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A Survey on Approximation of Common Fixed Point of Countably Infinite Family of Nonexpansive Mappings

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Abstract

In this paper, strong convergence theorems for approximation of a common fixed point of countably family of nonexpansive mappings in strictly convex reflexive real Banach space with uniformly Gâteaux differentiable norm are proved. Consequently, strong convergence theorems for approximation of a common fixed point of countably infnite family of k-strictly pseudo-contractive mappings are proved in strictly convex q-uniformly smooth real Banach spaces. Numerical example is generated to demonstrate workability of an algorithm developed. Our theorems extend, improve and generalize some existing results.

Keywords: Nonexpansive mappings, Reflexive and strictly convex Banach spaces, Uniformly Gâteaux differentiable norm, Weakly inward.
MSC2010: 47H06, 47H09, 47J05, 47J25.

1 Introduction

Let C be a nonempty subset of a real normed space, E. A mapping $T : C \to E$ is said to be *nonexpansive* if $\forall\, x, y \in C$, $\|Tx - Ty\| \le \|x - y\|$. Most published results on nonexpansive mappings centered on existence theorems for fixed points of these mappings; and iterative approximation of such fixed points. DeMarr [1] in 1963 studied the problem of existence of common fixed point for a family of nonlinear nonexpansive mappings. He proved the following theorem:

Theorem DM. Let E be a real Banach space and C be a nonempty compact convex subset of E. If Ω is a nonempty commuting family of nonexpansive mappings of C into itself, then the family Ω has a common fixed point in C.

In 1965, Browder [2] proved the result of DeMarr in a uniformly convex real Banach space, requiring that C be only bounded closed convex and nonempty. For other fixed point theorems for families of nonexpansive mappings, the reader may consult any of the following references: Belluce and

Kirk [3], Lim [4] and Bruck [5].

In 1973, Bruck [6] considered the study of structure of the fixed-point set $F(T) = \{x \in C : Tx = x\}$ of nonexpansive mapping T and established several results.

Kirk [7] introduced an iterative process given by

$$x_{n+1} = \alpha_0 x_n + \alpha_1 T x_n + \alpha_2 T^2 x_n + ... + \alpha_r T^r x_n, \tag{1.1}$$

where $\alpha_i \geq 0, \alpha_0 > 0$ and $\sum_{i=0}^{r} \alpha_i = 1$, for approximation of fixed points of nonexpansive mappings on convex subset of uniformly convex real Banach spaces. Maiti and Saha [8] worked and improved on the results of Kirk [7].

Considerable research efforts have been devoted to developing iterative methods for approximating common fixed points (when such fixed points exist) of families of several classes of nonlinear mappings (see e.g. [9–16]).

Let C be a bounded closed convex nonempty subset of a real Banach space E. Let $T_i : C \to C,\ i = 1, 2, ..., r$ be a finite family of nonexpansive mappings and let

$$S = \alpha_0 I + \alpha_1 T_1 + \alpha_2 T_2 + ... + \alpha_r T_r, \tag{1.2}$$

where $\alpha_i \geq 0,\ \alpha_1 > 0$ and $\sum_{i=0}^{r} \alpha_i = 1$. Then the family $\{T_i\}_{i=1}^r$ such that the common fixed point set $F := \cap_{i=1}^r F(T_i) \neq \emptyset$ is said to satisfy condition A (see e.g. [8, 17, 18]) if there exists a nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ with $\phi(0) = 0,\ \phi(\varepsilon) > 0\ \forall\ \varepsilon \in (0, +\infty)$, such that $\|x - Sx\| \geq \phi\Big(d(x, F)\Big)\ \forall\ x \in C$, where $d(x, F) = \inf\{\|x - z\| : z \in F\}$.

Liu *et al.* [17] introduced the following iteration process:

$$x_0 \in C, x_{n+1} = Sx_n,\ n \geq 0, \tag{1.3}$$

and showed that $\{x_n\}_{n\geq 0}$ defined by (1.3) converges to a common fixed point of $\{T_i\}_{i=1}^r$ in Banach spaces, provided that $\{T_i\}_{i=1}^r$ satisfy condition A. The result of Liu *et al.* [17] improves the corresponding results of Kirk [7], Maiti and Saha [8], Senter and Doston [14] and those of a host of other authors. However, the assumption that the family $\{T_i\}_{i=1}^r$ satisfies condition A is strong.

Let E be a reflexive and strictly convex real Banach space with a uniformly Gâteaux differentiable norm. Let $T_i : E \to E,\ i = 1, 2, ..., r$ be nonexpansive mappings and $\{x_n\}_{n\geq 0}$ be a sequence in E defined iteratively by (1.3) and suppose that $J^{-1} : E^* \to E$ is weakly sequentially continuous at 0, where J is the normalized duality mapping (see section 2 of this paper). If $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, then Jung [19] proved that, under this situation, $\{x_n\}_{n\geq 0}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^r$. In [20], Gossez and Lami Dozo proved that for any normed linear space E, the existence of a weakly sequentially continuous duality mapping implies that the space E satisfies Opial's condition (that is, for all sequences $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\liminf_{n\to\infty} \|x_n - y\| > \liminf_{n\to\infty} \|x_n - x\|$ holds for all $y \neq x$, see e.g. [21]). It is well known that L_p spaces, $1 < p < +\infty,\ p \neq 2$, do not satisfy Opial's condition. Consequently, the results of Jung [19] are not applicable in L_p spaces $1 < p < +\infty,\ p \neq 2$.

In our discussion so far, the operators considered remain *self mappings* of closed convex nonempty subset of real Banach spaces. If, however, the operators are *nonself mappings* (as is the case in several applications), the iterative methods discussed above may fail to be well defined. Several authors have studied various classes of nonself mappings. In most cases, authors dwell on the existence and iterative approximation of fixed points of nonself nonlinear mappings (see e.g., [21–37]). For more on recent research developments involving nonexpansive mappings, the reader may see Alam *et al.* [38], Aleomraninejad *et al.* [39], Berinde [40] and [41], Khatoon *et al.* [42], Pragadeeswarar and Gopi [43], Shrama *et al.* [44], and the references therein.

Moreover, Aibinu and Kim [45], and Zhao *et al.* [46] studied *i*mplicit viscosity approximation methods for solution of nonlinear operator equations invloving nonexpansive mappings. Bian *et al.* [47] studied generalized *implicit* iterative process for approximation of fixed points of nonexpansive mappings. Jiang *et al.* [48] introduced and studied hybrid *implicit* iteration process for a finite family of nonself nonexpansive mappings in unifromly convex Banach space. *It is important to note that in application, explicit iterative algorithms are preferred, and are easier to utilize than the implicit ones.* Assuming that C is closed convex nonempty subset of a real reflexive and strictly convex Banach space E which has a uniformly Gâteaux differentiable norm and suppose further that every nonempty closed convex bounded subset of C has the fixed point property for nonexpansive mappings, Chidume *et al.* [49] studied an explicit iterative process which converges strongly to a common fixed point of a finite family of non-self nonexpansive mappings.

It is our aim in this paper to introduce an explicit iterative sequence for approximation of a common fixed point of *a countably infinite family of nonself nonexpansive mappings* in Banach spaces. As a result, we obtain strong convergence theorems for approximation of a common fixed point of countably infnite family of nonself k-strictly pseudocontractive mappings in strictly convevex q-uniformly smooth real Banach spaces. The corresponding results of Chidume *et al.* [49] and that of a host of other authors are extended from consideration of *finite family* of nonexpansive mappings to the case of *countably infinite family* of nonexpansive mappings. Our theorems improve, generalize, unify and extend several results recently announced.

2 Preliminaries.

Let E be a real normed space with dual E^*. The *normalized duality mapping* is the mapping $J : E \to 2^{E^*}$ defined $\forall\, x \in E$ by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2,\ \|f^*\| = \|x\|\},$$

where $\langle ., . \rangle$ denotes the generalized duality pairing between members of E and E^*. It is well known that if E^* is strictly convex then J is single-valued. In what follows, the single-valued normalized duality mapping will be denoted by j.

Let $(E, \|.\|)$ be a real normed space. The norm $\|.\|$ is said to be *uniformly Gâteaux differentiable* if for each $y \in S = \{x \in E : \|x\| = 1\}$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists uniformly for $x \in S$. It is well known that L_p spaces, $1 < p < \infty$, have uniformly Gâteaux differentiable norm (see e.g. [12, 50]). Furthermore, if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded subsets of E.

Let C be a nonempty subset of a Banach space E. For $x \in C$, the inward set of x, $I_C(x)$, is defined by

$$I_C(x) = \{x + \lambda(u - x) : u \in C,\ \lambda \geq 1\}.$$

A mapping $T : C \to E$ is called *weakly inward* if $Tx \in \overline{I_C(x)}$, $\forall\ x \in C$, where $\overline{I_C(x)}$ denotes the closure of the inward set. Every self-map is trivially weakly inward. If C is convex (see e.g. [51]), then the weakly inward condition has been shown to be equivalent to the *flow invariance condition* in the theory of differential equations given by

$$\lim_{t\to 0}\frac{d\Big(x-t(I-T)x, C\Big)}{t}=0,\ x\in C \tag{2.1}$$

Let C be a nonempty closed convex subset of E and let P be a mapping of E onto C. Then P is said to be sunny if $P\Big(Px + t(x - Px)\Big) = Px\ \forall\ x \in E$ and $t > 0$. A mapping P of E into E is said to be a retraction if $P^2 = P$. If a mapping P is a retraction, then $Px = x\ \forall\ x \in R(P)$, where $R(P)$ denotes the range of P. A subset C of E is said to be sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto C and it is said to be a nonexpansive retract of E if there exists a nonexpansive retraction of E onto C. If $E = H$, where H denotes a Hilbert space, the metric projection P_C is a sunny nonexpansive retraction from H to any closed convex nonempty subset of H. This, however, is not true for larger Banach spaces since nonexpansivity of projections P_C characterizes Hilbert spaces. On the other hand, a sunny nonexpansive retraction can play a similar role in a Banach space as a projection does in Hilbert spaces. In [52] page 382, one will observe that if a real Banach space E is uniformly convex and an operator $A \subset E \times E$ is m-accretive, then $cl(D(A))$ is a nonexpansive retract of E, where $cl(D(A))$ denotes the closure of domain of the operator A. For more on nonexpansive retractions and what they look like outside Hilbert spaces, one may see for example [6, 49, 53–57].

In the sequel, the following Lemmas and Theorem shall be used.

Lemma 2.1 (see e.g., Xu [58–60])**.** *Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of non-negative real numbers satisfying the condition*

$$\lambda_{n+1} \leq (1-\alpha_n)\lambda_n + \sigma_n,\ n \geq 0,$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ are sequences of real numbers such that $\{\alpha_n\}_{n\geq 1} \subset [0, 1]$, $\sum_{n=1}^{\infty}\alpha_n = +\infty$. Suppose that $\sigma_n = o(\alpha_n)$, $n \geq 0$ (i.e., $\lim_{n\to\infty}\frac{\sigma_n}{\alpha_n} = 0$) or $\sum_{n=1}^{\infty}|\sigma_n| < +\infty$, then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.2. *Let E be a real normed linear space. Then the following inequality holds: For all $x, y \in E$, $\forall\ j(x + y) \in J(x + y)$ we have that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\big\langle y, j(x+y)\big\rangle.$$

Lemma 2.3 (Compare with Theorem 1 of Morales and Jung [61])**.** *Let C be a nonempty closed convex subset of a real reflexive and strictly convex Banach space E which has uniformly Gâteaux differentiable norm and $T : C \to E$ a weakly inward nonexpansive mapping with $F(T) \neq \emptyset$. Suppose $\{t_n\}_{n\geq 1}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$; and $u \in C$ is arbitrary, then there exists a path $\{z_n\}_{n\geq 1}$ satisfying*

$$z_n := t_n u + (1-t_n)Tz_n$$

which converges strongly to a fixed point of T.

Lemma 2.4 (Compare with Lemma 3 pg. 257 of Bruck [6])**.** *Let C be a nonempty closed and convex subset of a real strictly convex Banach space E. Let $\{T_j\}_{j\geq 1}$ be a sequence of nonself nonexpansive mappings $T_j : C \to E$ such that $F := \bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Then the mapping $T := \sum_{j=1}^{\infty}\xi_j T_j : C \to E$ is nonexpansive and $F(T) = \bigcap_{j=1}^{\infty} F(T_j)$, where $\{\xi_j\}_{j\geq 1}$ is a sequence in $(0, 1)$ such that $\sum_{j=1}^{\infty}\xi_j = 1$.*

Proof. Observe that the mapping T is well defined since for $x \in C$ and $x^* \in F$,

$$\|T_j x\| \leq \|T_j x - T_j x^*\| + \|T_j x^*\| \leq \|x - x^*\| + \|x^*\|;$$

thus, $\sum_{j=1}^{\infty} \xi_j T_j x$ converges absolutely for each $x \in C$. It is easy to see that T is nonexpansive and maps C into E. Next, we claim that $F(T) = \bigcap_{j=1}^{\infty} F(T_j)$. The inclusion $\bigcap_{j=1}^{\infty} F(T_j) \subset F(T)$ is obvious. We prove the reverse inclusion only. Suppose that $Tx_0 = x_0$. Then

$$\begin{aligned} \|x_0 - x^*\| = \|Tx_0 - x^*\| &= \left\|\sum_{j=1}^{\infty} \xi_j T_j x_0 - x^*\right\| \\ &= \left\|\sum_{j=1}^{\infty} \xi_j (T_j x_0 - x^*)\right\| \leq \sum_{j=1}^{\infty} \xi_j \|T_j x_0 - x^*\|. \end{aligned} \tag{2.2}$$

But $T_j x^* = x^*$ and T_j is nonexpansive $\forall\, j \geq 1$, so $\|T_j x_0 - x^*\| \leq \|x_0 - x^*\|$.
Since $\sum_{j=1}^{\infty} \xi_j = 1$, (2.2) implies that

$$\left\|\sum_{j=1}^{\infty} \xi_j T_j x_0 - x^*\right\| = \|x_0 - x^*\| \text{ and } \|T_j x_0 - x^*\| = \|x_0 - x^*\| \;\forall\, j \geq 1. \tag{2.3}$$

Since E is strictly convex and each $\xi_j > 0$ while $\sum_{j=1}^{\infty} \xi_j = 1$, (2.3) implies $T_j x_0 - x^* = T_m x_0 - x^*$ for all $j, m \in \mathbb{N}$, i.e., $T_j x_0 = T_m x_0$ for all $j, m \in \mathbb{N}$. Hence,

$$x_0 = Tx_0 = \sum_{j=1}^{\infty} \xi_j T_j x_0 = \sum_{j=1}^{\infty} \xi_j T_m x_0 = T_m x_0 \;\forall\, m \in \mathbb{N}.$$

Thus, $x_0 \in \cap_{m=1}^{\infty} F(T_m)$. This completes the proof.□

Lemma 2.5. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a sunny nonexpansive retract of E with P as a sunny nonexpansive retraction. Let $T_j : C \to E,\ j \geq 1$ be a sequence of weakly inward nonexpansive mappings such that $F = \bigcap_{i=1}^{\infty} F(T_j) \neq \emptyset$. Let $T := \sum_{j=1}^{\infty} \xi_j T_j$, then T is weakly inward nonexpansive mapping and $F(T) = F(PT)$.*

Proof. First, we show that T is weakly inward. Let $\varepsilon > 0$ be given and let $z \in C$. Then for any $j \in \{1, 2, ...\}$ we have from (4) that there exists $\delta_{j\varepsilon} > 0$ such that if $t \in (0, \delta_{j\epsilon})$ then $d\big(z - h(I - T_j)z, C\big) < \frac{1}{2} t\varepsilon$. For such t, there exists $\omega_{jt} \in C,\ j = 1, 2, ...$ such that

$$\|z - t(I - T_j)z - \omega_{jt}\| \leq d(z - t(I - T_j)z, C) + \frac{1}{2} t\varepsilon,$$

i.e., $\|z - t(I - T_j)z - \omega_{jt}\| \leq t\varepsilon$ and that for every $j \geq 1$ we have that

$\limsup_{t\to 0}\dfrac{||z-t(I-T_j)z-w_{jt}||}{t}\leq\varepsilon$. Thus, using the fact that $\sum_{j=1}^{\infty}\xi_j=1$ we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{d(z-t(I-T)z,C)}{t} &= \lim_{t\to 0^+}\frac{d\Big(\sum_{j=1}^{\infty}\xi_j(z-t(I-T_j)z),C\Big)}{t}\\ &\leq \limsup_{t\to 0^+}\frac{d\Big(\sum_{j=1}^{\infty}\xi_j\omega_{jt},C\Big)+\sum_{j=1}^{\infty}\xi_j\|z-t(I-T_j)z-\omega_{jt}\|}{t}\\ &\leq \limsup_{t\to 0^+}\frac{d\Big(\sum_{j=1}^{\infty}\xi_j\omega_{jt},C\Big)}{t}+\varepsilon\end{aligned}$$

Observe that $d\Big(\sum_{j=1}^{\infty}\xi_j\omega_{jt},C\Big)=0$ since by convexity of C, $\sum_{j=1}^{\infty}\xi_j\omega_{jt}\in C$. It, therefore, follows that $\lim_{t\to 0^+}\dfrac{d(z-t(I-T)z,C)}{t}=0\ \forall\ z\in C$. Hence, T is weakly inward. Next, we show that $F(T)=F(PT)$. Clearly, $F(T)\subset F(PT)$. It suffices to show that $F(PT)\subset F(T)$. Suppose for contradiction that $F(PT)$ is not a subset of $F(T)$. Let $x_0\in F(PT)\backslash F(T)$, where $F(PT)\backslash F(T)$ denotes the complement of $F(T)$ relative to $F(PT)$. Since T is weakly inward there exists $u\in C$ such that $Tx_0=x_0+\lambda(u-x_0)$ for some $\lambda>0$ and $x_0\neq u$. Observe that if $x_0=u$ then $Tx_0=x_0$, a contradiction. Now, since P is sunny nonexpansive, we have $P(PTx_0+t(Tx_0-PTx_0))=x_0\ \forall\ t\geq 0$. But $PTx_0=x_0$. This implies that $P(tTx_0+(1-t)x_0)=x_0\ \forall\ t\geq 0$. Since T is weakly inward, there exists $t_0\in(0,1)$ such that $u=t_0Tx_0+(1-t_0)x_0$. Besides, $Pu=u$ since $u\in C$, it implies that $u=Pu=x_0$, a contradiction, since $x_0\neq u$. Hence, $F(PT)\subset F(T)$. This completes the proof. □

Note that Lemma 2.5 is a modification of corresponding result of Chidume *et al.* [49]. It ia presented here to accommodate countably infinite family of nonself nonexpansive mappings which clearly generalizes a single nonself nonexpansive mapping as is the case in [49].

3 Main Results

For the rest of this paper, $\{\alpha_n\}_{n\geq 1}$ is a real sequence such that $\{\alpha_n\}_{n\geq 1}\subset[0,1]$ and satisfies (i) $\lim_{n\to\infty}\alpha_n=0$; (ii) $\sum_{n=1}^{\infty}\alpha_n=\infty$ and either (iii) $\lim_{n\to\infty}\dfrac{|\alpha_n-\alpha_{n-1}|}{\alpha_n}=0$ or $(iii)'$ $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty$. The sequence $\{\xi_j\}_{j=1}^{\infty}$ is a sequence of positive real numbers such that $\sum_{j=1}^{\infty}\xi_j=1$.

We now state and prove our main theorems.

Theorem 3.1. *Let C be a nonempty closed convex subset of a strictly convex reflexive real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a nonexpansive retract of E with P as the nonexpansive retraction. Let $T_j: C\to E$, $j\geq 1$ be a sequence of nonself nonexpansive mappings such that $F=\bigcap_{j=1}^{\infty}F(T_j)\neq\emptyset$. For arbitrary $u,x_1\in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1}=\alpha_n u+(1-\alpha_n)PTx_n,\ n\geq 1,\tag{3.1}$$

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where $T := \sum_{j\geq 1}\xi_j T_j$. *Suppose that* $F(T) = F(PT)$, *and there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, *then* $\{x_n\}_{n\geq 1}$ *converges strongly to a common fixed point of* $\{T_j\}_{j=1}^{\infty}$.

Proof. Clearly, by Lemma 2.4, T is well defined, nonexpansive and $F(T) = \bigcap_{j=1}^{\infty} F(T_j)$. Let $q \in F(T)$, then, from (3.1), we obtain by induction that

$$\|x_n - q\| \leq \max\{\|x_1 - q\|,\ \|u - q\|\}$$

for all $n \in \mathbb{N} \cup \{0\}$; hence $\{x_n\}_{n\geq 0}$ and $\{PTx_n\}_{n\geq 0}$ are bounded. This implies that for some $M_0 > 0$,

$$\|x_{n+1} - PTx_n\| = \alpha_n\|u - PTx_n\| \leq \alpha_n M_0 \to 0 \text{ as } n \to \infty.$$

Moreover, from (3.1) we abtain that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n u + (1-\alpha_n)PTx_n - \alpha_{n-1}u - (1-\alpha_{n-1})PTx_{n-1}\| \\
&= \|(\alpha_n - \alpha_{n-1})(u - PTx_{n-1}) + (1-\alpha_n)(PTx_n - PTx_{n-1})\| \\
&\leq (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_0.
\end{aligned}$$

Two cases arise:

Case 1: Condition (iii) of Remark 3 is satisfied. In this case, $\|x_{n+1} - x_n\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = \alpha_n\beta_n$; $\beta_n = \frac{|\alpha_n - \alpha_{n-1}|M_0}{\alpha_n}$, so that $\sigma_n = o(\alpha_n)$ (since $\lim_{n\to\infty} \dfrac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$).

Case 2: Condition $(iii)'$ of Remark 3 is satisfied. In this case, $\|x_{n+1} - x_n\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n$, where $\sigma_n = |\alpha_n - \alpha_{n-1}|M_0$, so that $\sum_{n=0}^{\infty}\sigma_n < \infty$.

In either case, we obtain (by Lemma 2.1), that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. This implies $\lim_{n\to\infty}\|x_n - PTx_n\| = 0$ (since $\|x_n - PTx_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - PTx_n\| \to 0$ as $n \to \infty$). Since there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then for all $n \geq n_0$, setting $t_n := \|x_n - PTx_n\|^{\frac{1}{2}}$, we obtain (since $\lim_{n\to\infty}\|x_n - PTx_n\| = 0$) that there exists $n_1 \geq n_0$ such that for all $n \geq n_1$, $t_n \in (0,1)$.

Now, for all $n \geq n_1$, define the mapping $G_n : C \to C$ (for $z \in C$) by

$$G_n z := t_n u + (1-t_n)PTz.$$

It is easy to see that for all $n \geq n_1$, G_n is a contraction; and so has a unique fixed point $z_n \in C$. Thus, for all $n \geq n_1$,

$$z_n := t_n u + (1-t_n)PTz_n.$$

Using Lemma 2.3, we obtain that there exists $z^* \in F(PT)$ such that $z_n \to z^*$ as $n \to \infty$.

Observe that for all $n \geq n_1$,

$$z_n - x_n = t_n(u - x_n) + (1-t_n)(PTz_n - x_n).$$

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Thus, by Lemma 2.2, we have that

$$\begin{aligned}
\|z_n - x_n\|^2 &\leq (1-t_n)^2\|PTz_n - x_n\|^2 + 2t_n\big\langle u - x_n, j(z_n - x_n)\big\rangle \\
&\leq (1-t_n)^2\Big(\|PTz_n - PTx_n\| + \|PTx_n - x_n\|\Big)^2 \\
&\quad +2\Big(\|z_n - x_n\|^2 + \big\langle u - z_n, j(z_n - x_n)\big\rangle\Big) \\
&\leq (1+t_n^2)\|z_n - x_n\|^2 + 2t_n\big\langle u - z_n, j(z_n - x_n)\big\rangle \\
&\quad +\|PTx_n - x_n\|\Big(2\|z_n - x_n\| + \|PTx_n - x_n\|\Big).
\end{aligned}$$

This implies that,

$$\big\langle u - z_n, j(x_n - z_n)\big\rangle \leq \Big[\frac{t_n}{2} + \frac{\|PTx_n - x_n\|}{2t_n}\Big]M.$$

for some $M > 0$. Thus,

$$\big\langle u - z_n, j(x_n - z_n)\big\rangle \leq Mt_n.$$

This implies that

$$\limsup_{n\to\infty}\big\langle u - z_n, j(x_n - z_n)\big\rangle \leq 0. \tag{3.2}$$

Moreover, we have that

$$\begin{aligned}
\big\langle u - z_n, j(x_n - z_n)\big\rangle &= \big\langle u - z^*, j(x_n - z^*)\big\rangle + \big\langle u - z^*, j(x_n - z_n) - j(x_n - z^*)\big)\big\rangle \\
&\quad +\big\langle z^* - z_n, j(x_n - z_n)\big\rangle.
\end{aligned}\tag{3.3}$$

Thus, since $\{x_n\}_{n\geq 0}$ is bounded, we have that $\big\langle z^* - z_n, j(x_n - z_n)\big\rangle \to 0$ as $n \to \infty$. Also, $\big\langle u - z^*, j(x_n - z_n) - j(x_n - z^*)\big\rangle \to 0$ as $n \to \infty$ since the normalized duality mapping j is norm-to-weak* uniformly continuous on bounded subsets of E. Thus as $n \to \infty$, we obtain from (3.2) and (3.3) that

$$\limsup_{n\to\infty}\big\langle u - z^*, j(x_n - z^*)\big\rangle \leq 0. \tag{3.4}$$

Now, put

$$\mu_n := \max\{0, \big\langle u - z^*, j(x_n - z^*)\big\rangle\}.$$

Then, $0 \leq \mu_n\ \forall\ n \geq 0$. It is easy to see that $\mu_n \to 0$ as $n \to \infty$ since by (3.4), if $\varepsilon > 0$ is given, there exists $n_\varepsilon \in \mathbb{N}$ such that $\big\langle u - z^*, j(x_n - z^*)\big\rangle < \varepsilon\ \forall\ n \geq n_\varepsilon$. Thus, $0 \leq \mu_n < \varepsilon\ \forall\ n \geq n_\varepsilon$. So, $\lim\limits_{n\to\infty} \mu_n = 0$.

Next, we obtain from the recursion formula (3.1) that

$$x_{n+1} - z^* = \alpha_n(u - z^*) + (1-\alpha_n)(PTx_n - z^*).$$

It follows that

$$\begin{aligned}
\|x_{n+1} - z^*\|^2 &\leq (1-\alpha_n)^2\|PTx_n - z^*\|^2 + 2\alpha_n\big\langle u - z^*, j(x_{n+1} - z^*)\big\rangle \\
&\leq (1-\alpha_n)\|x_n - z^*\|^2 + 2\alpha_n\mu_{n+1} \\
&= (1-\alpha_n)\|x_n - z^*\| + \gamma_n,
\end{aligned}$$

where $\gamma_n = 2\alpha_n\mu_{n+1}$. Therefore, $\gamma_n = o(\alpha_n)$ and by Lemma 1, we obtain that $\{x_n\}_{n\geq 0}$ converges strongly to $z^* \in F(PT)$. Again, by hypothesis, $F(PT) = F(T)$, and by Lemma 2.4, $F(T) = \bigcap\limits_{i=1}^{\infty} F(T_j)$. Hence, $\{x_n\}_{n\geq 1}$ converges strongly to the common fixed point of the family $\{T_j\}_{j=1}^{\infty}$. This completes the proof. □

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Corollary 3.1. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a nonexpansive retract of E with P as the nonexpansive retraction. Let $T_j : C \to E,\ j = 1, 2, ..., r$ be r nonself nonexpansive mappings such that $F = \bigcap_{i=1}^{r} F(T_j) \neq \emptyset$, for some $r \in \mathbb{N}$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)PTx_n,\ n \geq 1, \tag{3.5}$$

where $T = \sum_{j=1}^{r} \eta_j T_j$ and $\{\eta_j\}_{j=1}^{r}$ is a finite collection of positive numbers such that $\sum_{j=1}^{r} \eta_j = 1$. Suppose that $F(T) = F(PT)$, and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$.

Proof: The mapping $T = \sum_{j=1}^{r} \eta_j T_j$ is clearly nonexpansive. Following the argument of the proof of Lemma 2.4 we get that $F(T) = \bigcap_{i=1}^{r} F(T_j)$. The rest follows as in the proof of Theorem 3.1. This completes the proof. □

In the following theorem, the assumption $F(T) = F(PT)$ is dispensed with.

Theorem 3.2. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a sunny non-expansive retract of E with P as the sunny nonexpansive retraction. Let $T_j : C \to E,\ j \geq 1$ be a sequence of nonself nonexpansive mappings satisfying weakly inward condition with $\bigcap_{i=1}^{\infty} F(T_j) \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)PTx_n,\ n \geq 1, \tag{3.6}$$

where $T = \sum_{j=1}^{\infty} \xi_j T_j$. Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{\infty}$.

Proof. By Lemma 2.5, T is weakly inward and $F(T) = F(PT)$. The rest of the proof follows as in the proof of Theorem 3.1. □

If in Theorem 3.2, we assume that every nonempty closed convex subset of C has the fixed point property for nonexpansive mappings, the requirement that $\bigcap_{i=1}^{\infty} F(T_j) \neq \emptyset$ may not be required. In fact, we have the following theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a sunny non-expansive retract of E with P as the sunny nonexpansive retraction. Let $T_j : C \to E,\ j \geq 1$ be a sequence of nonself nonexpansive mappings satisfying weakly inward condition. Suppose that every nonempty closed convex subset of C has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)PTx_n,\ n \geq 1, \tag{3.7}$$

where $T = \sum_{j=1}^{\infty} \xi_j T_j$. Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of of $\{T_j\}_{j=1}^{\infty}$.

Proof. By Lemma 2.5, T is weakly inward and $F(PT) = F(T)$. Thus, Theorem 13.5 of Geobel and Kirk [68] implies that $F(T) \neq \emptyset$ and since by Lemma 2.4, $F(T) = \bigcap_{i=1}^{\infty} F(T_j)$, we obtain that $\bigcap_{i=1}^{\infty} F(T_j) \neq \emptyset$. The rest of the proof follows as in the proof of Theorem 3.1. □

If in Theorem 3.1, $\{T_j\}_{j\geq 1}$ is a collection of self mappings then the nonexpansive retraction P becomes the identity operator I defined on E. Moreover, each T_j, $j \geq 1$ is authomatically weakly inward. Thus, we have the following corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_j : C \to C$, $j \geq 1$ be a sequence of nonexpansive self mappings with $\bigcap_{i=1}^{\infty} F(T_j) \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \ n \geq 1, \tag{3.8}$$

where $T = \sum_{j=1}^{\infty} \xi_j T_j$. Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - Tx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{\infty}$.

Corollary 3.3. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Let $T_j : C \to C$, $j \geq 1$ be a sequence of nonexpansive self mappings. Suppose that every nonempty closed convex subset of C has the fixed point property for nonexpansive mappings. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \ n \geq 1, \tag{3.9}$$

where $T = \sum_{j=1}^{\infty} \xi_j T_j$. Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - Tx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{\infty}$.

If in Theorems 3.2 and 3.3, we consider finite family of nonexpansive mappings, then we get the following corollaries:

Corollary 3.4. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a sunny nonexpansive retract of E with P as the sunny nonexpansive retraction. Let $T_j : C \to E$, $j = 1, 2, ..., r$ be r nonself nonexpansive mappings satisfying weakly inward condition, for some $r \in \mathbb{N}$ with $\bigcap_{i=1}^{r} F(T_j) \neq \emptyset$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)PTx_n, \ n \geq 1, \tag{3.10}$$

where $T := \sum_{j=1}^{r} \eta_j T_j$; and $\{\eta_j\}_{j=1}^{r}$ a finite collection of positive real numbers such that $\sum_{j=1}^{r} \eta_j = 1$. Suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$.

Corollary 3.5. *Let C be a nonempty closed convex subset of a reflexive and strictly convex real Banach space E which has a uniformly Gâteaux differentiable norm. Suppose C is a sunny nonexpansive retract of E with P as the sunny nonexpansive retraction. Let $T_j : C \to E$, $j = 1, 2, ..., r$ be*

r nonself nonexpansive mappings satisfying weakly inward condition, for some $r \in \mathbb{N}$. For arbitrary $u, x_1 \in C$, let $\{x_n\}_{n\geq 1}$ be iteratively generated by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)PTx_n,\ n \geq 1, \tag{3.11}$$

where $T := \sum_{j=1}^{r} \eta_j T_j$; and $\{\eta_j\}_{j=1}^{r}$ a finite collection of positive real numbers such that $\sum_{j=1}^{r} \eta_j = 1$. Suppose that every nonempty closed convex subset of C has the fixed point property for nonexpansive mappings; and suppose that there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|x_n - PTx_n\| \neq 0$, then $\{x_n\}_{n\geq 1}$ converges strongly to a common fixed point of $\{T_j\}_{j=1}^{r}$.

4 Convergence theorem for countably infinite family of strictly pseudocontractive mappings

Let E be a normed space. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *k-strictly pseudocontractive* if there exists a real constant $k > 0$ such that for all $x, y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Tx - Ty, j(x-y)\big\rangle \leq \|x-y\|^2 - k\|x - y - (Tx - Ty)\|^2. \tag{4.1}$$

Without loss of generality we may assume that $k \in (0,1)$. If I denotes the identity operator, then (4.1) can be re-written as

$$\big\langle (I-T)x - (I-T)y, j(x-y)\big\rangle \geq k\|(I-T)x - (I-T)y)\|^2. \tag{4.2}$$

In Hilbert spaces, (4.1) (or equivalently (4.2)) is equivalent to the inequality

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \beta\|(I-T)x - (I-T)y\|^2, \text{ where } \beta = (1-k) < 1.$$

In the sequel, we shall need the following lemma.

Lemma 4.1. *(See e.g. [59]) Let E be a q-uniformly smooth real Banach space for some $q > 1$, then there exists some positive constant d_q such that*

$$\|x+y\|^q \leq \|x\|^q + q\big\langle y, j_q(x)\big\rangle + d_q\|y\|^q \ \forall\, x, y \in E,\ \forall\, j_q(x) \in J_q(x).$$

Now, if E is a q-uniformly smooth real Banach space; and $T : D(T) \to E$ is a k-strictly pseudocontractive mapping, then for the map $T_\lambda := (1-\lambda)I + \lambda T) : D(T) \to E$ (where I is the identity map of $D(T)$ and $\lambda > 0$), we obtain by Lemma 4.1 that:

$$\begin{aligned}
\|T_\lambda x - T_\lambda y\|^q &= \|x - y - \lambda\Big((I-T)x - (I-T)y\Big)\|^q \\
&\leq \|x-y\|^q - q\lambda\big\langle (I-T)x - (I-T)y, j_q(x-y)\big\rangle \\
&\quad + d_q\lambda^q\|(I-T)x - (I-T)y\|^q \\
&\leq \|x-y\|^q - \lambda(kq - d_q\lambda^{q-1})\|Ax - Ay\|^q,
\end{aligned}$$

where $A = (I-T)$. If λ is such that $0 < \lambda < \left(\frac{kq}{d_q}\right)^{\frac{1}{q-1}}$, we have that the mapping T_λ is a nonexpansive. It is also easy to see that the fixed point set of T_λ and T coincide.

Thus, we have the following theorem

Theorem 4.2. *Let C be a nonempty closed convex subset of a strictly convex q-uniformly smooth real Banach space E. Suppose C is a nonexpansive retract of E with P as the nonexpansive retraction. Let $T_j : C \to E$, $j \geq 1$ be a sequence of nonself k-stricltly pseudocontractive mappings*

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such that $F = \bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$*. Let* $\{\lambda_j\}_{j=1}^{\infty}$ *be such that* $0 < \lambda_j < \left(\frac{q\alpha}{d_q}\right)^{\frac{1}{q-1}}$*,* $j = 1, 2, \ldots$ *and define* $T_{\lambda_j} = (1-\lambda_j)I + \lambda_j T_j$*. For arbitrary* $u, x_1 \in C$*, let* $\{x_n\}_{n\geq 1}$ *be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)PTx_n, \ n \geq 1. \tag{4.3}$$

where $T := \sum_{j\geq 1} \xi_j T_{\lambda_j}$*. Suppose that* $F(T) = F(PT)$*; and suppose that there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*,* $\|x_n - PTx_n\| \neq 0$*, then* $\{x_n\}_{n\geq 1}$ *converges strongly to a common fixed point of* $\{T_j\}_{j=1}^{\infty}$*.*

Proof. Since q-uniformly smooth real Banach spaces are reflexive and have uniformly Gâteaux differentiable norm (see e.g., [66]), the proof follows as in the proof of Theorem 3.1 since T_{λ_j} is nonexpansive for each $j \in \mathbb{N}$ and $\bigcap_{j=1}^{\infty} F(T_{\lambda_j}) = \bigcap_{j=1}^{\infty} F(T_j)$.

Theorem 4.3. *Let* C *be a nonempty closed convex subset of strictly convex* q*-uniformly real Banach space* E*. Suppose* C *is a sunny nonexpansive retract of* E *with* P *as the sunny nonexpansive retraction. Let* $T_j : C \to E$*,* $j \geq 1$ *be a sequence of nonself* k*-strictlty pseudocontractive mappings satisfying weakly inward condition. Let* $\{\lambda_j\}_{j=1}^{\infty}$ *be such that* $0 < \lambda_j < \left(\frac{q\alpha}{d_q}\right)^{\frac{1}{q-1}}$*,* $j = 1, 2, \ldots$ *and define* $T_{\lambda_j} = (1-\lambda_j)I + \lambda_j T$*. For arbitrary* $u, x_1 \in C$*, let* $\{x_n\}_{n\geq 1}$ *be iteratively generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)PTx_n, \ n \geq 1. \tag{4.4}$$

where $T := \sum_{j\geq 1} \xi_j T_{\lambda_j}$*. Suppose that there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*,* $\|x_n - PTx_n\| \neq 0$*, then* $\{x_n\}_{n\geq 1}$ *converges strongly to a common fixed point of* $\{T_j\}_{j=1}^{\infty}$*.*

Proof. The proof follows as in the proof of Theorem 3.3 since T_{λ_j} is nonexpansive for each $j \in \mathbb{N}$ and $\bigcap_{j=1}^{\infty} F(T_{\lambda_j}) = \bigcap_{j=1}^{\infty} F(T_j)$ and it is well known that since E is q-uniformly smooth Banach space, every nonempty closed convex subset of C has the fixed point property for nonexpansive mappings (see e.g. [63, 64]).

The addition of bounded error terms to our recursion formulas leads to no further generalization.

If $f : K \to K$ is a contraction mapping and u is replaced by $f(x_n), n \in \mathbb{N}$ in the recursion formulas of theorems presented in this paper, then what some authors now call *viscosity* iteration process will be obtained. Observe that all the theorems in this paper carry over trivially to the so-called viscosity process. One simply replaces u by $f(x_n)$, repeats the argument of this paper, using the fact that f is a contraction map. So, consideration of viscosity iteration process in the cases under study lead to no further generalization.

It is of interest to note that the family $\{T_j\}_{j\in\mathbb{N}}$ of nonself nonexpansive mappings in the theorems obtained in this paper need not satisfy condition A. The main results of this paper are applicable, in particular, in L_p spaces, $1 < p < +\infty$. The theorems obtained extend and improve the correponding results of Liu *et al* [25], Maiti and Saha [8], Senter and Doston [14], Jung [47] and that of a host of other authors in the sense that the results of these authors were obtained for finite family of self nonexpansive mappings in uniformly convex real Banach spaces, while the main results of this paper (see Theorems 3.1, 3.2 and 3.3) took into consideration approximation of common fixed point of countably infinite family of nonself nonexpansive mappings in more general strictly convex and reflexive real Banach spaces with uniformly Gâteaux differentiable norm. It is note worthy that in all the results obtained in this paper, the so called condition A as imposed in the works of Liu *et al* [25], Maiti and Saha [8], Senter and Doston [14] is dispensed with. Moreover, Theorems 4.2 and 4.3 are of independent interest, in the sense that they provided means of obtaining strong

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convergence results in strictly convex q-uniformly smooth real Banach spaces for approximation of common fixed point of countably infinite family of nonself k-strickly pseudocontractive mappings - a class of mappings more general than the class of nonexpansive mappings. Furthermore, as will be seen in the numerical example provided below, choices of iterative parameters $\{\alpha_n\}_{n\geq 1}$ used in this paper are canonical, in the sense that for any $n \in \mathbb{N}$, $\alpha_n = \frac{1}{n+1}$ (for example) is a prototype satisfying the conditions on our iterative parameters.

5 Numerical Example

We now give the following numerical example to show the applicability of algorithm (3.1) and consider possible implications of remark 4.

Let $E = \mathbb{R}$ and $C = [-1, 1]$. We define the following sequence of nonexpansive mappings; for $j \in \mathbb{N}$, $\{T_j\}_{j=1}^{\infty}$ as;

$$\begin{cases} T_{3j-2} = j \sin x \\ T_{3j-1} = x \cos(\frac{x}{j}) \\ T_{3j} = \frac{\sin^j x}{2} \end{cases} \tag{5.1}$$

Clearly, $0 \in F = \bigcap_{j=1}^{\infty} F(T_j)$. Let $\{\xi_j\}_{j=1}^{\infty} = \{2^{-j}\}_{j=1}^{\infty}$, and $\{\alpha_n\}_{n=1}^{\infty} = \{\frac{1}{n+1}\}_{n=1}^{\infty}$. We have the tolerance of error given as $TOL_n := \|x_{n+1} - x_n\|^2 < 10^{-7}$. We will examine the convergence of algorithm (3.1) for 3 different cases; namely,

Case1 when u is arbitrarily chosen in (3.1).

Case2 when u in (3.1) is replaced with $f(x_n)$, $n \in \mathbb{N}$, where $f : [-1, 1] \to [-1, 1]$ is a contraction defined for all $x \in [-1, 1]$ by $f(x) = \sin x$.

Case3 when u in (3.1) is replaced with $g(x_n)$, $n \in \mathbb{N}$, where $g : [-1, 1] \to [-1, 1]$ is a contraction defined for all $x \in [-1, 1]$ by $g(x) = \frac{1}{1+x^2}$.

All the computations are performed using Spyder (Python 3.8) which is running on a personal computer with an Intel(R) Core(TM) i5-4300 CPU at 2.50GHz and 8.00 Gb-RAM.
Also, in Table 1, *CPU* means the time in seconds it takes the computer prepossessing unit for computation and *Iter* (n) means the number of iterations.

	$x_0 = 0.85$		$x_0 = 0.50$	
Algorithm (3.1)	CPU	iter (n)	CPU	iter (n)
$u = 3.0$	0.035978	84	0.037960	88
$f(x) = \sin x$	0.008992	13	0.004000	8
$g(x) = \frac{1}{1+x^2}$	0.005998	10	0.011991	31

Table 1: Comparison of algorithm (3.1) with fixed u, $f(x)$, and $g(x)$.

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0.40

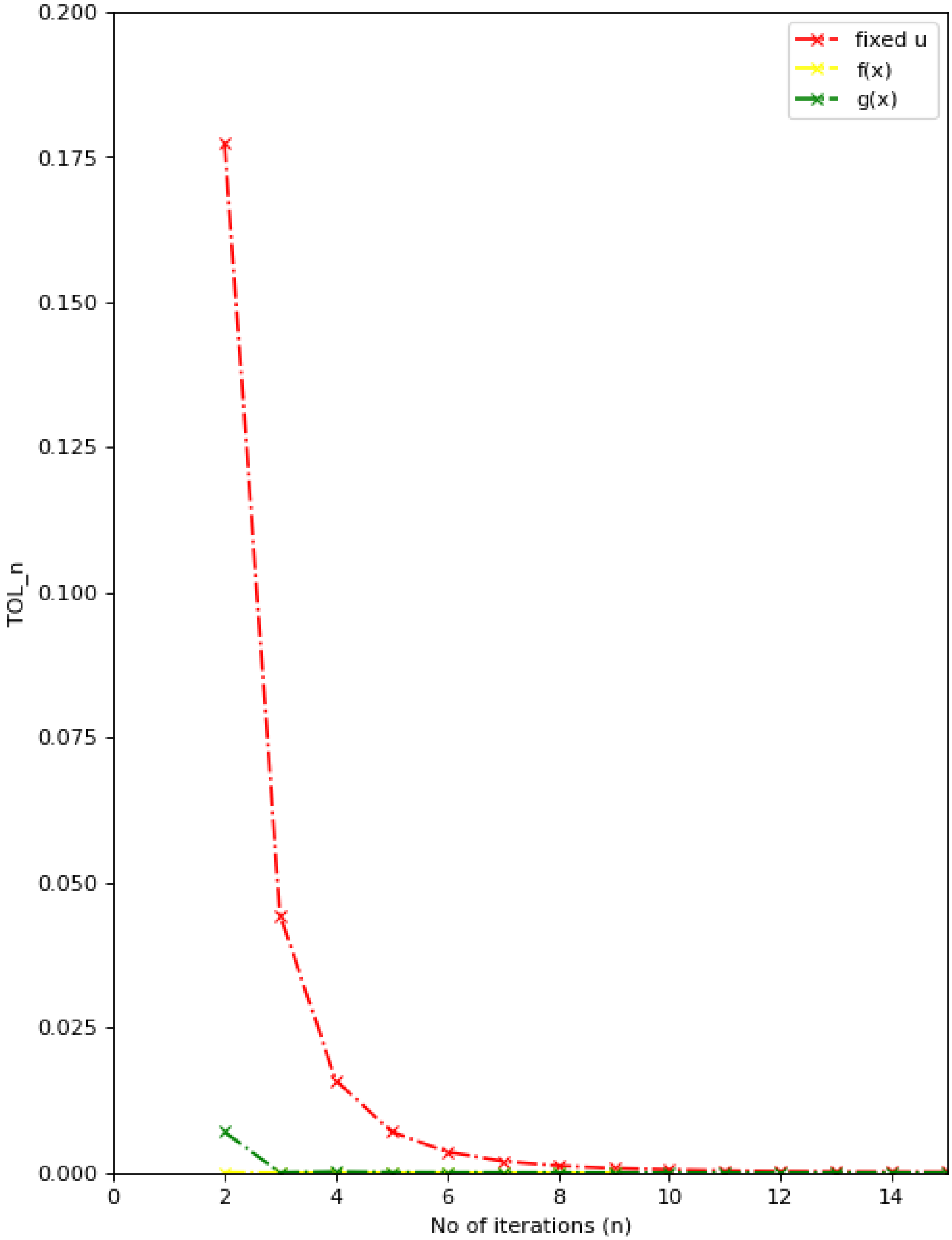

Figure 1: $x_0 = 0.85$

0.40

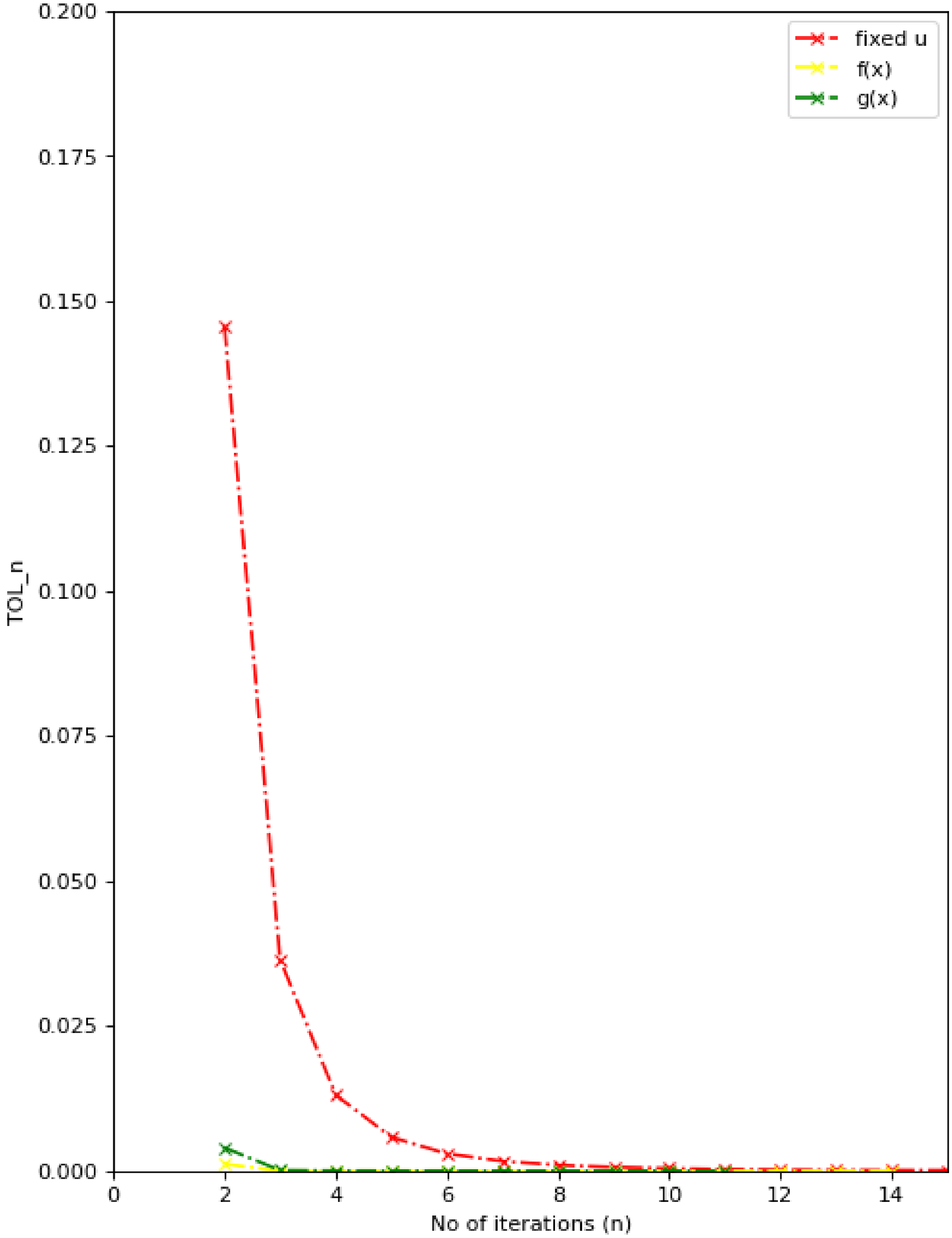

Figure 2: $x_0 = 0.5$

Figure 3: The Graphs of $\|x_{n+1} - x_n\|^2$ against n

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