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SOLUTION TO NONLINEAR WAVE AND EVOLUTION EQUATIONS BY DIRECT ALGEBRAIC METHOD

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Abstract

This work presents a dynamic and systematic step-by-step method for constructing solitary wave solutions for non-dissipative nonlinear evolution and wave equations from the real exponential solutions of the underlying linear equations. The work reviewed the direct algebraic method initially proposed by Hereman et. al. (1985) and employed the methodology in solving Benjamin - Bona - Mahony (RLW) equation and Joseph - Egri (TRLW) equation. By the method which involves using a traveling frame of reference to convert the PDE into an ODE and solving the ODE by algebraic processes; we obtained solutions for the Benjamin - Bona - Mahony and the Joseph - Egri equations. This method was found to be efficient in constructing a single solitary wave solution for non-dissipative evolution equations. The results obtained in this paper are in agreement with those obtained using other methods.

Keywords: Nonlinear wave equation, Nonlinear evolution equation, .
MSC2010: 76D07.

1 Introduction

Hereman et.al. (1985) proposed a direct algebraic method for solving nonlinear wave and evolution equations and demonstrated its applicability to the KdV and Burgers Equations. The method tries to construct single solitary wave solutions for non-dissipative nonlinear partial differential equations (PDE's). However, for over a decade, the method has been largely neglected, probably because it has been difficult to understand or due to the complexity of the methodology. This paper presents a more detailed physically transparent, straightforward step-by-step technique, with necessary justifications for every step for constructing such single solitary wave solutions for non-dissipative nonlinear partial differential equations (PDE's) and shows its applicability to other nonlinear wave and evolution equations using Benjamin - Bona - Mahony (RLW) and Joseph - Egri (TRLW) equations as examples. Some work has been done on nonlinear wave equations. The Regularized Long-Wave (RLW) equation of Benjamin - Bona - Mahony is a very important equation.

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As reported by Dereli [1] , the equation was first proposed by Peregrine to describe the undular bore development and later studied by Benjamin, Bona & Mahony [2] as an improvement of the Korteweg-de Vries equation (KdV equation) for modelling long surface gravity waves of small amplitude - propagating uni-directionally in 1+1 dimensions. Benjamin et.al. [2] showed the stability and uniqueness of solutions to the Benjamin - Bona - Mahony, BBM equation. This contrasts with the KdV equation, which is unstable in its high wavenumber components. Furthermore, while the KdV equation has an infinite number of integrals of motion, the BBM equation has only three [2]. The BBM equation possesses solitary wave solutions of a peculiar form. In 1977, Joseph and Egri proposed a further modification to the KdV equation. The equation became known as Joseph - Egri equation (or Time Regularized Long Wave equation - TRLWE) [3]. According to Hereman et.al. [4], the equation has a solitary wave solution. A number of research work has been done on non-linear wave equations using varying methods and approach [5–12]. Method of reduction of PDE to ODE to obtain solution has also been studied by [13,14]. The rest of the paper is organized as follows: Section 2 discusses equations (essential mathematical tools) used in the work. Section 3 presents the methodology. Section 4 is the application and results obtained. Section 5 concludes the work.

2 MATHEMATICAL TOOLS

In this section we present the RLW equation and the TRLW equation as Mathematical tools for this work. We also give a step-by-step approach to the improved methodology proposed in the work.

2.1 Benjamin - Bona - Mahony (RLW) Equation

The Benjamin - Bona - Mahony equation, also known as the Regularized Long-Wave (RLW) equation is the partial equation

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

A major mathematical tool is the solitary wave solution of the BBM equation which is of the form:

$$u = 3\frac{c^2}{1-c^2}sech^2\frac{1}{2}\left(cx - \frac{ct}{1-c^2} + \delta\right),$$

where *sech* is the hyperbolic secant function and δ is a phase shift (by an initial horizontal displacement)

2.2 Joseph - Egri (TRLW) Equation

Another important Mathematical tool is the TRLW equation, (a modification of the KdV equation) given as follows:

$$U + U_t + UU_x + U_{xtt} = 0.$$

This equation is useful for the following reasons:
The system is conservative since it can be derived from the Lagrangian density

$$L = \frac{1}{2}\theta_x\theta_t + \frac{1}{2}\theta_x^2 + \frac{1}{2}\theta_x^3 - \frac{1}{2}\theta_{xt}^2$$

$$\text{where} \quad \theta_x = U$$

For large wavenumbers $\mid k \mid$, the infinitesimal-wave phase speed falls off like $\frac{1}{|k|}$, in accord with physical intuition. Since the equation is of second order in t, both U and U_t can be independently specified for $t = 0$. The equation became known as Joseph - Egri (or Time Regularized Long Wave equation - TRLWE) [3]. According to Hereman et.al. [4] the equation has a solitary wave solution:

$$u(x,t) = \frac{3}{\alpha}(v-1)sech^2\left[\frac{\sqrt{v-1}}{2v}(x-vt)+\delta\right]$$

3 METHODOLOGY

Here, we present the methodology used in this work. The process of the solution is given below:

(i) Starting with the non-linear equation in 1+1 dimensions with x and t as the space and time co-ordinates respectively, we introduced a travelling frame of reference by $\varepsilon = x - vt$. This transforms the given non-linear PDE in $u(x,t)$ into an ODE in $\phi(\varepsilon) \triangleq u(x,t)$. Then,

(ii) We integrated the ODE with respect to ε as many times as possible avoiding integral equations.

(iii) We substituted $\phi = c_1 + \hat{\phi}$ to obtain the most general solitary solution, possibly having a constant term, c_1.

(iv) Then we considered the linear part of the equation in $\hat{\phi}$, by setting the coefficient(s) of the non-linear term(s) equal to zero or simply neglecting them. By setting $\hat{\phi} = e^{k\varepsilon}$ in the linear equation, the values of k and the constant term c_1 are obtained, by setting the $\hat{\phi}$ - *independent* part equal to zero. Then we substituted these values back into the equation formed in (iii) above.

(v) For mathematical convenience, we normalized a few coefficients of the nonlinear terms by a single scaling transformation of $\hat{\phi}$ into $\tilde{\phi}$.

(vi) Then we solved the nonlinear equation in $\tilde{\phi}$. First by expanding $\tilde{\phi}$ in terms of the harmonics of the decaying exponential solution of the linear equation. By setting $g(x) = e^{-k\varepsilon}$ and $\tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n(\varepsilon)$, and applying
Cauchy's rule for products, the recursion relation for a_n's is obtained.

(vii) We now solved the recursion relation to find the general form of the coefficients a_n. This we did through direct algebraic process/computation.

(viii) We substituted the coefficients, a_n, obtained from the recursion relation described above, into $\tilde{\phi}$ obtained above, and used the scaling factor $\phi = c_1 + \hat{\phi}$, to obtain the solution ϕ.

(ix) Finally, we returned to the original dependent variable u and the independent variables x and t. An exact and solitary wave solution of the nonlinear PDE is obtained. Further insight is given to the forgoing steps demonstrating its application. For the application we use both the RLW and the Joseph - Egri equation (TRLW).

4 APPLICATION

The application is in two parts. The method of solution described in section 3 are applied to both RLW and TRLW.

4.1 Benjamin - Bona - Mahony (RLW) Equation

The Benjamin - Bona - Mahony (RLW) equation is given below as:

$$u_t + u_x + \alpha u u_x - u_{2xt} = 0$$

We present a solution using the approach/methodology given in section 3.

$$u_t + u_x + \alpha u u_x - u_{2xt} = 0 \tag{4.1}$$

To solve equation (4.1), we begin by transforming it into an ODE using the travelling frame of reference

$$\varepsilon = x - vt, \quad \phi(\varepsilon) \triangleq u(x,t)$$
$$u_t = -v\phi_\varepsilon, \quad u_x = \phi_\varepsilon \tag{4.2}$$

Similarly,

$$u_{xx} = u_{2x} = \phi_{2\varepsilon}$$
$$u_{2xt} = (u_{2x})_t = -v\phi_{3\varepsilon}$$

Substituting (4.2) into (4.1), equation (4.3) is obtained.

$$-v\phi_\varepsilon + \phi_\varepsilon + \alpha\phi\phi_\varepsilon + v\phi_{3\varepsilon} = 0 \tag{4.3}$$

Integrating (4.3) with respect to ϵ we have

$$-v\int \phi_\varepsilon d\varepsilon + \int \phi_\varepsilon d\varepsilon + \alpha \int \phi\phi_\varepsilon d\varepsilon + v\int \phi_{3\varepsilon} d\varepsilon = 0$$

$$\Rightarrow \quad -v\phi + \phi + \frac{\alpha}{2}\phi^2 + v\phi_{2\varepsilon} + c_1 C = 0, \tag{4.4}$$

where $c_1 C$ is the constant of integration.
Substituting

$$\phi = c_1 + \hat{\phi} \tag{4.5}$$

into (4.4), we have

$$-v(c_1 + \hat{\phi}) + (c_1 + \hat{\phi}) + \frac{\alpha}{2}(c_1 + \hat{\phi})^2 + v(c_1 + \hat{\phi})_{2\varepsilon} + c_1 C = 0$$

$$(-v + 1 + \alpha c_1)\hat{\phi} + v\hat{\phi}_{2\varepsilon} + \frac{\alpha}{2}\phi^2 - vc_1 + c_1 + \frac{\alpha}{2}c_1^2 + c_1 C = 0 \tag{4.6}$$

Considering the linear part of equation (4.6), by ignoring the nonlinear part and setting the $\hat{\phi}$ - independent part equal to zero (0) as follows:

$$\text{i.e.} \quad (-v + 1 + \alpha c_1)\hat{\phi} + v\hat{\phi}_{2\varepsilon} = 0$$

$$\text{But,} \quad \hat{\phi} = e^{k\varepsilon}, \quad \hat{\phi}_{2\varepsilon} = k^2 e^{k\varepsilon}$$

$$\therefore \quad (-v + 1 + \alpha c_1)e^{k\varepsilon} + vk^2 e^{k\varepsilon} = 0$$

$$-v+1+\alpha c_1+vk^2=0$$
$$\therefore \quad k^2=\frac{v-1-\alpha c_1}{v}$$

From setting the $\hat{\phi}$ - independent part equal to zero, we get

$$-vc_1+c_1+\frac{\alpha}{2}c_1^2+c_1C=0$$
$$\therefore \quad -v+1+\frac{\alpha}{2}c_1+C=0$$
$$c_1=\frac{2(v-1-C)}{\alpha} \tag{4.7}$$

Hence,

$$k^2=\frac{1}{v}\left[v-1-\alpha\left(\frac{2(v-1-C)}{\alpha}\right)\right]$$
$$=\frac{1}{v}[v-1-2v+2+2C]$$
$$\therefore \quad k=\sqrt{\frac{1-v+2C}{v}} \tag{4.8}$$

Substituting (4.7) in (4.6), we get

$$\left(-v+1+\alpha\left[\frac{2(v-1-C)}{\alpha}\right]\right)\hat{\phi}+v\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0$$
$$(v-1-2C)\hat{\phi}+v\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0 \tag{4.9}$$

Here we normalize the coefficient of the nonlinear term by the scaling

$$\hat{\phi}=\frac{2}{\alpha}(1+2C-v)\tilde{\phi} \tag{4.10}$$

Then, putting (4.10) in (4.9) yields

$$\frac{2}{\alpha}(v-1-2C)(1+2C-v)\tilde{\phi}+v\left[\frac{2}{\alpha}(1+2C-v)\tilde{\phi}\right]_{2\varepsilon}...+\frac{\alpha}{2}\left[\frac{2}{\alpha}(1+2C-v)\tilde{\phi}\right]^2=0$$
$$\Rightarrow \quad (1+2C-v)\tilde{\phi}-v\tilde{\phi}_{2\varepsilon}-(1+2C-v)\tilde{\phi}^2=0 \tag{4.11}$$

From (4.8),

$$vk^2=1+2C-v$$

Substituting into (4.11) leads to

$$vk^2\tilde{\phi}-v\tilde{\phi}_{2\varepsilon}-vk^2\tilde{\phi}^2=0$$
$$\Rightarrow \quad k^2\tilde{\phi}-\tilde{\phi}_{2\varepsilon}-k^2\tilde{\phi}^2=0 \tag{4.12}$$

We now expand $\tilde{\phi}$ in terms of the harmonics of the decaying exponential solution of the linear equation.

$$\text{i.e.} \quad g(x)=e^{-k\varepsilon}$$
$$\tilde{\phi}=\sum_{n=1}^{\infty}a_n g^n(\varepsilon)=\sum_{n=1}^{\infty}a_n e^{-nk(\varepsilon)}$$

$$\tilde{\phi}_{\varepsilon} = -k\sum_{n=1}^{\infty} n a_n g^n(\varepsilon)$$
$$\tilde{\phi}_{2\varepsilon} = k^2\sum_{n=1}^{\infty} n^2 a_n g^n(\varepsilon)$$

$$\begin{aligned}\tilde{\phi}^2 &= \left[\sum_{n=1}^{\infty} a_n g^n(\varepsilon)\right]^2\\ &= \sum_{n=1}^{\infty}\sum_{l=1}^{n-1} a_l a_{n-l} g^n(\varepsilon)\end{aligned}$$

by Cauchy's rule.
Substituting these expansions into (4.12), it is easy to see that equation (4.12) becomes

$$k^2\sum_{n=1}^{\infty} a_n g^n(\varepsilon) - k^2\sum_{n=1}^{\infty} n^2 a_n g^n(\varepsilon) - k^2\sum_{n=1}^{\infty}\sum_{l=1}^{n-1} a_l a_{n-l} g^n(\varepsilon) = 0$$

$$\left[(1-n^2)a_n - \sum_{l=1}^{n-1} a_l a_{n-l}\right] k^2 \sum_{n=1}^{\infty} g^n(\varepsilon) = 0$$

$$(n^2-1)a_n + \sum_{l=1}^{n-1} a_l a_{n-l} = 0,\ n \geq 2, \tag{4.13}$$

The recursion relation (4.13) is solved to find the general form of the coefficient a_n.
For $n = 2$,

$$(2^2-1)a_2 + \sum_{l=1}^{1} a_l a_{2-l} = 0$$
$$\therefore \quad 3a_2 + a_1 a_1 = 0$$
$$a_2 = -\frac{1}{3}a_1^2 = -\frac{6^2}{3}\left(\frac{a_1}{6}\right)^2$$
$$a_2 = -12\left(\frac{a_1}{6}\right)^2 = 6(2)(-1)^{2+1}\left(\frac{a_1}{6}\right)^2$$

For $n = 3$,

$$(3^2-1)a_3 + \sum_{l=1}^{2} a_l a_{3-l} = 0$$
$$\therefore \quad 8a_3 + a_1 a_2 + a_2 a_1 = 0$$
$$a_3 = -\frac{1}{4}a_1\left(-\frac{1}{3}a_1^2\right) = \frac{1}{12}a_1^3 = 6(3)(-1)^{3+1}\left(\frac{a_1}{6}\right)^3$$

For $n = 4$,

$$\begin{aligned}a_4 &= -24\left(\frac{a_1}{6}\right)^4\\ &= 6(4)(-1)^{4+1}\left(\frac{a_1}{6}\right)^4\end{aligned}$$

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From the above, a_1 is arbitrary and pattern can be clearly seen as:

$$a_n = 6n(-1)^{n+1}\left(\frac{a_1}{6}\right)^n, \quad a_1 > 0$$

Then the coefficient a_n are substituted into the equation for ϕ to obtain the solution .

From (4.5),

$$\phi = c_1 + \hat{\phi}$$
$$\text{But,} \quad c_1 = \frac{2(v-1-C)}{\alpha}$$
$$\therefore \quad \phi = \frac{2}{\alpha}(v-1-C) + \hat{\phi}$$

Also from (4.10),

$$\hat{\phi} = \frac{2}{\alpha}(1+2C-v)\tilde{\phi}$$
$$\therefore \quad \phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\tilde{\phi}$$
$$\text{But,} \quad \tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n(\varepsilon)$$

$$\begin{aligned}\therefore \quad \phi &= \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)... \\ &\sum_{n=1}^{\infty} a_n g^n(\varepsilon) \\ &= \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)... \\ &\sum_{n=1}^{\infty} 6n(-1)^{n+1}\left(\frac{a_1}{6}\right)^n g^n\end{aligned}$$

Let $a = \frac{a_1}{6}$

$$\therefore \quad \phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\sum_{n=1}^{\infty} 6n(-1)^{n+1}(ag)^n \tag{4.14}$$

The power series $\sum_{n=1}^{\infty} 6n(-1)^{n+1}(ag)^n$ is convergent for $ag < 1$

Using the well-known power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |\, x \,| < 1$$

Differentiating both sides with respect to x we have

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \\ \frac{d}{dx}(1-x)^{-1} &= \frac{d}{dx}\sum_{n=0}^{\infty} x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} n x^{n-1}\end{aligned}$$

We multiply both sides by x

$$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty} nx^n$$

and substitute $(-ag)$ for x

$$\frac{-ag}{(1+ag)^2}=\sum_{n=1}^{\infty} n(-ag)^n$$

$$\frac{-ag}{(1+ag)^2}=\sum_{n=1}^{\infty} n(-1)^{n+1}(ag)^n \tag{4.15}$$

Substituting (4.15) into (4.14), we have

$$\phi=\frac{2}{\alpha}(v-1-C)+\frac{2}{\alpha}(1+2C-v)\frac{6ag}{(1+ag)^2}$$

Since $g(x)=e^{-k\varepsilon}$, this leads to equation (4.16)

$$\phi=\frac{2}{\alpha}(v-1-C)+\frac{2}{\alpha}(1+2C-v)\frac{6ae^{-k\varepsilon}}{(1+ae^{-k\varepsilon})^2} \tag{4.16}$$

Subsequently from (4.16), we can obtain equation (4.17) as follows:

$$\begin{aligned}
\phi &= \frac{2}{\alpha}(v-1-C)+\frac{12}{\alpha}(1+2C-v)\left[\frac{a}{e^{k\varepsilon}}\div\left(1+\frac{a}{e^{k\varepsilon}}\right)^2\right]\\
&= \frac{2}{\alpha}(v-1-C)+\frac{12}{\alpha}(1+2C-v)\times\frac{ae^{k\varepsilon}}{a^2(1+\frac{1}{a}e^{k\varepsilon})^2}\\
&= \frac{2}{\alpha}(v-1-C)+\frac{12}{\alpha}(1+2C-v)\times\frac{\frac{1}{a}e^{k\varepsilon}}{(1+\frac{1}{a}e^{k\varepsilon})^2}\\
&= \frac{2}{\alpha}(v-1-C)+\frac{12}{\alpha}(1+2C-v)\times\frac{e^{\ln(\frac{1}{a})+k\varepsilon}}{(1+e^{\ln(\frac{1}{a})+k\varepsilon})^2}
\end{aligned} \tag{4.17}$$

Also since

$$sechx=\frac{2e^x}{1+e^{2x}}$$

Hence,

$$\frac{1}{4}sech^2x=\frac{e^{2x}}{(1+e^{2x})^2}$$

$$\text{Let}\quad x=\frac{1}{2}\left(\ln\left(\frac{1}{a}\right)+k\varepsilon\right)$$

Then,

$$\frac{1}{4}sech^2\frac{1}{2}\left[\ln\left(\frac{1}{a}\right)+k\varepsilon\right]=\frac{e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}}{\left(1+e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}\right)^2}$$

Substituting this into (4.17), the solution $u(x,t)$ is obtained

$$\phi=\frac{2}{\alpha}(v-1-C)+\frac{3}{\alpha}(1+2C-v)sech^2\left[\frac{1}{2}\ln\left(\frac{1}{a}\right)+\frac{1}{2}k\varepsilon\right]$$

$$\text{Let}\quad \partial=\frac{1}{2}\ln\left(\frac{1}{a}\right)$$

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$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{3}{\alpha}(1+2C-v)sech^2\left[\partial + \frac{1}{2}k\varepsilon\right]$$

Also

$$k = \sqrt{\frac{1-v+2C}{v}} = \left(\frac{1+2C-v}{v}\right)^{\frac{1}{2}}$$

$$\text{and} \quad \varepsilon = x - vt$$

$$\therefore \quad u(x,t) = \frac{2}{\alpha}(v-1-C) + \frac{3}{\alpha}(1+2C-v)sech^2\left[\frac{1}{2}\left(\frac{1+2C-v}{v}\right)^{\frac{1}{2}}(x-vt) + \partial\right]$$

In a special case that $C = v - 1$,

$$u(x,t) = \frac{3}{\alpha}(v-1)sech^2\left[\frac{1}{2}\left(\frac{v-1}{v}\right)^{\frac{1}{2}}(x-vt) + \partial\right]$$

The solution is therefore given as: [box=]equation* u(x,t)=3 $\overline{\alpha(v-1)sech^2\left[\frac{1}{2}\left(\frac{v-1}{v}\right)^{\frac{1}{2}}(x-vt)+\partial\right]}$

4.2 Joseph - Egri (TRLW) Equation

In this section, we present a solution to the Joseph - Egri (TRLW) equation using the method described in section 3. The TRLW equation is given below:

$$u_t + u_x + \alpha u u_x + u_{x2t} = 0$$

Solution

$$u_t + u_x + \alpha u u_x + u_{x2t} = 0 \tag{4.18}$$

We transform (4.18) into an ODE using the travelling frame of reference

$$\varepsilon = x - vt, \quad \phi(\varepsilon) \triangleq u(x,t)$$

$$u_x = \phi_\varepsilon \tag{4.19}$$

$$\text{Similarly,} \quad u_{tt} = u_{2t} = v^2\phi_{2\varepsilon}$$

$$u_{x2t} = (u_x)_{2t} = v^2\phi_{3\varepsilon}$$

Substituting (4.19) in (4.18), we get

$$-v\phi_\varepsilon + \phi_\varepsilon + \alpha\phi\phi_\varepsilon + v^2\phi_{3\varepsilon} = 0. \tag{4.20}$$

Integrating (4.20) with respect to ε, we have

$$-v\int\frac{d\phi}{d\varepsilon}d\varepsilon + \int\frac{d\phi}{d\varepsilon}d\varepsilon + a\int\phi\frac{d\phi}{d\varepsilon}d\varepsilon + v^2\int\frac{d\phi_{2\varepsilon}}{d\varepsilon}d\varepsilon = 0$$

$$-v\phi + \phi + \frac{a}{2}\phi^2 + v^2\phi_{2\varepsilon} + c_1C = 0, \tag{4.21}$$

where c_1C is the constant of integration.

By substituting

$$\phi = c_1 + \hat{\phi} \tag{4.22}$$

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into equation (4.21), it can be seen that (4.21) leads to (4.23).

$$-v(c_1+\hat{\phi})+(c_1+\hat{\phi})+\frac{a}{2}(c_1+\hat{\phi})^2+v^2(c_1+\hat{\phi})_{2\varepsilon}+c_1C=0$$

$$-vc_1-v\hat{\phi}+c_1+\hat{\phi}+\frac{a}{2}c_1^2+ac_1\hat{\phi}+\frac{a}{2}\hat{\phi}^2+v^2\hat{\phi}_{2\varepsilon}+c_1C=0$$

$$(-v+1+\alpha c_1)\hat{\phi}+v^2\hat{\phi}_{2\varepsilon}+\frac{a}{2}\hat{\phi}^2-vc_1+c_1+\frac{\alpha}{2}c_1^2+c_1C=0 \tag{4.23}$$

Considering the linear part of the equation (4.23) by ignoring the nonlinear part and setting the $\hat{\phi}$-independent part equal zero as follows

$$\text{i.e.}\quad (-v+1+\alpha c_1)\hat{\phi}+v^2\hat{\phi}_{2\varepsilon}=0$$

$$\text{But}\quad \hat{\phi}=e^{k\varepsilon},\hat{\phi}_{2\varepsilon}=k^2e^{k\varepsilon}$$

$$\therefore\quad (-v+1+\alpha c_1)e^{k\varepsilon}+v^2k^2e^{k\varepsilon}$$

$$\therefore\quad k^2=\frac{v-1-\alpha c_1}{v^2}$$

From setting the $\hat{\phi}$ - independent part to zero, we get

$$-vc_1+c_1+\frac{\alpha}{2}c_1^2c_1C=0$$

$$c_1=2\frac{(v-1-C)}{\alpha} \tag{4.24}$$

$$k^2=\frac{1}{v^2}\left[v-1-\alpha\left(\frac{2(v-1-C)}{\alpha}\right)\right]$$
$$=\frac{1-v+2C}{v}$$

$$\therefore\quad k=\sqrt{\frac{1-v+2C}{v^2}} \tag{4.25}$$

Substituting (4.24) into (4.23), we get

$$\left(-v+1+\alpha\left(2\left[\frac{v-1-C}{\alpha}\right]\right)\right)\hat{\phi}+v^2\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0$$

$$(v-1-2C)\hat{\phi}+v^2\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0 \tag{4.26}$$

We normalize the coefficient of the nonlinear term by the scaling

$$\hat{\phi}=\frac{2}{a}(1+2C-v)\tilde{\phi} \tag{4.27}$$

It can be easily shown that substituting (4.27) into (4.26) will lead to equation (4.28)

$$\frac{2}{\alpha}(v-1-2C)(1+2C-v)\tilde{\phi}+v^2\left[\frac{2}{\alpha}(1+2C-v)\tilde{\phi}_{2\varepsilon}\right]+\frac{a}{2}\left[\frac{2}{\alpha}(1+2C-v)\tilde{\phi}^2\right]=0$$

$$(1+2C-v)\tilde{\phi}-v^2\tilde{\phi}_{2\varepsilon}-(1+2C-v)\tilde{\phi}^2=0 \tag{4.28}$$

From (4.25),

$$v^2k^2=1+2C-v$$

$$\therefore\quad v^2k^2\tilde{\phi}-v^2\tilde{\phi}_{2\varepsilon}=v^2k^2\tilde{\phi}^2=0$$

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$$\Rightarrow \quad k^2\tilde{\phi} - \tilde{\phi}_{2\varepsilon} - k^2\tilde{\phi}^2 = 0 \tag{4.29}$$

Expanding $\tilde{\phi}$ in terms of the harmonics of the decaying exponential solution of the linear equation, the following is obtained

$$\text{i.e} \quad g(x) = e^{-k\varepsilon}$$

$$\tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n(\varepsilon) = \sum_{n=1}^{\infty} a_n e^{-nk(\varepsilon)}$$

$$\tilde{\phi} = -k\sum_{n=1}^{\infty} na_n g^n(\varepsilon),$$

$$\tilde{\phi}_{2\varepsilon} = k^2\sum_{n=1}^{\infty} n^2 a_n g^n(\varepsilon)$$

$$\tilde{\phi}^2 = \left[\sum_{n=1}^{\infty} a_n g^n(\varepsilon)\right]^2 = \sum_{n=1}^{\infty}\sum_{l=1}^{n-1} a_l a_{n-l} g^n(\varepsilon)$$

by Cauchy's rule.

Substituting these expansions into (4.29), it is easy to see that equation (4.30) results

$$k^2\sum_{n=1}^{\infty} a_n g^2(\varepsilon) - k^2\sum_{n=1}^{\infty} n^2 a_n g^n(\varepsilon) - k^2\sum_{n=1}^{\infty}\sum_{l=1}^{n-1} a_l a_{n-l} g^n(\varepsilon) = 0$$

$$k^2 a_n\sum_{n=1}^{\infty} g^n(\varepsilon) - k^2 n^2 a_n\sum_{n=1}^{\infty} g^n(\varepsilon) - k^2\sum_{l=1}^{n-1} a_l a_{n-l}\sum_{n=1}^{\infty} g^n(\varepsilon) = 0$$

$$(n^2-1)a_n + \sum_{l=1}^{n-1} a_l a_{n-l}\sum_{n=1}^{\infty} g^n(\varepsilon) = 0, \quad n \geq 2 \tag{4.30}$$

The recursion relation (4.13) is solved to find the general form of the coefficient a_n

For $n = 2$,

$$(2^2-1)a_2 + \sum_{l=1}^{1} a_l a_{2-l} = 0$$

$$\therefore \quad a_2 = -12\left(\frac{a_1}{6}\right)^2$$

$$= 6(2)(-1)^{2+1}\left(\frac{a_1}{6}\right)^2$$

For $n = 3$,

$$(3^2-1)a_3 + \sum_{l=1}^{2} a_l a_{3-l} = 0$$

$$a_3 = \frac{1}{12}a_1^3 = 6(3)(-1)^{3+1}\left(\frac{a_1}{6}\right)^3$$

For $n = 4$,

$$(4^2-1)a_4 + \sum_{l=1}^{3} a_l a_{4-l} = 0$$

$$a_4 = -24\left(\frac{a_1}{6}\right)^4 = 6(4)(-1)^{4+1}\left(\frac{a_1}{6}\right)^4$$

The pattern can be clearly seen as:

$$a_n = 6n(-1)^{n+1}\left(\frac{a_1}{6}\right)^n, \quad a_1 > 0$$

Substituting the coefficient of a_n into the equation for ϕ, the solution is obtained.
Using equation (4.22),

$$\phi = c_1 + \hat{\phi}$$

$$\text{and for, } \ c_1 = \frac{2(v-1-c)}{\alpha},$$

$$\text{we have } \ \phi = \frac{2}{\alpha}(v-1-C) + \hat{\phi}$$

Also, from (4.27)

$$\hat{\phi} = \frac{2}{\alpha}(1+2C-v)\tilde{\phi}$$

Substituting into the equation above, we have

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\tilde{\phi}$$

$$\text{But, } \ \tilde{\phi} = \sum_{n=1}^{\infty} a_n g^n(\varepsilon)$$

$$\therefore \quad \phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\sum_{n=1}^{\infty} 6(n)(-1)^{n+1}\left(\frac{a_1}{6}\right)^n g^n$$

Let $a = \frac{a_1}{6}$

$$\text{Then } \ \phi = \frac{2}{a}(v-1-C) + \frac{2}{a}(1+2C-v)\sum_{n=1}^{\infty} 6(n)(-1)^{n+1}(ag)^n \tag{4.31}$$

The power series $\sum_{n=1}^{\infty} 6(n)(-1)^{n+1}(ag)^n$ is convergent for $ag < 1$

Using well-known power series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

And differentiating both sides with respect to x yields equation (4.32)

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

$$= 0x^{0-1} + \sum_{n=1}^{\infty} nx^{n-1}$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Multiplying both sides by x

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$$

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Substituting (ag) for x, we have

$$\frac{ag}{(1-ag)^2} = \sum_{n=1}^{\infty} n(ag)^n \tag{4.32}$$

Substituting (4.32) in (4.31), we obtain

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\frac{6ag}{(1+ag)^2}$$

Putting $g(x) = e^{-k\varepsilon}$ in the equation above, it is easily seen that (4.33) is obtained

$$\therefore\ \phi = \frac{2}{\alpha}(v-1-C) + \frac{2}{\alpha}(1+2C-v)\frac{6ae^{-k\varepsilon}}{(1+ae^{-k\varepsilon})^2} \tag{4.33}$$

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{12}{\alpha}(1+2C-v)\frac{\frac{1}{a}e^{k\varepsilon}}{\left(1+\frac{1}{a}e^{k\varepsilon}\right)^2}$$

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{12}{\alpha}(1+2C-v)\frac{e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}}{\left(1+e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}\right)^2} \tag{4.34}$$

Recalling that

$$sech x = \frac{2e^x}{1+e^{2x}}$$

and squaring both sides, we have

$$\frac{1}{4}sech^2 x = \frac{e^{2x}}{(1+e^{2x})^2}$$

$$\text{Let} \quad x = \frac{1}{2}\left(\ln\left(\frac{1}{a}\right) + k\varepsilon\right)$$

Then,

$$\frac{1}{4}sech^2\frac{1}{2}\left[\ln\left(\frac{1}{a}\right) + k\varepsilon\right] = \frac{e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}}{(1+e^{\ln\left(\frac{1}{a}\right)+k\varepsilon})^2}$$

Substituting this into (4.34), to obtain

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{12}{\alpha}(1+2C-v)\frac{1}{4}sech^2\frac{1}{2}\left[\ln\left(\frac{1}{\alpha}\right) + k\varepsilon\right]$$

$$\phi = \frac{2}{a}(v-1-C) + \frac{3}{a}(1+2C-v)sech^2\left[\frac{1}{2}\ln\left(\frac{1}{a}\right) + \frac{1}{2}k\varepsilon\right]$$

$$\text{Let} \quad \partial = \frac{1}{2}\ln\left(\frac{1}{a}\right)$$

Then,

$$\phi = \frac{2}{\alpha}(v-1-C) + \frac{3}{\alpha}(1+2C-v)sech^2\left[\partial + \frac{1}{2}k\varepsilon\right]$$

$$\text{Recall that} \quad k = \sqrt{\frac{1-v+2C}{v^2}}$$

$$\text{and} \quad \varepsilon = x - vt$$

$$\therefore \quad u(x,t) = \frac{2}{\alpha}(v-1-C) + \frac{3}{\alpha}(1+2C-v)sech^2\left[\frac{1}{2}\left(\frac{1+2C-v}{v^2}\right)^{\frac{1}{2}}(x-vt) + \partial\right]$$

In a special case that $C = v-1$, we have [box=]equation* u(x,t)= $3\frac{}{\alpha(v-1)sech^2\left[\frac{1}{2}\left(\frac{v-1}{2v}\right)^{\frac{1}{2}}(x-vt)+\partial\right]}$

5 CONCLUSION

We have expanded the methodology proposed by Hereman et.al. [15] to include necessary details that simplifies the process of obtaining solutions for easy comprehension. We applied this method to other equations not discussed by Hereman et.al. [15]. We found the method very efficient in the construction of solitary wave solution for non-dissipative nonlinear evolution and wave equations. Benjamin-Bona-Mahony (RLW) equation and Joseph Egri (TRLW) equation were used as examples.

The results obtained are similar to the ones obtained by other methods. For the Benjamin -Bona-Mahony (RLW) equation, our solution is the same as the one indicated by Hereman et.al. [4] and similar to the ones obtained by Irk [16] and Bona et.al. [17] through numerical methods. The solution of the Time Regularized Long-Wave (Joseph-Egri) equation is the same as presented by Hereman et.al. [4].

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