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Gamma Generalized Power Lomax: A Novel Distribution in Modelling Covid-19 Data

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Abstract

In this study, we developed a flexible form of Power Lomax distribution called Gamma Power Lomax distribution. This distribution was obtained by combining the Power Lomax and Gamma distributions to generates a novel life-time distribution with three parameters. Different structural characteristics of the new distribution have been determined and analysed which includes moments, moment generating function, order statistics, and incomplete moments. Distinct plots depict the behaviour of the density and the distribution function. The maximum likelihood estimation method was applied to estimate the distribution parameters. The practicability of the stated distribution is proven by Covid-19 data set. The objective of this study is to establish an optimum statistical model considering current covid-19 pandemic.

Keywords: Moments, generating function, covid-19 data, incomplete moments, maximum likelihood function.
MSC2010: 26A18.

1 Introduction

Recently, several attempts have been made to develop new families of distribution to provide great flexibility and enhance the scope of applications of lifetime models in modeling skewed lifetime data in practice. The addition of parameter(s) to a classical distribution is a well-established device for obtaining more flexible new distributions, example includes Alpha power-g, Marshall and Olkin-g by [8], Weibull-g by [3], Inverse-Weibull-g by [1], Exponentiated-g by [11], Truncated Inverted Kumaraswamy-g by [4] , Nadarajah Haghighi Topp Leone-g by [7], beta–g, gamma-g [13], odd-weibull family by [4], alpha power transformed Weibull g family by [19], Gompertz-g by [10], Cauchy-g by [20], type II power Topp-Leone-g by [21], and many more others. The focus of this study is to develop a four-parameter gamma power Lomax distribution which is an extension of

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Power Lomax distribution developed and studied by Rady [18]. Various variant of the power Lomax distribution has been studied and this includes Transmuted power Lomax by [11], inverse PL distribution studied by [6], type II Topp-Leone power Lomax distribution proposed and studied by [2], Kumaraswamy Generalized Power Lomax Distribution by [14], Marshall–Olkin power Lomax Distribution developed by [5]. Harris power Lomax by [15], Bivariate power Lomax distribution studied by [16], Alpha Power Power-Lomax was introduced by [17], among many others. The Cumulative Distribution Function (CDF) is given by;

$$G(x;\alpha,\lambda,\beta) = 1-(1+\frac{x^{\lambda}}{\theta})^{-\alpha} \qquad ;x>0;\alpha,\lambda,\theta>0 \tag{1.1}$$

Where α and λ are the shape parameters and θ is a scale parameter.

With corresponding probability density function (PDF) given as;

$$g(x,\frac{\alpha}{\theta},\lambda) = \frac{\alpha^{\lambda}}{\theta}x^{\lambda-1}(1+\frac{x^{\lambda}}{\theta})^{\alpha-1} \qquad ;x>0;\alpha,\lambda,\theta>0 \tag{1.2}$$

It should be noted that the PDF of the Power Lomax distribution decreases if $\theta \leq 1$ and increases if $\theta > 1$.

2 Gamma Power Lomax distribution.

For nay baseline CDF f(x), x $\in$ R, [13] define the gamma-g distribution with PDF f(x) and CDF F(x) which added a shape parameter (v) is given by

$$f(x) = \frac{1}{\Gamma(v)}\left\{-\log[1-G(x)]\right\}^{v-1} g(x) \tag{2.1}$$

and

$$G(x) = \frac{\gamma(v,-\log[1-G(x)])}{\Gamma(v)} \tag{2.2}$$

Respectively, where $g(x) = \frac{dG(x)}{dx}.\Gamma(v) = \int_0^{\infty} t^{v-1}e^{-t}dt$,and$\gamma(v,y) = \int_0^{y} t^{v-1}e^{-t}dt$ are the gamma and the incomplete gamma functions. The shape parameter v is meant to controls kurtosis and skewness through the weights.

We define the Power Gamma Lomax (GPL) density function by inserting (1) and (2) into (3). Then,

$$f(x) = \frac{1}{\Gamma(v)}\frac{\alpha\lambda}{\theta}x^{\lambda-1}\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha-1}\left\{-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{\alpha}\right\}^{v-1}, x>0;\alpha,\lambda,\theta,v \tag{2.3}$$

The corresponding CDF can be obtained by integrating (5) to obtain

$$F(x) = \frac{1}{\Gamma}\gamma\left(v,-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right), x>0. \tag{2.4}$$

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For positive shape parameters α, λ, v and scale parameter θ. For a random variable having density function G(x), we write $X \sim GPL(\alpha, \lambda, \theta, v)$. The corresponding survival and hazard rate function is respectively, given by

$$S(x) = \frac{1}{\Gamma(v)}\gamma\left(v, -\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right), x > 0, \tag{2.5}$$

and

$$h(x) = \frac{\frac{\alpha\lambda}{\theta}x^{\lambda-1}\left(1+\frac{x^{-\lambda}}{\theta}\right)^{\alpha-1}\left\{-\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right\}^{v-1}}{\Gamma(v)-\gamma\left(v, -\log\left(1+\frac{x^{\lambda}}{\theta}\right)^{-\alpha}\right)} \tag{2.6}$$

Figures 1 and 2 shows some possible shapes for CDF (5), PDF (7) and hazard (8) for some hypothetical parameters values, respectively. The density function can take different forms of shapes depending on the parameter values displaying its flexibility as a lifetime distribution.

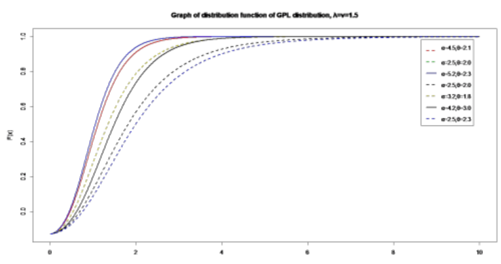

Figure 1: Graph of the *CDF* OF *GPL* distribution.

- The graph indicates that *GPL* distribution has a proper PDF

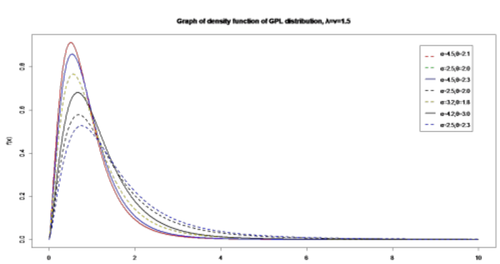

Figure 2: Graph of the PDF OF *GPL* distribution.

- The graph indicates that PDF of *GPL* distribution is non-monotone

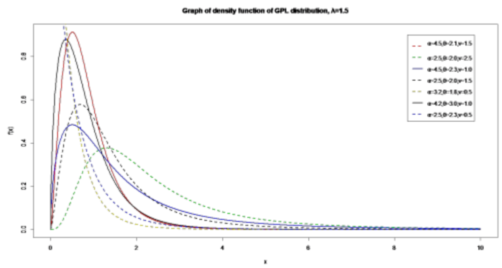

Figure 3: Graph of the PDF OF *GPL* distribution.

- The graph indicates that PDF of GPL distribution is non-monotone

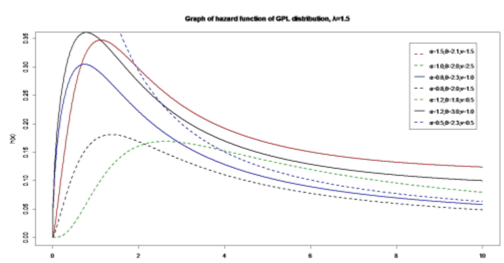

Figure 4: Graph of the PDF OF GPL distribution.

- Figure 4 shows that the hazard function of GPL model exhibits different shapes of the hazard rate, which may be increasing, decreasing, and inverted bathtub.

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3 Properties of GPL distribution

The binomial coefficient generalized to real arguments is given by $\binom{y}{z} = \frac{\Gamma(y+1)}{\Gamma(y+1)\Gamma(y-z+1)}$ For any real parameter $z > 0$ the following formula holds (http://functions.wolfram.com/ElementaryFunctions/Log/06/01/04/03/)

$$\{-\log[1-G(x)]\}^{z-1} = (z-1)\sum_{l=0}^{\infty}\sum_{p=0}^{l}\binom{l+1-z}{l}\frac{(-1)^{l+p}\binom{l}{p}Q_{p,l}}{(z-1-p)}G(x)^{z+l-1} \tag{3.1}$$

Where the constant $Q_{l,p}$ can be estimated recursively by

$$Q_{l,p} = l^{-}1\sum_{m=1}^{l}(-1)^{m+1}\frac{[l-m(p+1)]}{m+1}Q_{p,l-m}, \tag{3.2}$$

For l = 1, 2, ... , $Q_{l,0} = 1$
Using the expression given in (3.1), we can summarize the PDF in (2.3) as

$$f(x) = (\alpha-1)\frac{1}{\Gamma(v)}\frac{\alpha\lambda}{\theta}\sum_{l=0}^{\infty}H^{p,q}x^{\lambda-1}\left(1+\frac{x^{\lambda}}{\theta}\right)^{-[(q\alpha+\alpha)+1]} \tag{3.3}$$

Where,

$$H^{p,q} = \sum_{q}^{\infty}\sum_{p=0}^{l}\binom{l+1-v}{l}\binom{v+1-1}{q}\frac{(-1)^{l+p+q}\binom{l}{p}Q_{p,l}}{v-1-p} \tag{3.4}$$

3.1 Ordinary and Incomplete Moments of GPL distribution

The r^{th} moment of GPL distribution can be obtained using the relation

$$E(x^r) = \mu_r' = \int_{\infty}^{-\infty} x^r f(x)d(x) \tag{3.5}$$

Plugging (3.3) in (3.5), we have;

$$\mu_r^{'} = (z-1)\frac{1}{\Gamma(v)}\frac{\alpha\lambda}{\theta}\sum_{i=0}^{\infty}H^{p,q}\int_{-\infty}^{\infty}x^{r+\lambda-1}\left(1+\frac{x^{\lambda}}{\theta}\right)^{-[(q\alpha+\alpha)+1]}dx \tag{3.6}$$

Now, letting $w = \frac{x^{\lambda}}{\theta}, x = (w\theta)^{\frac{1}{\lambda}}, dx = \frac{1}{\lambda}\theta^{\frac{1}{\lambda}}w^{\frac{1}{\lambda-1}}dw$ and putting it in (3.6), we have

$$\mu_r' = (z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{l=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}\int_{-\infty}^{\infty}w^{\frac{r-\lambda}{\lambda}}(1+w)^{-[(q\alpha+\alpha)+1]}dw \tag{3.7}$$

Consequently, taking $w = \frac{y}{1-y}$, $dw = \frac{1}{(1-y)^2}$, and putting in (3.7), we obtain

$$\mu_r' = (z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{l=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}\int_{-\infty}^{\infty}y^{\frac{r-\lambda}{\lambda}}(1-y)^{[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-2}dy \tag{3.8}$$

Finally, we have

$$\mu_r' = (z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{l=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}B\left(\frac{r-\lambda}{\lambda}+1,[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-1\right) \tag{3.9}$$

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Where $B(a,n)$ represents the standard beta function defined by $B(a,n)=\int_0^1 v^{\alpha-1}(1-v)^{n-1}dv$. The mean of GPL distribution is given by taking $r=1$ in (3.9), thus, we have

$$\mu_1^{'}=\mu=(z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{l=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}B\left(\frac{1-\lambda}{\lambda}+1,[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-1\right) \tag{3.10}$$

An expression for the incomplete moments is obtained by

$$\phi_r^{'}=\int\limits_0^t x^r f(x)dx \tag{3.11}$$

Putting (3.3) in (3.11), we have

$$\phi_r^{'}=(z-1)\frac{1}{\Gamma(v)}\frac{\alpha\lambda}{\theta}\sum_{i=0}^{\infty}H^{p,q}\int\limits_0^t x^{r+\lambda-1}\left(1+\frac{x^\lambda}{\theta}\right)^{-[(q\alpha+\alpha)+1]}dx \tag{3.12}$$

Now, letting $w=\frac{x^\lambda}{\theta}$, $x=(w\theta)^{\frac{1}{\lambda}}$, $dx=\frac{1}{\lambda}\theta^{\frac{1}{\lambda}}w^{\frac{1}{\lambda-1}}dw$ and plugging it in (3.12), we have

$$\phi_r^{'}=(z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{i=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}\int\limits_0^{\frac{t^\lambda}{\theta}}\theta^{\frac{r-\lambda}{\lambda}}(1+w)^{-[(q\alpha+\alpha)+1]}dw \tag{3.13}$$

Also, taking $w\frac{y}{1-y}$, $dw=\frac{1}{(1-y)^2}$, and putting it in (21), we obtain

$$\phi_r^{'}=(z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{i=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}\int\limits_0^{\frac{t^\lambda}{\theta+t^\lambda}}y^{\frac{r-\lambda}{\lambda}}(1-y)^{[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-2}dy \tag{3.14}$$

Finally, we obtain

$$\phi_r^{'}=(z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{i=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}B\left(\frac{t^\lambda}{\theta+t^\lambda};\frac{r-\lambda}{\lambda}+1,[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-1\right) \tag{3.15}$$

where $B(x;p,q)=\int_0^x y^{p-1}(1-y)^{q-1}dy$ is a beta function
From the incomplete moment of X, other important quantities and functions of X can be obtained. This may include the Bonferroni and Lorenz curve, the mean deviation of X about the mean and the median can be obtained.

3.2 Moment generating function of GPL model

The moment generating function of X can be obtained from the relation.

$$M_x(t)=E(e^{tx})=\int\limits_0^\infty e^{tx}f(x)dx=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r' \tag{3.16}$$

Putting (3.9) in (3.16), we obtain the moment generating function of GPL distribution as

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}(z-1)\frac{1}{\Gamma(v)}\frac{\alpha}{\theta}\sum_{i=0}^{\infty}H^{p,q}\theta^{\frac{r-\lambda}{\lambda}}B\left(\frac{r-\lambda}{\lambda}+1,[(q\alpha+\alpha)+1]-\frac{r-\lambda}{\lambda}-1\right) \tag{3.17}$$

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Order Statistics

let $x_{1:n} < x_{2:n} < \ldots x_{2:n}$ be an order statistics obtained from the GPL distribution. Then the PDF, $f_{r:n}(x)$, of the r^{th} order statistics $x_{r:n}$ is:

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x), \tag{3.18}$$

using binomial series expansion given by

$$(1-J)^v = \sum_{i=1}^{\infty}\binom{v}{1}(-1)^i J^i \tag{3.19}$$

Then, we have

$$f_{r:n}(x) = \frac{1}{B(s, n-r+1)}\sum_{i=1}^{n}\binom{n-r}{i}(-1)^i[F(x)]^{r+i-1}f(x) \tag{3.20}$$

,
Where F(x) and f(x) are the CDF and PDF of the GPL distribution respectively, and B(.,.) is the beta function. Putting the CDF and the PDF of the GPL distribution in (5) and (6), follows by algebraic manipulation using (9) gives:

$$f_{r:n}(x) = \frac{(\alpha-1)\frac{1}{\Gamma(v)\frac{\alpha\lambda}{\theta}}}{B(s, n-r+1)}\sum_{i=1}^{n}\sum_{l=0}^{\infty}H^{p,q}\binom{n-r}{i}(-1)^i\left[\frac{1}{\Gamma(v)}\gamma\left(v, -log\left(1+\frac{x^\lambda}{\theta}\right)^{-\alpha}\right)\right]^{r+i-1}$$
$$\times\, x^{\lambda-1}\left(1+\frac{x^\lambda}{\theta}\right)^{[}(q\alpha+\alpha)+1] \tag{3.21}$$

The 1^st and the n^{th} order statistics can be obtained from (3.21) by taking $r = 1$ and $r = n$ respectively.

4 Parameter Estimation

The parameters of the GPL distribution are estimated using the maximum-likelihood estimation technique. Given a random sample $x_1, x_2, x_3, \ldots, x_n$ of size n from the GPL distribution with parameter vector $\psi = (\alpha, \lambda, \theta, v)'$, then the log-likelihood function is given by:

$$l = \log\left(\frac{1}{\Gamma(v)}\frac{\alpha\lambda}{\theta}\right) - (\lambda-1)\sum_{i=1}^{n}x_i - (\alpha+1)\sum_{i=1}^{n}\log\left(1+\frac{x^\lambda}{\theta}\right) - (v-1)\sum_{i=1}^{n}\log\left[\log\left(1+\frac{x^\lambda}{\theta}\right)^{-\alpha}\right] \tag{4.1}$$

The maximum likelihood (ML) method and its procedures exist in the literature with details.

4.1 Application

This section examines the applicability of the GPL distribution using covid-19 data set. We fitted the GPL distribution and many other models for comparative purposes, including Gamma Lomax (GL), Gamma Frechet (GF), Lomax (L), and the Gamma Inverse Exponential (GIE) distributions. We will use certain measures of goodness of fits to evaluate which of the competitive models is the strongest, including AIC (Akaike Information Criterion), CAIC (Consistent Akaike Information Criterion), BIC (Bayesian Information Criterion) and HQIC (Hannan-Quinn Information Criterion). Such criteria can be represented mathematically by

$$AIC = 2z - 2\ln(l), \quad CAIC = \frac{2zn}{n-z-1} - 2\ln(l)$$

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$$BIC = z\ln(n) - 2\ln(l), and\ HQIC\ = 2z\ln(\ln(n)) - 2\ln(l)$$

Where n is the sample size, z is the number of parameters, l is the log-likelihood value. The model with the lowest value of these indicators is considered the best among the competing models. Covid-19 Data Set: The data shows the mortality rate due to covid-19 of Canada country which has been recorded from first November to 26 December 2020 [https://covid-19.who.int/]. The data follows

0.1622, 0.1159, 0.1897, 0.1260, 0.3025,0.2190, 0.2075, 0.2241, 0.2163, 0.1262, 0.1627, 0.2591, 0.1989, 0.3053, 0.2170, 0.2241, 0.2174, 0.2541, 0.1997, 0.3333,0.2594, 0.2230, 0.2290, 0.1536, 0.2024, 0.2931, 0.2739,0.2607, 0.2736, 0.2323, 0.1563, 0.2677, 0.2181, 0.3019,0.2136, 0.2281, 0.2346, 0.1888, 0.2729, 0.2162, 0.2746, 0.2936, 0.3259, 0.2242, 0.1810, 0.2679, 0.2296, 0.2992, 0.2464,0.2576, 0.2338, 0.1499, 0.2075, 0.1834, 0.3347, 0.2362.

Table 1 shows that the covid-19 data is negatively skewed (skewed to the left), under-dispersed, and mesokurtic. The graph of empirical density indicates that the data is positively skewed and the graph of Total Time on Test (TTT) plot shows that the data exhibits increasing failure rate.

Table 1: **The descriptive statistics for the Covid-19 data**

Min	Q_1	Med.	Mean	Q_3	Kurt.	Skew.	variance	Max
0.1159	0.2017	0.2261	0.2305	0.2677	2.6537	-0.0872	0.0027	0.3347

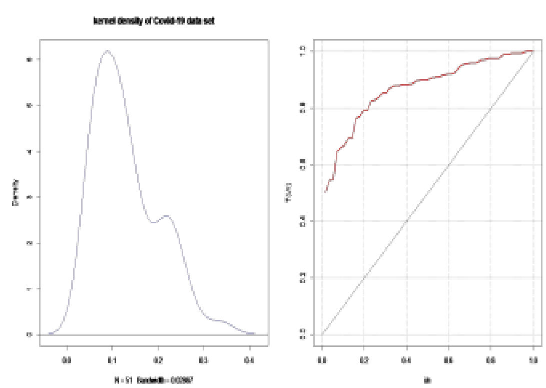

Figure 5: Graph of the empirical density and TTT Plot for Covid-19 data.

Table 2: MLEs of the model parameters, and selection criteria

Model	α	λ	θ	v	l	*AIC*	*BIC*	*HQIC*	*CAIC*
GPL	4.21 (1.43)	4.8616 (0.52)	0.0034 (0.01)	1.1704 (0.37)	-86.13	-164.25	-156.15	-161.11	-163.47
GL	20.71 (5.23)	(—)	0.28 (0.09)	12.4523 (2.45)	-75.3	-144.61	-138.53	-142.25	-144.14
GF	24.38 (7.51)	-2.62 (0.04)	-3.16 (0.24)	(—)	-77.51	-149.02	-142.96	-146.67	-148.56
L	15.37 (8.50)	(—)	3.56 (2.05)	(—)	-24.49	-44.75	-40.93	-43.41	-44.75
GIE	0.07 (0.01)	1.46 (0.14)	(—)	(—)	-22.18	-40.38	-36.31	-38.79 & -40.13	

From Table 2, it is obvious that the Gamma Power Lomax provides a better fit than all other competing models used in this study because it possess that minimum value of AIC, BIC,CAIC and HQIC.

5 Concluding Remarks

The focus of this work is to investigate the ravaging effect of covid-19 pandemic-19 mortality data. In this study, we developed and studied a novel flexible distribution known as the Gamma Power Lomax (GPL). Mathematical characteristics are examined for this distribution, including moments, incomplete moments, moment generating functions, order statistics. The maximum likelihood estimation approach was used to estimate the distribution's parameters. From Table 2 it is evident that the new distribution has a better fit than the comparable ones.

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