\begin{aligned} y(x) = y_{n+1} &+ \left(-\frac{5}{12}h - x_n + x - \frac{3}{4}\frac{(-x_n+x)^2}{h} + \frac{1}{6}\frac{(-x_n+x)^3}{h^2}\right) f_n \\ &+ \left(-\frac{2}{3}h + \frac{(-x_n+x)^2}{h} - \frac{1}{3}\frac{(-x_n+x)^3}{h^2}\right) f_{n+1} + \left(\frac{1}{12}h - \frac{1}{4}\frac{(-x_n+x)^2}{h} + \frac{1}{6}\frac{(-x_n+x)^3}{h^2}\right) f_{n+2} \end{aligned} \tag{2.10}$$

collocating (2.10) at $x = x_{n+2}$, one obtain the main method given by

$$y_{n+2} - y_{n+1} = \frac{h}{12}\left(5f_{n+2} + 8f_{n+1} - f_n\right) \tag{2.11}$$

The complementary method is obtained from the same continuous form given by (2.10) and evaluating at $x = x_n$, yields

$$y_n - y_{n+1} = \frac{h}{12}\left(f_{n+2} - 8f_{n+1} - 5f_n\right) \tag{2.12}$$

Hence (2.11) and (2.12) produce the Block Method called 2 Step Block Implicit Adams Method (2-BIAM). The next step is to consider equation (2.3) for $k = 3$.

2.2.3 3-Step BIAM

Applying (2.1) for $k = 3$ and imposing the conditions in (2.4) and (2.5) respectively give the following system of 5 equations and 5 unknown parameters $a_j's$ to be determined

$$\begin{pmatrix} 1 & x_{n+2}-\frac{1}{2} & \frac{1}{2}x_{n+2}^2-\frac{1}{2}x_{n+2} & \frac{1}{6}x_{n+2}^3-\frac{1}{4}x_{n+2}^2+\frac{1}{24} & \frac{1}{24}x_{n+2}^4-\frac{1}{12}x_{n+2}^3+\frac{1}{24}x_{n+2} \\ 0 & 1 & x_n-\frac{1}{2} & \frac{1}{2}x_n^2-\frac{1}{2}x_n & \frac{1}{6}x_n^3-\frac{1}{4}x_n^2+\frac{1}{24} \\ 0 & 1 & x_{n+1}-\frac{1}{2} & \frac{1}{2}x_{n+1}^2-\frac{1}{2}x_{n+1} & \frac{1}{6}x_{n+1}^3-\frac{1}{4}x_{n+1}^2+\frac{1}{24} \\ 0 & 1 & x_{n+2}-\frac{1}{2} & \frac{1}{2}x_{n+2}^2-\frac{1}{2}x_{n+2} & \frac{1}{6}x_{n+2}^3-\frac{1}{4}x_{n+2}^2+\frac{1}{24} \\ 0 & 1 & x_{n+3}-\frac{1}{2} & \frac{1}{2}x_{n+3}^2-\frac{1}{2}x_{n+3} & \frac{1}{6}x_{n+3}^3-\frac{1}{4}x_{n+3}^2+\frac{1}{24} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{pmatrix} \tag{2.13}$$

Solving for $a_j's$, $j = 0\,(1)\,4$ and substituting the values of $a_j's$ into equation (2.1) to obtain the continuous form given by

$$\begin{aligned} y(x) = y_{n+2} &+ \left(-\frac{1}{3}h + (-x_n + x) - \frac{11}{12}\frac{(-x_n+x)^2}{h} + \frac{1}{3}\frac{(-x_n+x)^3}{h^2} - \frac{1}{24}\frac{(-x_n+x)^4}{h^3}\right) f_n \\ &+ \left(-\frac{4}{3}h + \frac{3}{2}\frac{(-x_n+x)^2}{h} - \frac{5}{6}\frac{(-x_n+x)^3}{h^2} + \frac{1}{8}\frac{(-x_n+x)^4}{h^3}\right) f_{n+1} \\ &+ \left(-\frac{1}{3}h - \frac{3}{4}\frac{(-x_n+x)^2}{h} + \frac{2}{3}\frac{(-x_n+x)^3}{h^2} - \frac{1}{8}\frac{(-x_n+x)^4}{h^3}\right) f_{n+2} \\ &+ \left(\frac{1}{6}\frac{(-x_n+x)^2}{h} - \frac{1}{6}\frac{(-x_n+x)^3}{h^2} + \frac{1}{24}\frac{(-x_n+x)^4}{h^3}\right) f_{n+3} \end{aligned} \tag{2.14}$$

The desired main method for $k = 3$ is obtained by evaluating (2.14) at $x = x_{n+3}$ to obtain

$$y_{n+3} - y_{n+2} = \frac{h}{24}\left(9f_{n+3} + 19f_{n+2} - 5f_{n+1} + f_n\right) \tag{2.15}$$

The two complementary methods is also obtained from equations (2.13) upon evaluating at $x = x_n$ and $x_n = x_{n+1}$ respectively, yield

$$y_n - y_{n+2} = \frac{h}{3}\left(-f_{n+2} - 4f_{n+1} - f_n\right) \tag{2.16}$$

$y_{n+1}-y_{n+2}=\frac{h}{24}\left(f_{n+3}-13f_{n+2}-13f_{n+1}+f_n\right)$ (2.17)

Hence (2.15)-(2.17) produce the block method called 3-Step BIAM.

2.1.3 4- Step BIAM

By following same procedure in $k=2$ and $k=3$, the following continuous form is obtained for $k=4$

$$\begin{aligned} y(x) = y_{n+3} &+ \left(-\frac{27}{80}h + -x_n + x - \frac{25}{24}\frac{(-x_n+x)^2}{h} + \frac{35}{72}\frac{(-x_n+x)^3}{h^2} - \frac{1}{1152}\frac{(-x_n+x)^9}{h^7}\right) f_n \\ &+ \left(-\frac{51}{40}h + \frac{2(-x_n+x)^2}{h} - \frac{13}{9}\frac{(-x_n+x)^3}{h^2} + \frac{3}{8}\frac{(-x_n+x)^4}{h^3} - \frac{1}{30}\frac{(-x_n+x)^5}{h^4}\right) f_{n+1} \\ &+ \left(-\frac{9}{10}h - \frac{3}{2}\frac{(-x_n+x)^2}{h} + \frac{19}{12}\frac{(-x_n+x)^3}{h^2} - \frac{1}{2}\frac{(-x_n+x)^4}{h^3} + \frac{1}{20}\frac{(-x_n+x)^5}{h^4}\right) f_{n+2} \\ &+ \left(-\frac{21}{40}h + \frac{2}{3}\frac{(-x_n+x)^2}{h} - \frac{7}{9}\frac{(-x_n+x)^3}{h^2} + \frac{7}{24}\frac{(-x_n+x)^4}{h^3} - \frac{1}{30}\frac{(-x_n+x)^5}{h^4}\right) f_{n+3} \\ &+ \left(-\frac{3}{80}h - \frac{1}{8}\frac{(-x_n+x)^2}{h} + \frac{11}{72}\frac{(-x_n+x)^3}{h^2} - \frac{1}{16}\frac{(-x_n+x)^4}{h^3} + \frac{1}{120}\frac{(-x_n+x)^5}{h^4}\right) f_{n+4} \end{aligned} \tag{2.18}$$

Evaluating (2.18) at $x=\{x_{n+4}\ ,\ x_n\ ,\ x_{n+1}\ ,\ x_{n+2}\}$ gives the following discrete methods

$y_{n+4}-y_{n+3}=\frac{h}{720}\left(251f_{n+4}+646f_{n+3}-264f_{n+2}+106f_{n+1}-19f_n\right)$ (2.19)

$y_n-y_{n+3}=\frac{h}{80}\left(3f_{n+4}-42f_{n+3}-72f_{n+2}-102f_{n+1}-27f_n\right)$ (2.20)

$y_{n+1}-y_{n+3}=\frac{h}{90}\left(f_{n+4}-34f_{n+3}-114f_{n+2}-34f_{n+1}+f_n\right)$ (2.21)

$y_{n+2}-y_{n+3}=\frac{h}{720}\left(19f_{n+4}-346f_{n+3}-456f_{n+2}+74f_{n+1}-11f_n\right)$ (2.22)

where (2.18) and (2.19) - (2.22) are respectively the main method and the complementary methods of the 4-Step Block Implicit Adams Method (4-BIAM).

2.2 Analysis of the K-Step BIAM, K = 2,3,4

In this section of the paper, the analysis of the proposed method is discussed based on its local truncation error, order, consistency, A-Stability and convergence of the k-step BIAM.

2.2.1 Local Truncation Error of BIAM

Based on Lambert [1], we define the local truncation error associated with BIAM by the linear difference operator defined by

$\mathcal{L}\left[y\left(x\right);h\right]=\sum_{j=k-1}^{k}\alpha_j\ y\left(x+jh\right)-\sum_{j=0}^{k}h\beta_j y'\left(x+jh\right)$ (2.23)

Assume that$y(x)$ is sufficiently differentiable, then by Taylor Series expansion of $y(x+jh)$ and $y'(x+jh)$, about x, we have

For $k=1$

$\mathcal{L}\left[y(x);h\right]=\left(\alpha_0+\alpha_1\right)y(x)+\left(\alpha_1-\beta_0-\beta_1\right)hy'(x)+\frac{1}{2!}\left(\alpha_1-\beta_1\right)h^2y''(x)+\left(\frac{1}{3!}\alpha_1-\frac{1}{2!}\beta_1\right)h^3y'''(x)+O\left(h^4\right)$

For $k=2$,

$\mathcal{L}\left\{y\left(x\right);h\right\}=\left(\alpha_1+\alpha_2\right)y(x)+\left(\alpha_1+2\alpha_2-\beta_0-\beta_1-\beta_2\right)hy'(x)+\left(\frac{1}{2!}\alpha_1+2\alpha_2-2\beta_1-2\beta_2\right)h^2y''(x)+\left(\frac{1}{6}\alpha_1+\frac{4}{3}\alpha_2-\frac{1}{2!}\beta_1-2\beta_2\right)h^3y'''(x)+O\left(h^4\right)$

$=\left(\alpha_1+\alpha_2\right)y(x)+\left(\alpha_1+2\alpha_2-\beta_0-\beta_1-\beta_2\right)hy'(x)+\left[\frac{1}{2!}\left(\alpha_1+2^2\alpha_2\right)-\left(\beta_1+2\beta_2\right)\right]h^2y''(x)+\left[\frac{1}{3!}\left(\alpha_1+2^3\alpha_2\right)-\frac{1}{2!}\left(\beta_1+2^2\beta_2\right)h^3y'''(x)+O\left(h^4\right)\right.$

For $k=3$,

$\mathcal{L}\left\{y(x);h\right\}=\left(\alpha_2+\alpha_3\right)y(x)+\left(2\alpha_2+3\alpha_3-\beta_0-\beta_1-\beta_2-\beta_3\right)hy'(x)$
$+\left[\frac{1}{2!}\left(2^2\alpha_2+3^2\alpha_3\right)-\beta_1+2\beta_2+2^2\beta_3\right]h^2y''(x)+\left[\frac{1}{3!}\left(2^3\alpha_2+3^3\alpha_3\right)-\frac{1}{2!}\left(\beta_1+2^2\beta_2+2^3\beta_3\right)\right]h^3y'''(x)+O\left(h^4\right)$

In general, for $j=k$, we have

$\mathcal{L}\left\{y(x);h\right\}=\sum_{j=k-1}^{k}\alpha_j y(x)+\sum_{j=k-1}^{k}j\alpha_j y'(x)-h\sum_{j=0}^{k}\beta_j y'(x)$
$+\left\{\sum_{j=2}^{k}\frac{1}{q!}j^q\alpha_j-\sum_{j=1}^{k}\frac{1}{(q-1)!}j^{q-1}\beta_j\right\}h^q y^q(x)+\dots$

$$\mathcal{L}\left[y\left(x\right);h\right]=c_0y\left(x\right)+c_1hy^{'}\left(x\right)+c_2\mathrm{h}^2y^{''}\left(x\right)+\cdots+c_qh^qy^{(q)}\left(x\right)+\dots \tag{2.24}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 1 - 14
https://doi.org/10.6084/m9.figshare.21505032

where $c_q,\ q = 0, 1, 2, \ldots$ are constants in terms of α_j *and* β_j

and
$$\left.\begin{array}{c} c_0 = \sum_{j=k-1}^{k} \alpha_j \\ c_1 = \sum_{j=k-1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \\ \vdots \\ c_q = \left\{ \sum_{j=2}^{k} \frac{1}{q!} j^q \alpha_j - \sum_{j=1}^{k} \frac{1}{(q-1)!} j^{q-1} \beta_j \right\},\ q = 2, 3 \end{array}\right\} \tag{2.25}$$

By Lambert [1], a numerical method is said to have order q if $c_0 = c_1 = c_2 = \cdots = c_q = 0$ and $c_{q+1} \neq 0$. c_{q+1} is the error constant and $c_{q+1} h^{q+1} y^{q+1}(x_n)$ is the principal local truncation error of BIAM at the point x_n.
Following the description in (2.25), the error constant c_{q+1} and the order q of the class of BIAM are presented in table 1

Table 1: Error constants and orders of BIAM

k	c_{q+1} **for BIAM**	q **of BIAM**
2	$\left(-\frac{5}{18}\ ,\ -\frac{2}{9}\right)^T$	$(3\ ,\ 3)^T$
3	$\left(\frac{1}{90}\ ,\ -\frac{11}{720}\ ,\ -\frac{19}{720}\right)^T$	$(4\ ,\ 4\ ,\ 4)^T$
4	$\left(-\frac{3}{160}\ ,\ -\frac{1}{756}\ ,\ -\frac{11}{1440},\ -\frac{3}{160}\right)^T$	$(5\ ,\ 5\ ,\ 5\ ,\ 5)^T$

2.2.2 Consistency of Class of BIAM

Since the order q of the class of BIAM is greater than 1, then it is consistent.

2.2.3 Zero Stability of K-Step BIAM

The class of BIAM is represented by block matrix finite difference equation given by
$$P^{(1)} Y_{m+1} = P^{(0)} Y_m + hQ^{(1)} F_{m+1} + hQ^{(0)} F_m \tag{2.26}$$
where $P^{(0)},\ \ P^1\ ,\ Q^{(0)},\ \ Q^1$are $k \times k$ matrices and $Y_{m+1} = (y_{n+1}\ ,\ y_{n+2}, dots, y_{n+k})^T$,
$Y_m = (y_{n-k+1}\ ,\ \ \ldots, y_{n-1},\ y_n)^T,\ \ F_{m+1} = (f_{n+1}\ ,\ f_{n+2},\ \ldots, f_{n+k})^T,\ F_m = (f_{n-k+1}\ ,\ \ \ldots, f_{n-1},\ f_n)^T$

Definition 2.1 (Lambert [1]) A block multi step method is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple.
That is $\mu(R) = det\left[RA^{(i)} - A^{(0)}\right] = 0$ and $|R_i| \leq 1$
When $k = 2$,
We have
$P^{(0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\ ,\ \ P^{(1)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\ ,\ Q^{(0)} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{12} \\ -\frac{2}{3} & -\frac{5}{12} \end{bmatrix}\ ,$
$Q^{(1)} = \begin{bmatrix} 0 & \frac{5}{12} \\ 1 & \frac{1}{12} \end{bmatrix},$
Now $RP^{(1)} - P^{(0)} = \begin{bmatrix} -R & -1 \\ -R & R \end{bmatrix}$ det $\left[RP^{(1)} - P^{(0)}\right]\ = -R - R^2$ so that,
$det\left[RP^{(1)} - P^{(0)}\right] = 0 \longrightarrow R = -1.$
Consequently, $|R| = 0$ or $|R| = 1$ is simple. Hence, the 2-step BIAM is zero stable.

When $k = 3$;
The following Matrices are obtained
$$P^{(1)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad P^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^{(0)} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & 0 & -\frac{1}{24} \\ 0 & 0 & -\frac{1}{24} \end{bmatrix} \quad Q^{(1)} = \begin{bmatrix} \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{13}{24} & \frac{13}{24} & -\frac{1}{24} \\ \frac{5}{24} & -\frac{19}{24} & -\frac{3}{8} \end{bmatrix}$$
So that $RP^{(1)} - P^{(0)} = \begin{bmatrix} 0 & -R & -1 \\ R & -R & 0 \\ 0 & -R & R \end{bmatrix}$ and $det\ \left[RP^{(1)} - P^{(0)}\right] = 0 \quad \Longrightarrow \quad R = 0,\ \ 0,\ -1.$
Consequently, $|R| = 0,\ 0,\ $ or $|R| = 1$. Therefore, the 3-step BIAM is zero stable.
When $k = 4$,
we obtained the following matrices of order 4

$$P^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad P^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$Q^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 27/80 \\ 0 & 0 & 0 & -1/90 \\ 0 & 0 & 0 & 11/720 \\ 0 & 0 & 0 & 19/720 \end{bmatrix} \quad Q^{(1)} = \begin{bmatrix} -\frac{51}{40} & -\frac{9}{10} & -\frac{21}{40} & \frac{3}{80} \\ -\frac{17}{40} & -\frac{19}{15} & -\frac{17}{45} & \frac{1}{90} \\ \frac{37}{360} & -\frac{19}{30} & -\frac{173}{40} & \frac{19}{1720} \\ \frac{53}{360} & -\frac{11}{30} & \frac{323}{360} & \frac{251}{720} \end{bmatrix}$$

So that $det\left[RP^{(1)} - P^{(0)}\right] = 0 \implies R = 0,\ 0,\ 0,\ -1$ and consequently $|R| = 0,\ 0,\ 0$ or $|R| = 1$ is simple. Therefore, the 4-step BIAM is zero stable

2.2.4 Convergence of BIAM for k = 2, 3, 4

The k-Step block implicit Adams method is consistency based on Table 1, since it has order $q = 3 > 1$ for $k = 2, q = 4 > 1$ for $k = 3$ and $q = 5 > 1$ for $k = 4$. Since the necessary and sufficient conditions for convergence are consistency and zero stability, it follows that k-BIAM is convergent for $k = 2, 3, 4$ based on Henrici [22] assertion.

2.2.5 Stability of BIAM, for k = 2,3,4

Definition 2.2 (Enright[12]): A numerical method is said to be A stable iff its region of absolute stability contains the whole of the left hand of the complex plane. To analyze the linear stability of 2 step BIAM, the block method is applied to the test equation $y' = \lambda y$ and by letting $\tau = \lambda h$, we obtain

$$Y_{m+1} = \mu(\tau) Y_m \tag{2.27}$$

where

$$\mu(\tau) = \frac{P^{(0)} + \tau Q^{(0)}}{P^{(1)} - \tau Q^{(1)}}$$

is called a stability matrix whose latent roots are 0 and $-\frac{\tau^2+\tau-3}{\tau^2+3\tau+3}$. The region of absolute stability of 2 step BIAM is plotted using the non-zero latent root of the stability matrix as shown in Figure 1

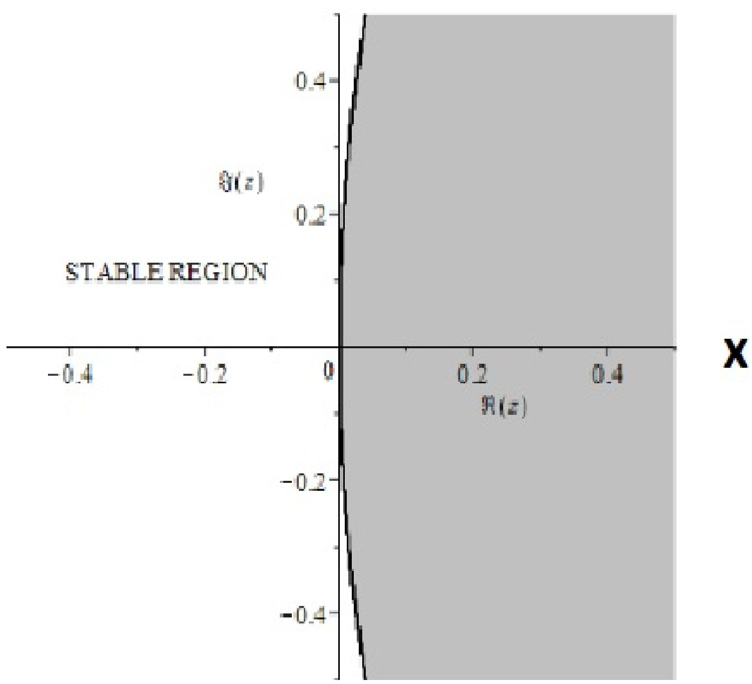

Figure 1: Region of Stability of 2 step BIAM

Hence, the 2 step BIAM is A stable.
By similar approach for k = 3, we obtain $Y_{m+1} = \mu(\tau) Y_m$
where non zero latent root of the matrix $\mu(\tau) = \frac{P^{(0)}+\tau Q^{(0)}}{P^{(??)}-\tau Q^{(??)}}$ is given by

$$\mu() = \frac{3\tau^3 + 2\tau^2 - 9\tau + 12}{3\tau^3 + 11\tau^2 + 18\tau + 12}$$

which is used to plot the region of absolute stability as shown in figure 2

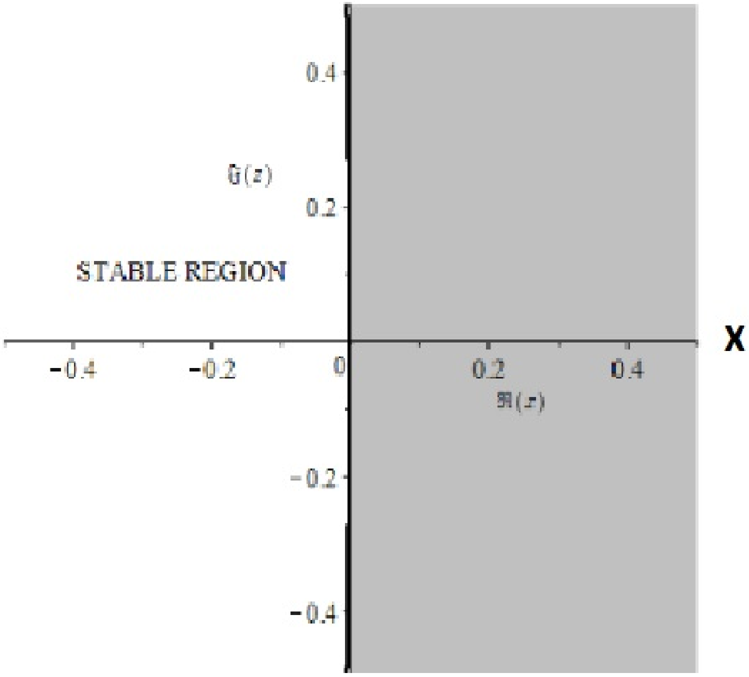

Figure 2: Region of Absolute Stability (RAS) for 3 step BIAM
Thus, the 3-step BIAM is A stable.
For k = 4
we obtain $Y_{m+1} = \mu(\tau) Y_m$
where non zero latent root of the matrix $\mu(\tau) = \frac{P^{(0)}+\tau Q^{(0)}}{P^{(??)}-\tau Q^{(??)}}$ is given by

$$\mu(\tau) = -\frac{36\tau^4 + 18\tau^3 - 91\tau^2 + 248\tau - 180}{3\left(12\tau^4 + 50\tau^3 + 105\tau^2 + 120\tau + 60\right)}$$

which is used to plot the region of absolute stability as shown in figure 3

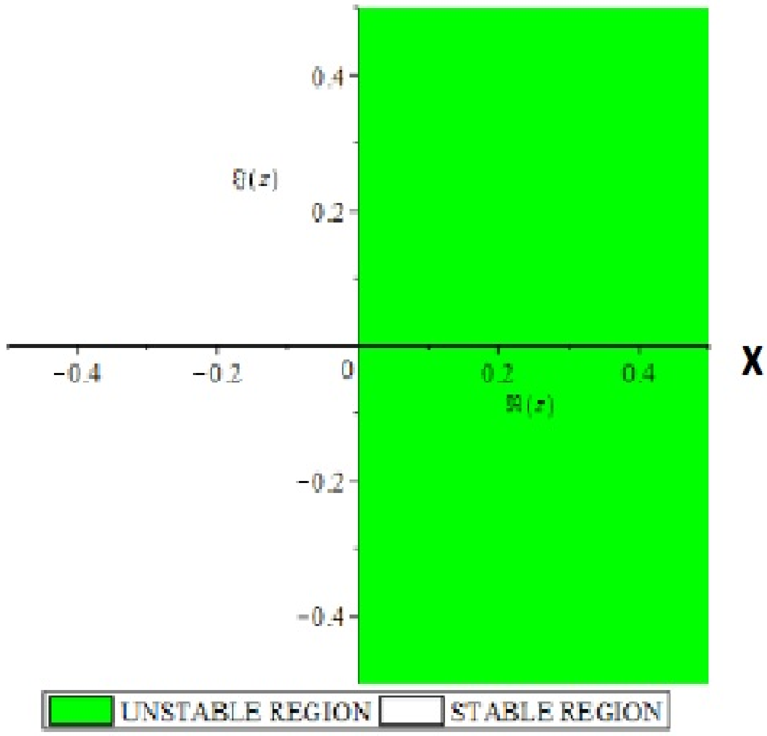

Figure 3: Region of Absolute Stability (RAS) for 4 step BIAM

3.0 Test Problems

In this section of the paper, the k-step BIAM, k= 2,3,4 are tested on a variety of initial value problems of ordinary differential equations. The efficiency and accuracy of the proposed methods on tested problems are shown in tabular form. All the absolute error of the approximate solution are given by $e_j = \max\limits_{a \le x \le b} |y_j(x) - y_j(x_j)|$, $j = 1, 2...., n$

Where $y_j(x)$ and $y_j(x_j)$ are the exact solution and the computed value at each grid point.

Problem 1

Consider the following system of nonlinear initial value problem [24]

$$\left.\begin{array}{l} y_1' = -1002y_1 + 1000y_2^2, \qquad 0 \le x \le 1 \\ y_2' = y_1 - y_2\,(1 + y_2) \end{array}\right\} \tag{3.2}$$

with initial conditions

$$y_1\,(0) = y_2\,(0) = 1 \tag{3.3}$$

whose solution in closed form is given by

$$y_1 = e^{-2x} \quad , \quad y_2 = e^{-x} \tag{3.4}$$

The results for this problem are displayed in Tables 2 and 3.

Table 2: Results of Problem 1 with $h = 0.01$

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 1 - 14
https://doi.org/10.6084/m9.figshare.21505032

h	Exact y_1 y_2	2-Step BIAM y_1 y_2	3-Step BIAM y_1 y_2	4-Step BIAM y_1 y_2
0.1	0.81873075 0.90483742	0.81872370 0.90483379	0.81873031 0.90483720	0.81873072 0.90483740
0.2	0.67032005 0.81873075	0.67032500 0.81873112	0.67032000 0.81873068	0.67032004 0.81873075
0.3	0.54881164 0.74081822	0.54880652 0.74081559	0.55881089 0.74081793	0.54811606 0.74081821
0.4	0.44932896 0.67032005	0.44933276 0.67032064	0.44932848 0.67031962	0.44932902 0.67032005
0.5	0.36787944 0.60653066	0.36787550 0.60652879	0.36787903 0.60653037	0.36787942 0.60653065
0.6	0.30119421 0.54881164	0.3011987 0.54881237	0.30119387 0.54881121	0.30119422 0.54881163
0.7	0.24659696 0.49658530	0.24659375 0.49658400	0.24659641 0.49658450	0.24659694 0.49658530
0.8	0.20189652 0.44932896	0.20190123 0.44932977	0.20189623 0.44932858	0.20189660 0.44932897
0.9	0.16529889 0.40656966	0.16529613 0.40656877	0.16529829 0.40656918	0.16529887 0.40656966
10	0.13533528 0.36787944	0.13534002 0.36788026	0.13533497 0.36787892	0.13533530 0.36787944

IJMSO

INTERNATIONAL JOURNAL OF MATHEMATICAL SCIENCES AND OPTIMIZATION: THEORY AND APPLICATIONS
VOL. 8, NO. 2, PP. 1 - 14
HTTPS://DOI.ORG/10.6084/M9.FIGSHARE.21505032

Table 3: Absolute Error of Problem 1 with $h = 0.01$

h	2-Step BIAM		3-Step BIAM		4-Step BIAM	
	Err1	Err2	Err1	Err2	Err1	Err2
0.1	7.05×10^{-6}	3.63×10^{-6}	4.39×10^{-7}	2.21×10^{-7}	3.15×10^{-8}	1.53×10^{-8}
0.2	2.45×10^{-6}	3.67×10^{-7}	4.17×10^{-8}	7.15×10^{-8}	7.57×10^{-9}	7.59×10^{-9}
0.3	5.12×10^{-6}	2.67×10^{-6}	7.47×10^{-7}	2.88×10^{-7}	3.00×10^{-8}	1.31×10^{-8}
0.4	3.79×10^{-6}	6.00×10^{-7}	4.85×10^{-7}	4.23×10^{-7}	5.80×10^{-8}	3.53×10^{-10}
0.5	3.94×10^{-6}	1.87×10^{-6}	4.10×10^{-7}	2.88×10^{-7}	2.30×10^{-8}	6.83×10^{-9}
0.6	4.46×10^{-6}	7.37×10^{-6}	3.44×10^{-7}	4.25×10^{-7}	8.41×10^{-9}	2.08×10^{-9}
0.7	3.22×10^{-6}	1.31×10^{-6}	5.53×10^{-7}	5.05×10^{-7}	2.32×10^{-8}	6.00×10^{-9}
0.8	4.72×10^{-6}	8.04×10^{-7}	2.86×10^{-7}	3.86×10^{-7}	7.72×10^{-8}	4.69×10^{-9}
0.9	2.76×10^{-6}	8.86×10^{-7}	5.94×10^{-7}	4.71×10^{-7}	2.05×10^{-8}	2.36×10^{-9}
1.0	4.74×10^{-6}	8.22×10^{-7}	3.17×10^{-7}	5.15×10^{-7}	1.31×10^{-8}	5.76×10^{-10}

From Table 2 and 3, it is observed that the 4-step BIAM of order 5 is more accurate than other members of the derived class. This is expected since 4 step-BIAM has the highest order.

Test Problem 2

Consider the system of differential equations [25]

$$y'(x) = A \ . y(x) \tag{3.5}$$

Where $y(0) = (1,\ 0,\ -1)^T$, $\quad A = \begin{pmatrix} -20 & -0.25 & -19.75 \\ 20 & -20.25 & 0.25 \\ 20 & -19.75 & -0.25 \end{pmatrix}$

and its exact solution is

$$y_1 = \frac{1}{2}\left(e^{-0.5x} + e^{-20x}\left(\cos 20x \ + \sin 20x \ \right)\right)$$

$$y_2 = \frac{1}{2}\left(e^{-0.5x} - e^{-20x}\left(\cos 20x \ - \sin 20x \ \right)\right)$$

$$y_3 = -\frac{1}{2}\left(e^{-0.5x} + e^{-20x}\left(\cos 20x \ - \sin 20x \ \right)\right)$$

Then, (3.5) is integrated by each member of the class of BIAM constructed and the results are presented in Tables 4 and 5

Table 4 Results of Problem 2

x		Exact Solution	2-Step BIAM	3-Step BIAM	4-Step BIAM
10	y_1	3.36897×10^{-3}	3.368976×10^{-3}	3.368976×10^{-3}	3.368974×10^{-3}
	y_2	3.36897×10^{-3}	3.368976×10^{-3}	3.368976×10^{-3}	3.368974×10^{-3}
	y_3	-3.36897×10^{-3}	-3.368976×10^{-3}	-3.368976×10^{-3}	-3.368974×10^{-3}
20	y_1	2.26999×10^{-5}	2.269999×10^{-5}	2.269997×10^{-5}	2.269997×10^{-5}
	y_2	2.26999×10^{-5}	2.269999×10^{-5}	2.269997×10^{-5}	2.269997×10^{-5}
	y_3	-2.26999×10^{-5}	-2.269999×10^{-5}	-2.269997×10^{-5}	-2.269997×10^{-5}
30	y_1	1.52951×10^{-7}	1.529515×10^{-7}	1.529510×10^{-7}	1.529512×10^{-7}
	y_2	1.52951×10^{-7}	1.529515×10^{-7}	1.529510×10^{-7}	1.529512×10^{-7}
	y_3	-1.52951×10^{-7}	-1.529515×10^{-7}	-1.529510×10^{-7}	-1.529512×10^{-7}

Table 5 Absolute Error of Problem 2

x		2-Step BIAM	3-Step BIAM	4-Step BIAM
10	y_1	2.44×10^{-9}	1.23×10^{-9}	9.00×10^{-11}
	y_2	2.44×10^{-9}	1.23×10^{-9}	9.10×10^{-11}
	y_3	2.44×10^{-9}	1.23×10^{-9}	8.90×10^{-11}
20	y_1	3.28×10^{-11}	1.63×10^{-11}	1.10×10^{-12}
	y_2	3.28×10^{-11}	1.63×10^{-11}	1.09×10^{-12}
	y_3	3.28×10^{-11}	1.62×10^{-11}	1.12×10^{-12}
30	y_1	3.33×10^{-13}	1.65×10^{-13}	1.13×10^{-14}
	y_2	3.33×10^{-13}	1.65×10^{-13}	1.13×10^{-14}
	y_3	3.33×10^{-13}	1.65×10^{-13}	1.13×10^{-14}

From Tables 4 and 5, it can be seen that the class of 4-step BIAM is more accurate as its absolute error is the least .

Test Problem 3

Lastly, we consider the first order oscillatory initial value problem of the form [26]

$$y' = -sinx - 200\,(y - cosx)\,, y\,(0) = 0, \qquad 0 \le x \le 1 \tag{3.6}$$

whose solution in closed form is given by

$$y\,(x) = \cos x\ - \exp(-200x). \tag{3.7}$$

The use of block integrator to solve some stiff and oscillatory problem gives a solution that oscillates and grows exponentially in x. Thus,each member of the class of BIAM constructed is used to integrate (3.6) for $h = 0.001$ in the interval $0 \le x \le 1$and the results are presented in Tables 6 and 7.

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 8, No. 2, pp. 1 - 14
https://doi.org/10.6084/m9.figshare.21505032

Table 6 Results of Problem 3 for $h = 0.001$

h	Exact Solution	2-Step BIAM	3-Step BIAM	4-Step BIAM
0.001	0.1812687469	0.1813181814	0.1812744	0.1812694732
0.002	0.3296779540	0.3296683299	0.329679274	0.3296795
0.003	0.4511838639	0.4512091218	0.4511907192	0.451184402
0.004	0.5506630359	0.5506501334	0.5506717508	0.550662735
0.005	0.6321080588	0.6321197082	0.6321133789	0.6321081381
0.006	0.6987877882	0.6987748149	0.6987953126	0.6987877176
0.007	0.7533785362	0.7533828048	0.7533863985	0.7533786146
0.008	0.7980714822	0.7980598874	0.7980769238	0.7980712111
0.009	0.8346606121	0.8346611004	0.8346668062	0.8346605366
0.010	0.8646147172	0.864605002	0.864620722	0.8646145949

Table 7 Absolute Error of Problem 3 for $h = 0.001$

h	2-Step BIAM	3-Step BIAM	4-Step BIAM
0.001	4.94×10^{-5}	4.94×10^{-5}	7.26×10^{-7}
0.002	9.62×10^{-6}	9.62×10^{-6}	2.96×10^{-7}
0.003	2.52×10^{-5}	2.53×10^{-5}	5.43×10^{-7}
0.004	1.29×10^{-5}	1.29×10^{-5}	3.01×10^{-7}
0.005	1.16×10^{-5}	1.16×10^{-5}	7.93×10^{-8}
0.006	1.30×10^{-5}	1.30×10^{-5}	7.06×10^{8}
0.007	4.27×10^{-6}	4.27×10^{-6}	7.84×10^{-8}
0.008	1.16×10^{-7}	1.16×10^{-5}	2.71×10^{-7}
0.009	4.88×10^{-7}	4.88×10^{-7}	1.22×10^{-7}
0.01	9.72×10^{-6}	9.72×10^{-6}	1.76×10^{-12}

Table 6 and 7 show better approximate solution by the class of BIAM for Eq. (3.6).

5.0 Conclusion

It is shown in this paper that a class of A − stable block implicit Adams method based on the interpolation and collocation of Euler polynomial as basis function is efficient and more accurate. The derived schemes of k - step BIAM all preserves the properties of linear multistep methods with good convergence, consistency and stability properties. Numerical experiments showed that the newly developed methods are accurate when compared to the exact solution and some existing methods mentioned in the literature. It must however be noted that 4-step BIAM is most accurate among the other members of the derived class. All the derivations and investigations are carried out with the guide of composed codes in Maple 2016.1 Programming worked in Windows 8.1

Acknowledgment

The authors wish to appreciate the referees for their contributions.

Funding
Non received

Conflict of Interest The authors declare that there is no conflict of interest regarding the publication of the paper.

References

[1] Lambert, J. D. Computational Methods in Ordinary Differential Equations. John Wiley and Sons, New York.1973.

[2] Hirzebrush, F, Eulerian Polynomials: Musister Journal of Mathematics, **vol. 1**, pp. 9-14, 2008.

[3] Anake, T. A., Awoyemi, D. O.& Adesanya, A. O. One step Implicit Hybrid Block Method for the Direct Solution of General Second Order Ordinary Differential Equations IAENG International Journal of Applied Mathematics, **vol. 42(4)**, pp. 1-5, 2012..

[4] Fox, L. & Parker, I.B. "Chebyshev polynomials in numerical analysis". Addison-Wesley, Phillipines. 1968

[5] Adeniyi, R. B. & Alabi, M. O, Derivation of Continuous Multistep Method Using Polynomial basis function, Abacus, **vol. 33**, pp. 351-361, 2006.

[6] Sunday, J; James, A.A; Odekunle, M.R & Adesanya, A.O. Chebyshevian basis function - type block method for the solution of first order initial value problem with oscillatory solution. Journal of Mathematics and Computer Science **5(4)**, 462 - 472 (2015).

[7] Okunuga, A. & Ehigie, J. A New Derivative of Continuous Multistep Methods Using Power Series as Basis Function. Journal of Modern Mathematics and Statistics, **vol. 3(2)**, pp. 43-50, 2009.

[8] Akinfenwa, O., Jator, S.& Yao, N. An Eight order backward differentiation formula with continuous coefficients for stiff ordinary differential equations. Int. J. Math. Comput. Sci. **vol. 17(4)**, pp. 160-165, 2011.

[9] Abioyar, T., Luga, T, & Iyerter, B. V. (Derivation of Continuous Linear Multistep Methods Using Hermite Polynomials as Basis Functions. American Journal of Applied Mathematics and Statistics, **vol. 3(6)**, pp. 220-225, 2015.

[10] Ramos,H,& Singh, G. A tenth order A - stable two step hybrid block method for solving initia value problems of ODE. Applied Mathematics and Computation, **vol. 310**, pp. 75-88, 2017.

[11] Osilagun, J.A., Adesanya, A.O., Anake, T.A ,& Oghoyon, G.J, Four Steps Implicit Method for the Solution of General Second Order Ordinary Differential Equations. Journal of Natural Sciences, Engineering and Technology, **vol. 8(1)**, pp. 52-61, 2009.

[12] Enright,W.H. Second derivative multi-step method for shift Ode's. SIAM J. Numerical Analysis, **vol.11(2)**, pp. 321-331, 1974.

[13]]Adeniyi, R.B, Adeyesa, E.O & Alabi, M.O. A continuous formulation of some classical initial value solvers by non-perturbed multi-step. Collocation approach using Chebyshev polynomials as basis function. J. of Nig. Association of mathematical physics, **vol. 10**, pp. 261-274, 2006

[14] Yahaya, Y.A & Kumleng, G.M. Construction of two-step block Simpson type method with large region of absolute stability, **vol. 11**, pp. 261-268, 2007

[15] Abdulganiy, R.I., Akinfewa, O.A & Okunuga S.A. A family of Lo-Stable third derivative block methods for solving system of first order initial value problems. Journal of Nigerian Association of Mathematical Physics, **vol. 36**, pp. 47-54, 2016.

[16] Ngwane, F.F & Jator, S.N .Block Hybrid-Second Derivative Method For Stiff Systems. International Journal of Pure and Applied Mathematics, **vol. 80**, pp. 543-559, 2012.

[17] Akinfenwa, O.A., Abdulganiy, R.I., Akinnukawe,B.I & Okunuga, S.A (2020), Seventh Order Hybrid Block Method for Solution of First Order Stiff System of Initial Value Problems. Journal of the Egyptian Mathematical Society, **vol. 28**, pp. 1-11, 2020.

[18] Akinfenwa, O.A., Okunuga, S.A., Akinnukawe, B.I., Rufai, U.O. & Abdulganiy, R.I, Multiderivative Hybrid Implicit Runge-Kutta Method for Solving Stiff System of a First Order Differential Equations. Pushpa Publishing House, **vol. 106(2)**, pp. 543-562, 2018.

[19] Akinfenwa,O.A., Abdulganiy, R.I., Okunuga, S.A , & Irechukwu, V. Simpson's $\frac{3}{8}$ Type Block Method for stiff systems of Ordinary Differential Equations. Journal of the Nigeria Mathematical society, **vol. 36(3)**, pp. 503-514, 2017.

[20] Sahi, R.K, Jator,S.N & Khan, N.A .A Simpson's-Type Second Derivative Method For Stiff Systems. International Journal of Pure and Applied Mathematics, **vol. 81**, pp. 619-633, 2012.

[21] Alabi, M.O, Adesanya, A.O & Babatunde, O.Z. .Formulations of Continuous Method for Direct Solution of Third and Fourth order ODE Using Chebyshev polynomials as the Basis Function. Journal of Modern Mathematics and Statistics, **vol. 2(4)**, pp. 146-152, 2008.

[22] Henric, P .Discrete variable methods in ordinary differential equations. Wiley & Sons, New York (1962)

[23] Adeniyi, R.B.,& Alabi, M.O."Derivation of continuous multistep methods using Chebyshev basis functions". Journal of the Mathematical Association of Nigeria, **Vol. 33**, pp. 351-361, 2006.

[24] Vaquero, J.M,& Vigo-Agular; Exponential fitted Runge - Kutta methods of collocation type based on Gauss, Radan, and Labatto traditional methods. Proceedings of the International Conference on Computational and Mathematical methods in Science and Engineering (CMMSE' 07) 289

[25] Ezzeddine, A.K & Hojjatti, G. Third Derivative multistep methods for stiff systems. International Journal of Nonlinear Sciences, **4(4)**, 443 - 450 (2012).

[26] Sunday, J; Skwane, Y & Odekunle, M.R (2013). Acontinuous Block Integrator for the Solutionof the Stiff and Oscillatory Differential Equations. IOSR Journal of Mathematics, **8(3)**, 75 - 80 (2013).