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Trigonometrically-Fitted Simpson's Method for Solving Volterra Integro-Differential Equations

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Abstract

In this study, a third derivative trigonometrically fitted Simpson's method is developed and applied to approximate the solution of Volterra Integro-Differential Equations (VIDEs) via the multistep collocation method. The VIDEs are first transformed to IVPs by the Leibnitz rule of differentiating integral. A continuous third derivative trigonometrically fitted method is constructed with the trigonometric basis function from which both the main and the complementary discrete formulas are generated. The two discrete formulas are then applied as simultaneous integrators in a block by block form to solve the VIDEs. Whereas numerical properties of the proposed method are investigated, its accuracy is demonstrated through some standard examples.

Keywords: Volterra integro-differential equations (VIDEs), block methods, continuous method,Leibnitz rule, Trigonometrically fitted methods.
MSC2010: 35Q20.

1 Introduction

The Volterra Integro-Differential Equation (VIDE) is an equation, in which the unknown function $y(x)$ appears on one side as an ordinary derivative, and on the other side under the integral sign. It is of the general form:

$$y^n(x) = f(x) + \lambda \int_0^x k(x,t)y(t)dt \tag{1.1}$$

where
$k(x,t)$ is the kernel or nucleus of the integral equation, $f(x)$ is the source function, λ is a constant and $n \geq 1$ with initial conditions $y(0), y'(0), ..., y^{n-1}(0) = y_0, y_1, ..., y_{n-1}$ for the determination of the particular solution $y(x)$.

Since its establishment by Vito Volterra, an italian mathematician and physicist , Volterra integro-differential differential equation has been explored by many scientific researchers in numerous fields such as heat transfer, neurosciences, diffusion process, neutron diffusion, biological species,

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biomechanics, economics, electrical engineering, electrodynamics, game theory, oscillation theory, queuing theory, airfoil theory, elastic contact problems, fracture mechanics, combined infrared radiation, molecular conduction nano-hydrodynamics, and glass-forming process. [1], [2]. There is a host of solution techniques available in the literature, for solving Volterra Integro-differential equations ranging from the traditional methods such as successive approximations and successive substitutions to recently developed methods namely the Adomian decomposition method and its modifications that are gaining popularity among scientists and engineers [3]. The traditional methods have shortcomings of huge computation, a high order of convergence, and problems of unrealistic series solutions when dealing with physical problems [4]. The variational iteration method is a modified Lagrange multiplier method and was proposed by He (1997) to facilitate computational work and yields solution faster than the Adomian method [5]. The Tau method introduced by Lanczos in 1938 has been employed in solving some nonlinear VIDEs of the first and second kind by [6]. There the nonlinear VIDEs were converted to linear Volterra Integral Equations (VIEs) of the second kind which were in turn solved by using a forward substitution method. Another method employed in solving VIDEs is the Sinc-collocation method which involves the reduction of VIDEs to an explicit system of algebraic equations as can be seen in [7], [8] .
Numerically, VIDEs are popularly solved by adapting quadrature methods for solving an integral part and then applying numerical methods to solve the resulting differential equations. This is mostly done when the kernel of integration is of separable type. The integral part of equation (1.1) is then replaced by the calculation of the integral sum(see [9], [10], [11], [12], [13], [14], and [15]).
Despite the success achieved by the above-mentioned methods, they suffer from the increasing volume of calculations from one point to another. This study is motivated by the fact that Volterra integro-differential equations can be transformed to initial value problems and thus develop a new third derivative of the Simpson's method with trigonometric coefficients to solve the equation (1.1) wholly in a block by block fashion.
The paper is organized as follows: section 2 discusses the derivation of the proposed method, numerical properties of the method, and related discussions are presented in section 3, section 4 presents some numerical results, and section 5 highlights the conclusions.

2 Derivation of Method

This section considers the construction of a continuous block third derivative trigonometrically fitted Simpson's method (CTDTFSM) on the interval $[x_n, x_{n+2}]$ to produce a discrete formula TDTFSM and its complementary method in the form:

$$y_{n+2} - y_n = h\sum_{j=0}^{2}\beta_j(v)f_{n+j} + h^2\sum_{j=0}^{2}\gamma_j(v)g_{n+j} + h^3\tau_2(v)l_{n+2} \tag{2.1}$$

where $v = \omega h$, ω is the frequency, h is the stepsize, x_n is a node point and $\beta_j, j = 0, 1, 2$ and γ_j are coefficients to be uniquely obtained.
Traditionally, $y_{(n+j)}$ is the approximate solution to the exact solution $y(x_{n+j}), j = 1, 2,$ and $f_{n+j} = y'(x_n + jh)$, $g_{n+j} = y''(x_n + jh)$, and $l_{n+j} = y'''(x_n + jh)$.

Assuming $y(x)$ which is the exact solution within the interval $[x_n, x_{n+2}]$ can be approximated by a fitting function $I(x, v)$ i.e.

$$y(x) \approx I(x, v)$$

where $I(x, v)$ is of the form

$$I(x, v) = \sum_{j=0}^{5} a_j x^j + a_6 \sin\omega x + a_7 \cos\omega x \tag{2.2}$$

where $a_j, j = 0, 1, ..., 7$ are coefficients to be uniquely determined.
It is required that the following eight conditions be satisfied by equation (2.2)

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$$\begin{cases} I(x_{n+j}) = y_{n+j}, & \text{if } j = 0 \\ I'(x_{n+j}) = f_{n+j}, & \text{if } j = 0, 1, 2; \\ I''(x_{n+j}) = g_{n+j}, & \text{if } j = 0, 1, 2; \\ I'''(x_{n+j}) = l_{n+j}, & \text{if } j = 2. \end{cases} \tag{2.3}$$

The system of eight equations from equation (2.3) is solved to obtain a_j using a computer algebraic system (CAS) such as the Maple 16 package. The continuous equation is obtained by substituting the values of $a_j, j = 0, 1, ..., 7$ into equation (2.2). After some algebraic simplification, the continuous equation takes the form of equation(2.4) given by

$$I(x, v) = y_n + h \sum_{j=0}^{2} \beta_j(v, x) f_{n+j} + h^2 \sum_{j=0}^{2} \gamma_j(v, x) g_{n+j} + h^3 \tau_2(v, x) l_{n+2} \tag{2.4}$$

Equation (2.4) is thereafter evaluated at $x = x_{n+2}$ yielding the main discrete formula for the TDTFBSM in the form:

$$y_{n+2} = y_n + h \sum_{j=0}^{2} \beta_j(\cos(v), \sin(v)) f_{n+j} + h^2 \sum_{j=0}^{2} \gamma_j(\cos(v), \sin(v)) g_{n+j} + h^3 \tau_2(\cos(v), \sin(v)) l_{n+2} \tag{2.5}$$

The complementary discrete formula given in equation (2.6) is obtained by evaluating equation (2.4) at $x = x_{n+1}$ to get

$$y_{n+1} = y_n + h \sum_{j=0}^{2} \beta_j(\cos(v), \sin(v)) f_{n+j} + h^2 \sum_{j=0}^{2} \gamma_j(\cos(v), \sin(v)) g_{n+j} + h^3 \tau_2(\cos(v), \sin(v)) l_{n+2} \tag{2.6}$$

For emphasis, we write equations (2.5) and(2.6) in block form called the third derivative trigonometrically fitted block Simpson's method (TDTFBSM) as follows

$$\begin{aligned} y_{n+2} = y_n &+ h(\beta_0(\cos(v), \sin(v)) f_n + \beta_1(\cos(v), \sin(v)) f_{n+1} \\ &+ \beta_2(\cos(v), \sin(v)) f_{n+2}) + h^2(\gamma_0(\cos(v), \sin(v)) g_n \\ &+ \gamma_1(\cos(v), \sin(v)) g_{n+1} + \gamma_2(\cos(v), \sin(v)) g_{n+2}) \\ &+ h^3 \tau_2(v, x) l_{n+2}, \\ y_{n+1} = y_n &+ h(\hat{\beta}_0(\cos(v), \sin(v)) f_n + \hat{\beta}_1(\cos(v), \sin(v)) f_{n+1} \\ &+ \hat{\beta}_2(\cos(v), \sin(v)) f_{n+2}) + h^2(\hat{\gamma}_0(\cos(v), \sin(v)) g_n \\ &+ \hat{\gamma}_1(\cos(v), \sin(v)) g_{n+1} + \hat{\gamma}_2(\cos(v), \sin(v)) g_{n+2}) \\ &+ h^3 \hat{\tau}_2(v, x) l_{n+2}. \end{aligned} \tag{2.7}$$

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The coefficients of equation (2.7) are as given in the equations (2.8) and (2.9)

$$
\begin{cases}
\beta_0 = \dfrac{\left(\begin{array}{c} -360\cos(2v) + 28v^4\cos(v) - 72v^3\sin(v) + 252v^2\cos(v) - 1020v\sin(v) + 360 \\ +10v^4 + 462v^2 - 21v^3\sin(2v) - 210v\sin(2v) + 4v^4\cos(2v) + 6v^2\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\beta_1 = \dfrac{\left(\begin{array}{c} 64v^4\cos(v) - 384v^3\sin(v) - 720 - 960v^2\cos(v) + 960v\sin(v) - 144v^3\sin(2v) \\ +960v\sin(2v) + 16v^4 + 48v^2 - 528v^2\cos(2v) + 16v^4\cos(2v) + 720\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\beta_2 = \dfrac{\left(\begin{array}{c} 28v^4\cos(v) - 144v^3\sin(v) + 360 + 228v^2\cos(v) - 1380v\sin(v) - 75v^3\sin(2v) \\ -30v\sin(2v) + 4v^4 + 660v^2 - 168v^2\cos(2v) + 10v^4\cos(2v) - 360\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\gamma_0 = \dfrac{\left(\begin{array}{c} \left(36v^2 - 135\right)\cos(2v) + \left(4v^3 - 114v\right)\sin(2v) + \left(4v^4 + 120v^2\right)\cos(v) \\ + \left(4v^3 - 312v\right)\sin(v) + 2v^4 + 114v^2 + 135 \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\gamma_1 = \dfrac{\left(\begin{array}{c} 64v^3\sin(v) + 192v^2\cos(v) - 192v\sin(v) + 180 + 40v^3\sin(2v) \\ -264v\sin(2v) + 4v^4 + 12v^2 - 4v^4\cos(2v) + 156v^2\cos(2v) - 180\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\gamma_2 = \dfrac{\left(\begin{array}{c} -4v^4\cos(v) + 4v^3\sin(v) - 288v^2\cos(v) + 864v\sin(v) - 315 \\ +10v^3\sin(2v) + 198v\sin(2v) - 2v^4\cos(2v) - 324v^2 - 18v^2\cos(2v) + 315\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)} \\
\tau_2 = \dfrac{\left(\begin{array}{c} 90 + 8v^3\sin(v) - 216v\sin(v) + 2v^3\sin(2v) - 72v\sin(2v) \\ +88v^2\cos(v) + 72v^2 + 20v^2\cos(2v) - 90\cos(2v) \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 20v^2\sin(v) - 8v^2\sin(2v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) - 48\sin(v) + 24\sin(2v) + 39v \end{array}\right)}
\end{cases}
\tag{2.8}
$$

$$
\begin{cases}
\hat{\beta}_0 = \dfrac{\left(\begin{array}{c} 31v^4\cos(2v) + 262v^4\cos(v) - 159v^3\sin(2v) - 1038v^3\sin(v) + 55v^4 + 84v^2\cos(2v) \\ -312v^2\cos(v) - 1740v\sin(2v) - 2280v\sin(v) + 3108v^2 - 2880\cos(2v) + 5760\cos(v) - 2880 \end{array}\right)}{120v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\beta}_1 = \dfrac{\left(\begin{array}{c} 32v^4\cos(v) + 8v^4\cos(2v) - 192v^3\sin(v) - 72v^3\sin(2v) + 8v^4 - 480v^2\cos(v) \\ -264v^2\cos(2v) + 480v\sin(v) + 480v\sin(2v) + 24v^2 + 360\cos(2v) - 360 \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\beta}_2 = \dfrac{\left(\begin{array}{c} \left(25v^4 - 732v^2\right)\cos(2v) + \left(-225v^3 + 780v\right)\sin(2v) + \left(-38v^4 + 2232v^2 - 5760\right)\cos(v) \\ + \left(174v^3 - 7320v\right)\sin(v) + v^4 + 1380v^2 + 5760 \end{array}\right)}{120v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\gamma}_0 = \dfrac{\left(\begin{array}{c} \left(279v^2 - 1020\right)\cos(2v) + \left(31v^3 - 876v\right)\sin(2v) + \left(46v^4 + 120v^2 + 1920\right)\cos(v) \\ + \left(-134v^3 - 408v\right)\sin(v) + 8v^4 + 681v^2 - 900 \end{array}\right)}{120v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\gamma}_1 = \dfrac{\left(\begin{array}{c} -23v^4\cos(2v) + 128v^3\sin(v) + 230v^3\sin(2v) - 7v^4 - 96v^2\cos(v) + 927v^2\cos(2v) \\ +2016v\sin(v) - 1728v\sin(2v) - 111v^2 + 3840\cos(v) - 1320\cos(2v) - 2520 \end{array}\right)}{60v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\gamma}_2 = \dfrac{\left(\begin{array}{c} -2700 + 14v^4\cos(v) - 134v^3\sin(v) - 1512v^2\cos(v) + 4296v\sin(v) + 55v^3\sin(2v) \\ +372v\sin(2v) - 8v^4\cos(2v) + 780\cos(2v) + 63v^2\cos(2v) + 1920\cos(v) - 1071v^2 \end{array}\right)}{120v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)} \\
\hat{\tau}_2 = \dfrac{\left(\begin{array}{c} 4v^3\sin(v) + v^3\sin(2v) + 44v^2\cos(v) + 10v^2\cos(2v) \\ -108v\sin(v) - 36v\sin(2v) + 36v^2 - 45\cos(2v) + 45 \end{array}\right)}{15v\left(\begin{array}{c} v^3\cos(2v) + 4v^3\cos(v) - 8v^2\sin(2v) - 20v^2\sin(v) + v^3 \\ -23v\cos(2v) - 16v\cos(v) + 24\sin(2v) - 48\sin(v) + 39v \end{array}\right)}
\end{cases}
\quad .
\tag{2.9}
$$

According to [16], the coefficients in equations (2.8) and (2.9) are subject to heavy cancellations as $v \to 0$. In this case, the Taylor series expansions of the coefficients which are stated to be up to

$o(v^8)$ in equations (2.10) and (2.11) respectively, are preferred.

$$\begin{cases} \beta_0 = \frac{2}{5} + \frac{17\,v^2}{8820} + \frac{949\,v^4}{27165600} + \frac{215\,v^6}{339026688} \\ \beta_1 = \frac{32}{35} - \frac{4\,v^2}{2205} - \frac{53\,v^4}{1697850} - \frac{2113\,v^6}{3708104400} \\ \beta_2 = \frac{24}{35} - \frac{v^2}{8820} - \frac{101\,v^4}{27165600} - \frac{347\,v^6}{5393606400} \\ \gamma_0 = \frac{1}{21} + \frac{v^2}{1470} + \frac{167\,v^4}{13582800} + \frac{6623\,v^6}{29664835200} \\ \gamma_1 = -\frac{16}{105} + \frac{4\,v^2}{2205} + \frac{19\,v^4}{565950} + \frac{41\,v^6}{67420080} \\ \gamma_2 = -\frac{19}{105} - \frac{v^2}{2205} - \frac{v^4}{138600} - \frac{73\,v^6}{549348800} \\ \tau_2 = \frac{2}{105} + \frac{v^2}{4410} + \frac{53\,v^4}{13582800} + \frac{2113\,v^6}{29664835200} \end{cases} \quad (2.10)$$

and

$$\begin{cases} \hat{\beta}_0 = \frac{31}{80} + \frac{241\,v^2}{141120} + \frac{3053\,v^4}{108662400} + \frac{625\,v^6}{1356106752} \\ \hat{\beta}_1 = \frac{16}{35} - \frac{2\,v^2}{2205} - \frac{53\,v^4}{3395700} - \frac{2113\,v^6}{7416208800} \\ \hat{\beta}_2 = \frac{87}{560} - \frac{113\,v^2}{141120} - \frac{1357\,v^4}{108662400} - \frac{41759\,v^6}{237318681600} \\ \hat{\gamma}_0 = \frac{5}{112} + \frac{83\,v^2}{141120} + \frac{117\,v^4}{12073600} + \frac{37867\,v^6}{237318681600} \\ \hat{\gamma}_1 = -\frac{17}{70} + \frac{67\,v^2}{35280} + \frac{841\,v^4}{27165600} + \frac{1961\,v^6}{3955311360} \\ \hat{\gamma}_2 = -\frac{39}{560} + \frac{v^2}{47040} - \frac{v^4}{15523200} - \frac{4393\,v^6}{237318681600} \\ \hat{\tau}_2 = \frac{1}{105} + \frac{v^2}{8820} + \frac{53\,v^4}{27165600} + \frac{2113\,v^6}{59329670400} \end{cases} \quad (2.11)$$

3 Properties of the TDTFBSM

In this section, the basic properties of the proposed method are examined. Such properties include Local truncation error, Order, convergence, zero stability, and region of absolute stability.

3.1 Order and Local truncation error

The TDTFBSM as constructed in section 2 above can be written as

$$A_1 Y_{\delta+1} = A_2 Y_\delta + h(A_3 F_{\delta+1} + A_4 F_\delta) + h^2(A_5 G_{\delta+1} + A_6 G_\delta) + h^3 A_7 L_{\delta+1} \quad (3.1)$$

where
$Y_{\delta+1} = (y_{n+1}, y_{n+2})^T$,$Y_\delta = (y_{n-1}, y_n)^T$,$F_{\delta+1} = (f_{n+1}, f_{n+2})^T$,$F_\delta = (f_{n-1}, f_n)^T$ $G_{\delta+1} = (g_{n+1}, g_{n+2})^T$, $G_\delta = (g_{n-1}, g_n)^T$ and $L_{\delta+1} = (l_{n+1}, l_{n+2})^T$.

The coefficients of $A_i, i = 1, ..., 7$ are 2×2 matrices whose entries are functions of the frequency and the stepsize.
According to [16], the linear difference operator $\mathcal{L}[y(x); \omega, h] = y(x_{n+2}) - y_{n+2}$ associated with equation (3.1) is

$$\mathcal{L}[y(x); \omega, h] = A_1 \bar{Y}_{\delta+1} - A_2 \bar{Y}_\delta - h(A_3 \bar{F}_{\delta+1} - A_4 \bar{F}_\delta) - h^2(A_5 \bar{G}_{\delta+1} - A_6 \bar{G}_\delta) - h^3 A_7 \bar{L}_{\delta+1} \quad (3.2)$$

where $\bar{Y}_{\delta+1} = (y(x_n + h), (x_n + 2h))^T$, $\bar{Y}_\delta = (y(x_n - h), (x_n))^T$, $\bar{F}_{\delta+1} = (y'(x_n + h), y'(x_n + 2h))^T$,

$\bar{F}_\delta = (y'(x_n - h), y'(x_n))^T$ $\bar{G}_{\delta+1} = (y''(x_n + h), y''(x_n + 2h))^T$ $\bar{G}_\delta = (y''(x_n - h), y''(x_n))^T$ $\bar{L}_{\delta+1} = (y'''(x_n + h), y'''(x_n + 2h))^T$.
The coefficients of $A_i, i = 1, ..., 7$ remains as in the equation (3.1)

Equation (3.2) is then expanded with Taylor series of all its components about the point x_n and we have the local truncation error as given below:

$$LTE1 = -\frac{h^8}{33075}(y^8(x) + \omega^2 y^6(x)),$$

$$LTE2 = -\frac{11h^8}{470400}(y^8(x) + \omega^2 y^6(x)).$$

The block method is of the uniform order of accuracy seven and the error constants are $-\frac{1}{33075}$ and $-\frac{11}{470400}$ respectively.

3.2 Zero Stability

Zero stability is the stability of the difference system (3.1) when the stepsize $h \to 0$ and this gives

$$A_1 Y_{\delta+1} = A_2 Y_\delta \tag{3.3}$$

The block method (3.1) is zero stable if the first characteristic polynomial $\rho(R)$ stated by

$$\rho(R) = \det(RA_1 - A_2) = 0$$

Using equation (3.1), $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$$|RA_1 - A_2| = 0$$

$$\begin{vmatrix} R & -1 \\ 0 & R-1 \end{vmatrix} = 0$$

.
$\Rightarrow R(R-1) = 0$. Hence $R = 0, 1$ thus satisfying the condition:
$|R_j| \leq 1$, $j = 1 and 2$ with the roots $|R_j| = 1$ the multiplicity does not exceed one, see [16]. Therefore, TDTFBSM is zero stable.

3.3 Consistency and Convergency

The block method (3.1) is consistent because each of the methods has order p of seven which is greater than one. i.e. $p \geq 1$ and is also convergent as it has satisfied the conditions necessary for convergence which are Zero stability and consistency according to [16].

3.4 Linear Stability

To analyze the linear stability of TDTFBSM, we apply the TDTFBSM to the test equations $Y' = \lambda Y$, $Y'' = \lambda^2 Y$, $Y''' = \lambda^3 Y$, taking $z = \lambda h$, and $v = \omega h$.

$$Y_{\delta+1} = M(z; v) Y_\delta$$

where

$$M(z; v) = \frac{A_2 + zA_4 + z^2 A_6}{A_1 - zA_3 - z^2 A_5 - z^3 A_7}.$$

The shaded part of the diagram below shows the region of stability of the TDTFBSM method

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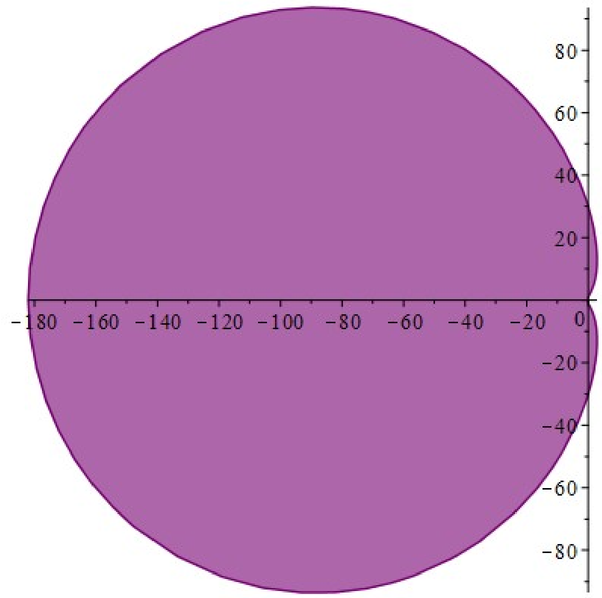

Figure 1: Region of Absolute Stability of TDTFBSM

4 Numerical Examples

This section deals with the presentation of solutions of some VIDEs that have been solved in the literature with the mind of comparing the efficiency and accuracy of the new TDTFBSM and other methods. All computations are carried out using written codes in Maple 16.

Example 4.1. *Consider a first-order Volterra integro-differential equation:*

$$y'(x) = 1 - \int_0^x y(t)dt, y(0) = 0, 0 \leq x \leq 1 \tag{4.1}$$

whose exact solution is given as $y(x) = \sin x$
Given that the Leibnitz rule of differentiating an integral as

$$\frac{d}{dx}\int_{g(x)}^{h(x)} F(x,t)dt = F(x,h(x))\frac{dh}{dx} - F(x,g(x))\frac{dg}{dx} + \int_{g(x)}^{h(x)} \frac{\partial F(x,t)}{\partial x} \tag{4.2}$$

where
$F(x,t)$ and $\frac{\partial F(x,t)}{\partial x}$ are continuous functions of x and t in the domain of $\alpha \leq x \leq \beta$, $t_0 \leq x \leq t_1$; and the limits of integration, $h(x)$ and $g(x)$ are defined functions having continuous derivatives for $\alpha \leq x \leq \beta$. see [1]
Then the VIDE in example (4.1) is converted by the Leibnitz rule (4.2), to an initial value problem given as

$$y''(x) = -y(x), y(0) = 0, y'(0) = 1, 0 \leq x \leq 1. \tag{4.3}$$

Equation (4.3) is then reduced to a system of first-order equations and solved by using the new TDTFBSM.The results obtained are compared with the Runge-Kutta-Fehberg and Boole's rules (RKF) in [9] and the two-point block one-step method with trapezoidal and Simpson's 1/3 rule (2PVIDE) in [10].

The results are as displayed in Table 1 and Figure 2 below:

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Table 1: Data for Example 4.1

Step length (h)	2PVIDE (Max. Err.)	RKF FILIZ(Max. Err.)	TDTFBSM(Max. Err.)
$\frac{1}{40}$	1.4219×10^{-6}	4.1910×10^{-10}	5.6861×10^{-27}
$\frac{1}{80}$	1.8018×10^{-7}	1.2697×10^{-11}	9.1059×10^{-24}
$\frac{1}{160}$	2.2675×10^{-8}	3.9047×10^{-13}	1.0154×10^{-19}

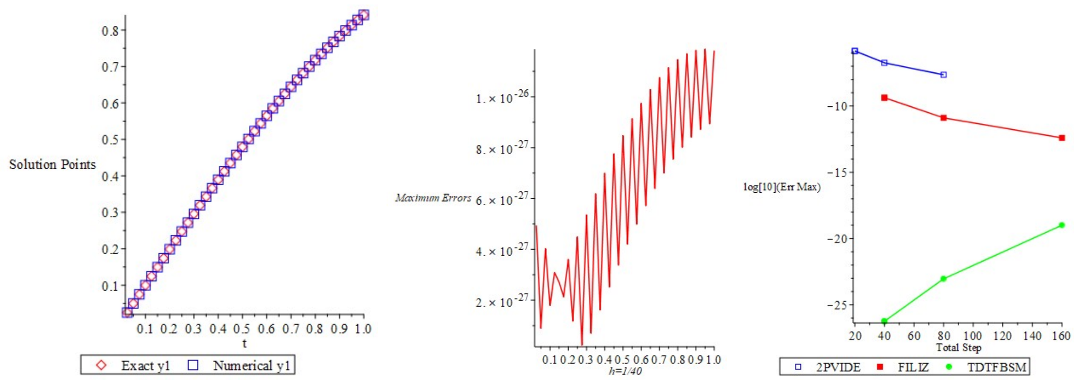

Figure 2: Discrete solution using TDTFBSM (Left), Maximum Error using TDTFBSM (Middle), Efficiency curves using 2PVIDE,RKF(FILIZ),and TDTFBSM (Right)

Example 4.2. *Consider a first-order Volterra integro-differential equation:*

$$y'(x) = -\int_0^x y(t)dt, y(0) = 1, 0 \leq x \leq 1 \tag{4.4}$$

whose exact solution is given as $y(x) = \cos x$
Equation (4.4) is converted by (4.2), to an initial value problem given as

$$y''(x) = -y(x), y(0) = 1, y'(0) = 0, 0 \leq x \leq 1. \tag{4.5}$$

Equation (4.5) is then reduced to a system of first-order and solved by using the TDTFBSM. We integrate using step lengths $h = \frac{1}{2^i}, i = 3(1)7$ and the errors are compared with those of the fifth-order Adams-Bashforth-Moulton predictor-corrector (ABM5) and two point three-step block (2P3BVIDE) methods as can be seen in [15].
The Table 2 and Figure 3 below display the comparison:

Table 2: Data for Example 4.2

Step length (h)	ABM5 (Max. Err.)	2P3BVIDE(Max. Err.)	TDTFBSM(Max. Err.)
$\frac{1}{40}$	2.8951×10^{-7}	5.7323×10^{-8}	8.9548×10^{-27}
$\frac{1}{80}$	3.6127×10^{-8}	5.5893×10^{-9}	2.4244×10^{-23}
$\frac{1}{160}$	4.3953×10^{-9}	2.2443×10^{-10}	2.0910×10^{-19}
$\frac{1}{320}$	5.4213×10^{-10}	1.3908×10^{-11}	1.8968×10^{-15}
$\frac{1}{640}$	6.7325×10^{-11}	8.6930×10^{-13}	1.0489×10^{-11}

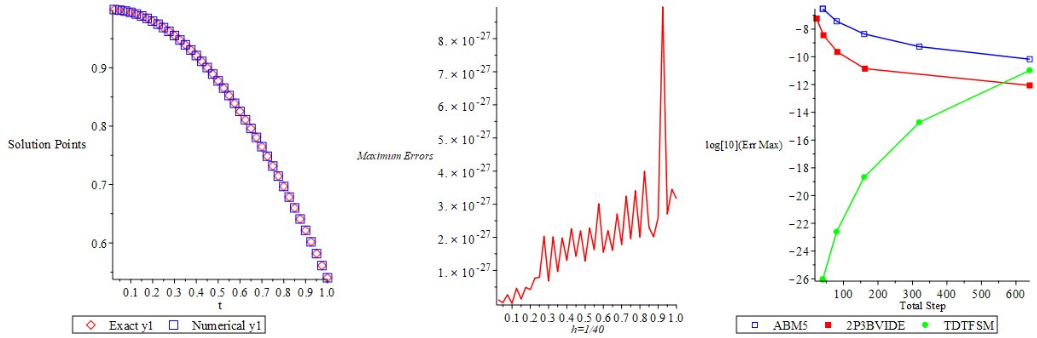

Figure 3: Discrete solution using TDTFBSM (Left), Maximum Error using TDTFBSM (Middle), Efficiency curves using ABM5,2P3BVIDE, and TDTFBSM (Right)

Example 4.3. *Consider a first-order Volterra integro-differential equation:*

$$y'(x) = 1 + \int_0^x y(t)dt, y(0) = 0, 0 \le x \le 1 \tag{4.6}$$

whose exact solution is given as $y(x) = \sinh x$

Equation (4.6) is converted by equation (4.2), to an initial value problem given as

$$y''(x) = y(x), y(0) = 0, y'(0) = 1, 0 \le x \le 1. \tag{4.7}$$

Equation (4.7) is then reduced to a system of first-order equations, and solved using the proposed method. A comparison of error is made with the results obtained by using the General Linear Method (GLM) and Runge-Kutta (RK) methods both of third order in [14]. The results are as displayed in Table 3 and Figure 4 below:

Table 3: Data for Example 4.3

h	GLM		RK		TDTFBSM		[t]
	Error	NFE	Error	NFE	Error	NFE	[t]
$\frac{1}{10}$	2.4606×10^{-6}	34	6.9906×10^{-6}	34	3.7864×10^{-12}	28	[t]
$\frac{1}{40}$	1.6319×10^{-8}	124	1.0137×10^{-7}	124	2.2030×10^{-16}	103	[t]
$\frac{1}{100}$	8.3870×10^{-10}	304	6.4622×10^{-9}	304	3.5789×10^{-19}	253	[t]
$\frac{1}{200}$	9.7077×10^{-11}	604	8.0749×10^{-10}	604	8.0866×10^{-18}	503	[t]

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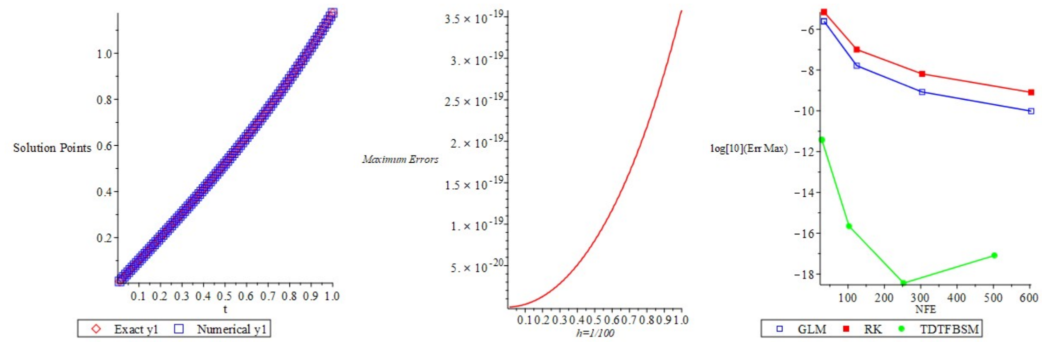

Figure 4: Discrete solution using TDTFBSM(Left), Maximum Error using TDTFBSM (Middle), Efficiency curves using GLM,RK, and TDTFBSM (Right)

5 Conclusion

The proposed third derivative trigonometrically fitted block Simpson's method is suitable for solving linear VIDEs of the second kind. It is self-starting and performs better in comparison with the existing methods in the reviewed literature in terms of having the lowest efficiency curves as can be seen in the figures above.

Competing Interests

The authors declare that they have no competing interests.

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