International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

Fixed point of multivalued $\rho-$contractive mappings and their comparison in modular function spaces

H. Akewe[1*], A. A. Mogbademu[2], H. O. Olaoluwa[3]

1-3. Department of Mathematics, University of Lagos, Lagos, Nigeria.
* Corresponding author: hakewe@unilag.edu.ng, amogbademu@unilag.edu.ng, holaoluwa@unilag.edu.ng

Article Info
Received: 22 May 2023 Revised: 14 August 2023
Accepted: 16 August 2023 Available online: 30 September 2023

Abstract

This paper presents a comprehensive survey on some multivalued $\rho-$contractive type mappings in the framework of modular function spaces, in which case, existence of fixed points are proved and the various $\rho-$contractive type mappings considered are compared in the framwork of modular function spaces. Our results complement the results obtained on normed and metric spaces in literature. Also, our method of proof serves as a guide to obtain several similar improved results for nonexpansive, pseudo-contractive and accretive type mappings.

Keywords: comparison, fixed point, modular function spaces, multivalued $\rho-$contractve mappings.
MSC2010: 47H09, 47H10.

1 Introduction and Preliminaries

The concept of modular space emanated in 1950 from Nakano [1]. Musielak and Orlicz [2] in 1959 created a theoretical premise of Nakano [1]. The modular function space was introduced as a more general space because it helps one with the apparatus to go much further, the operator itself is used for the construction of a function modular and thus of a space in which this operator has required properties (Khamsi [3]). This unique method is needful in solving several mathematical and fixed point related problems to bring out many complementary results. The modular function spaces have wide range of applications in the field of integral and differential equations and also in machine learning. They are useful tools for constructing the existence and approximation of fixed points and the results complement those in normed and metric spaces.

Many authors have contributed to the development of modular function spaces, chief among them are the results of: Chistyakov [4], Chistyakov [5], Okeke and Khan [6], Okeke et al. [7], Okeke and Francis [8], Okeke et al. [9] and many others, like [10–16]. For instance, in 2010, Chistyakov [4], defined the notion of a modular on an arbitrary set, developed the theory of metric spaces generated by modulars, called modular metric spaces and, on the basis of it, defined new metric spaces

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

of multivalued functions of bounded generalized variation of a real variable with values in metric semigroups and abstract convex cones. In Chistyakov [5], the author gave some applications to superposition operators. Okeke and Khan [6] proved fixed point results for the class of multivalued ρ-quasi-contractive mappings in modular function spaces. Okeke et al. [7] further contributed to the theory of modular function spaces by proving useful fixed point theorems for a class of $\alpha - \nu$ - Meir – Keeler – type contraction mapping in modular extended b-metric spaces. For other relevant and comprehensive research work done by Okeke, Francis and Abbas, see [8], [9] and several references therein. Numerous fixed point results involving contractive and contractive-type mappings in metric and normed linear spaces have been proved in literature, some that are relevant to this study are: [17–50]. For example, the author [49], proved the existence of a unique fixed point of a mapping satisfying rational type contractive inequality conditions in complex-valued Banach spaces while in [50], Okeke et al. established $\rho - T$ stable results with respect to a generalized operator satisfying contractive conditions of integral type in modular function spaces. The main aim of the present paper is to consider some fixed point results on multivalued $\rho-$contractive mappings and compare these mappings in the framework of spaces of measurable functions, known in literature as modular function spaces.

The following definitions will be helpful in proving our main results.
Let Ω be a nonempty set and Σ be a nontrival $\sigma-$algebra of subsets of Ω, P be a $\delta-$ring of subsets of Ω such that $E \cap A \in P$ for any $E \in P$ and $A \in \Sigma$. Assume there exists an increasing sequence $(K_n)_{n\in N} \subset P$ such that $\Omega = \bigcup_{n\in N} K_n$, let I_A represent the characteristic function of the set A in Ω, ϵ represent the linear space of all simple functions with supports from P and M_∞ represent the space of all extended measurable functions, that is all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \epsilon, |g_n| \le |f|$ and $g(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 1 ([1]). Let ρ be a nonzero regular convex function modular defined on Ω.
(i) Let $r > 0$, $\epsilon > 0$. Define $D_1(r, \epsilon) = \{(f, g) : f, g \in X_\rho, \rho(f), \rho(g) \le r, \rho(f - g) \ge \epsilon r\}$. Suppose, $\delta_1(r, \epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f, g) \in D_1(r, \epsilon)\}$ if $D_1(r, \epsilon) \ne \emptyset$ and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \emptyset$. We say that ρ satisfies $(UC1)$ if for every $r > 0$, $\epsilon > 0$, $\delta_1(r, \epsilon) > 0$. Observe that for every $r > 0$, $D_1(r, \epsilon) \ne \emptyset$, $\epsilon > 0$ small enough.

(ii) We say that ρ satisfies $(UUC1)$ if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_1(s, \epsilon) > 0$ depending only on s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(s, \epsilon) > 0$ for any $r > s$.

(iii) Let $r > 0$, $\epsilon > 0$. Define $D_2(r, \epsilon) = \{(f, g) : f, g \in X_\rho, \rho(f), \rho(g) \le r, \rho(\frac{f-g}{2}) \ge \epsilon r\}$. Suppose, $\delta_2(r, \epsilon) = \inf\{1 - \frac{1}{r}\rho(\frac{f+g}{2}) : (f, g) \in D_2(r, \epsilon)\}$ if $D_2(r, \epsilon) \ne \emptyset$ and $\delta_2(r, \epsilon) = 1$ if $D_2(r, \epsilon) = \emptyset$. We say that ρ satisfies $(UC2)$ if for every $r > 0$, $\epsilon > 0$, $\delta_2(r, \epsilon) > 0$. Observe that for every $r > 0$, $D_2(r, \epsilon) \ne \emptyset$, $\epsilon > 0$ small enough.

(iv) We say that ρ satisfies $(UUC2)$ if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_2(s, \epsilon) > 0$ depending only on s and ϵ such that $\delta_2(r, \epsilon) > \eta_2(s, \epsilon) > 0$ for any $r > s$.

(v) We say that ρ is strictly convex (SC), if for every $f, g \in X_p$ such that $\rho(f) = \rho(g)$ and $\rho(\frac{f+g}{2}) = \frac{\rho(f)+\rho(g)}{2}$, there holds $f = g$.

Definition 2 ([3]). Let $\rho : M_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if
(1) $\rho(0) = 0$;
(2) ρ is monotone, that is, $|f(\omega)| \le g|(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \le \rho(g)$, where $f, g \in M_\infty$;
(3) ρ is orthogonally subadditive, that is, $\rho(fI_{A\cup B}) \le \rho(fI_A) + \rho(fI_B)$ for any $A, B \in \Sigma$ such that $A \cap B \ne \phi$, $f \in M_\infty$;
(4) ρ has Fatou property, that is, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in M_\infty$;

(5) ρ is order continuous in ϵ, that is, $g_n \in \epsilon$ and $|g_n(\omega)| \downarrow 0$ for all $\omega \in \Omega$ implies $\rho(g_n) \downarrow 0$.

Definition 3 ([3]). Let ρ be a regular function pseudomodular;
(a) we say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0\ \ \rho$ -a.e.
(b) we say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0\ \ \rho$ -a.e.

ρ also satisfies the following properties ([13]):
(1) $\rho(0) = 0$ iff $f = 0\ \ \rho$ -a.e.
(2) $\rho(\alpha f) = \rho(f)$ for every scalar with $|\alpha| = 1$ and $f \in M$.
(3) $\rho(\alpha f + \beta g) \leq \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in M$.
ρ is called a convex modular if, in addition, the following property is satisfied:
(4) $\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $f, g \in M$.
The class of all nonzero regular convex function modulars on Ω is denoted by $\Re$.

Definition 4 ([3]). The convex function modular ρ defines the modular function space X_ρ as: $X_\rho = \{f \in M : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}$. In general terms, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. Nevertheless, the modular space X_ρ can be furnished with an $F-$norm defined thus: $\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq \alpha\}$.
In the instance ρ is convex modular, $\|f\|_\rho = \inf\{\alpha > 0 : \rho(\frac{f}{\alpha}) \leq 1\}$ defines a norm on the modular space X_ρ. This type of norm is known as the Luxemburg norm.

Definition 5 ([6]). A nonzero regular convex function ρ is said to satisfy the Δ_2- condition, if $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to $\emptyset$ and $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$. If ρ is convex and satisfies Δ_2-condition, then $X_\rho = E_\rho$.

Definition 6 ([6]). Let X_ρ be a modular space. The sequence $\{f_n\} \in X_\rho$ is called:
(1) ρ-convergent to $f \in X_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;
(2) ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as $n, m \to \infty$.

Remark 7. ρ-convergent sequence implies ρ-Cauchy sequence if and only if ρ satisfies the Δ_2-condition. However, ρ does not satisfy the triangle inequality.

Definition 8 ([6]). Let X_ρ be a modular space. A function $f \in X_\rho$ is called a fixed point of a multivalued mapping $T : X_\rho \to P_\rho(D)$ if $f \in Tf$. The set of all fixed points of T is represented by $F_\rho(T)$.

Definition 9 ([13]). Let X_ρ be a modular space. A subset $D \subset X_\rho$ is called:
(1) $\rho-$closed if the $\rho-$limit of a $\rho-$convergent sequence of D always belongs to D;
(2) $\rho-$a.e. closed if the $\rho-$a.e. limit of a $\rho-$a.e. convergent sequence of D always belongs to D;
(3) $\rho-$compact if every sequence in D has a $\rho-$convergent subsequence in D;
(4) $\rho-$a.e. compact if every sequence in D has a $\rho-$a.e. convergent subsequence in D;
(5) $\rho-$bounded if $diam_\rho(D) = \sup\{\rho(f - g) : f, g \in D\} < \infty$.

Several authors have used different contractive-type mappings in the framework of metric spaces to prove fixed point results with lots of applications in Mathematics and mathematical sciences. We shall give the definition of multitude of them in this section. Let X be a complete metric space and $T : X \to X$ be a self map. T is called:
Banach contraction map if there exists δ_1 satisfying $\delta_1 \in [0, 1)$ such that

$$d(Tx, Ty) \leq \delta_1 d(x, y),\ \forall x, y \in X. \tag{1}$$

Stefan Banach in 1922, employed the contraction condition (1) to obtain unique fixed point in the celebrated Banach contraction principle which is remarkable in its simplicity, but it is perhaps the

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

most widely applied fixed point theorem in all of analysis with special applications to the theory of differential and integral equations.

Rakotch contraction map ([17]), if there exists a monotone decreasing function $\phi : (0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \phi(d(x, y)), \ \forall x, y \in X, \ x \neq y. \tag{2}$$

Edelstein contractive map ([18]), if

$$d(Tx, Ty) < d(x, y), \ \forall x, y \in X, \ x \neq y. \tag{3}$$

Kannan contraction map ([20]), if there exists δ_2 satisfying $\delta_2 \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \delta_2[d(x, Tx) + d(y, Ty)], \ \forall x, y \in X. \tag{4}$$

Kannan [20] proved that if X is complete, then a Kannan mapping has a fixed point.

Bianchini contraction map ([22]), if there exists δ_2 satisfying $\delta_2 \in [0, 1)$ such that

$$d(Tx, Ty) \leq \delta_2 \ \max\{d(x, Tx), d(y, Ty)\}, \ \forall x, y \in X. \tag{5}$$

modified Bianchini contraction map ([22]), if

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty)\}, \ \forall x, y \in X, \ x \neq y. \tag{6}$$

Reich contraction map ([23]), if there exists $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ such that

$$d(Tx, Ty) \leq \delta_1 d(x, y) + \delta_2 d(x, Tx) + \delta_3 d(y, Ty), \ \forall x, y \in X. \tag{7}$$

Geraghty-Reich contraction map ([24]), if there exist monotonically decreasing functions ϕ_1, ϕ_2, ϕ_3 satisfying $\phi_1(t) + \phi_2(t) + \phi_3(t) < 1$ such that

$$\begin{aligned} d(Tx, Ty) \leq \ & \phi_1(d(x, y))d(x, y) + \phi_2(d(x, y))d(x, Tx) \\ & + \phi_3(d(x, y))d(y, Ty), \ \forall x, y \in X. \end{aligned} \tag{8}$$

Chatterjea contraction map ([26]), if there exists δ_3 satisfying $\delta_3 \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \delta_3[d(x, Ty) + d(y, Tx)], \ \forall x, y \in X. \tag{9}$$

modified Chatterjea contraction map ([26]), if there exists h satisfying $h \in [0, 1)$ such that

$$d(Tx, Ty) \leq h \ \max\{d(x, Ty), d(y, Tx)\}, \ \forall x, y \in X. \tag{10}$$

Chatterjea contractive-like map ([26]), if

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}, \ \forall x, y \in X, \ x \neq y. \tag{11}$$

extended Chatterjea contraction map ([26]), if there exist $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ such that

$$d(Tx, Ty) \leq \delta_1 d(x, y) + \delta_2 d(x, Ty) + \delta_3 d(y, Tx), \ \forall x, y \in X. \tag{12}$$

A map $T : X \to X$, is called Sehgal contraction map ([28]), if

$$d(Tx, Ty) < \max\{d(x, y), d(x, Ty), d(y, Tx)\}, \ \forall x, y \in X, \ x \neq y. \tag{13}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

Zamfirescu contraction map ([29]), if there exists $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 \in [0,1)$, $\delta_2 \in [0, \frac{1}{2})$, $\delta_3 \in [0, \frac{1}{2})$ such that one of the following is true

$$\begin{aligned} (i)\ d(Tx,Ty) &\leq \delta_1 d(x,y) \\ (ii)\ d(Tx,Ty) &\leq \delta_2[d(x,Tx)+d(y,Ty)] \\ (iii)\ d(Tx,Ty) &\leq \delta_3[d(x,Ty)+d(y,Tx)],\ \forall x,y \in X. \end{aligned} \tag{14}$$

Combining conditions (i), (ii) and (iii) in (14), we obtain

$$d(Tx,Ty) \leq \delta d(x,y) + 2\delta d(x,Tx),\ \forall x,y \in X. \tag{15}$$

Where $\delta = \max\{\delta_1, \frac{\delta_2}{1-\delta_2}, \frac{\delta_3}{1-\delta_3}\}$.

modified Zamfirescu contraction map ([29]), if

$$\begin{aligned} d(Tx,Ty) &< \max\{d(x,y), \frac{1}{2}[d(x,Tx)+d(y,Ty)], \\ &\quad \frac{1}{2}[d(x,Ty)+d(y,Tx)],\ \forall x,y \in X,\ x \neq y. \end{aligned} \tag{16}$$

Osilike contraction map ([31]), if there exists δ, L, satisfying $\delta \in [0,1)$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + L d(x,Tx),\ \forall x,y \in X. \tag{17}$$

Imoru and Olatinwo contraction map ([33]), if there exists δ, satisfying $\delta \in [0,1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$d(Tx,Ty) \leq \delta d(x,y) + \varphi(d(x,Tx)),\ \forall x,y \in X. \tag{18}$$

Hardy and Rogers contraction map ([34]), if there exist $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ satisfying $\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 < 1$ such that

$$\begin{aligned} d(Tx,Ty) &\leq \delta_1 d(x,y) + \delta_2 d(x,Tx) + \delta_3 d(y,Ty) + \delta_4 d(x,Ty) \\ &\quad + \delta_5 d(y,Tx),\ \forall x,y \in X. \end{aligned} \tag{19}$$

Ciric contraction map ([35]), if there exists h satisfying $h \in [0,1)$ such that

$$\begin{aligned} d(Tx,Ty) &\leq h\ \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), \\ &\quad d(y,Tx)\},\ \forall x,y \in X. \end{aligned} \tag{20}$$

modified Ciric contraction map ([35]), if

$$\begin{aligned} d(Tx,Ty) &< h\ \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), \\ &\quad d(y,Tx)\},\ \forall x,y \in X,\ x \neq y. \end{aligned} \tag{21}$$

The corresponding contraction definitions (1) to (21) can be written in the framewok of a Banach space as follows: Let $(E, \|.\|)$ be a Banach space and $T : E \to E$ be a self map of E.
T is called Banach contraction map if there exists δ_1 satisfying $\delta_1 \in [0,1)$ such that

$$\|Tx - Ty\| \leq \delta_1 \|x - y\|,\ \forall x,y \in E. \tag{22}$$

T is called Rakotch contraction map ([17]), if there exists a monotone decreasing function $\phi : (0,\infty) \to [0,1)$ such that

$$\|Tx - Ty\| \leq \phi(\|x - y\|),\ \forall x,y \in E,\ x \neq y. \tag{23}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

T is called Edelstein contraction map ([18]), if

$$\|Tx - Ty\| \quad < \quad \|x - y\|, \ \forall x, y \in E, \ x \neq y. \tag{24}$$

T is called Kannan contraction map ([20]), if there exists δ_2 satisfying $\delta_2 \in [0, \frac{1}{2})$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta_2[\|x - Tx\| + \|y - Ty\|], \ \forall x, y \in E. \tag{25}$$

T is called Bianchini contraction map ([22]), if there exists δ_2 satisfying $\delta_2 \in [0, 1)$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta_2 max\{\|x - Tx\|, \|y - Ty\|\}, \ \forall x, y \in E. \tag{26}$$

T is called modified Bianchini contraction map ([22]), if

$$\|Tx - Ty\| \quad < \quad max\{\|x - Tx\| - \|y - Ty\|\}, \ \forall x, y \in E, \ x \neq y. \tag{27}$$

T is called Reich contraction map ([23]), if there exists $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta_1\|x - y\| + \delta_2\|x - Tx\| + \delta_3\|y - Ty\|, \ \forall x, y \in E. \tag{28}$$

T is called Geraghty- Reich contraction map ([24]), if there exist monotonically decreasing functions ϕ_1, ϕ_2, ϕ_3 satisfying $\phi_1(t) + \phi_2(t) + \phi_3(t) < 1$ such that

$$\begin{aligned} \|Tx - Ty\| \quad \leq \quad & \phi_1(\|x - y\|)\|x - y\| + \phi_2(\|x - y\|)\|x - Tx\| \\ & + \phi_3(\|x - y\|)\|y - Ty\|, \ \forall x, y \in E. \end{aligned} \tag{29}$$

T is called Chatterjea contraction map ([26]), if there exists δ_3 satisfying $\delta_3 \in [0, \frac{1}{2})$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta_3[\|x - Ty\| + \|y - Tx\|], \ \forall x, y \in E. \tag{30}$$

T is called modified Chatterjea contraction map ([26]), if there exists h satisfying $h \in [0, 1)$ such that

$$\|Tx - Ty\| \quad \leq \quad h \ \max\{\|x - Ty\|), \|y - Tx\|\}, \ \forall x, y \in E. \tag{31}$$

T is called Chatterjea contractive-like map ([26]), if

$$\|Tx - Ty\| \quad < \quad \max\{\|x - Ty\|, \|y - Tx\|\}, \ \forall x, y \in E, \ x \neq y. \tag{32}$$

T is called extended Chatterjea contraction map ([26]), if there exist $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ such that

$$\|Tx - Ty\| \quad \leq \quad \delta_1\|x - y\| + \delta_2\|x - Ty\| + \delta_3\|y - Tx\|, \ \forall x, y \in E. \tag{33}$$

T is called Sehgal contraction map ([28]), if

$$\|Tx - Ty\| < \max\{\|x - y\|, \|x - Ty\|, \|y - Tx\|\}, \ \forall x, y \in E, \ x \neq y. \tag{34}$$

T is called Zamfirescu contraction map ([29]), if there exists δ satisfying $\delta \in [0, 1)$, such that

$$\|Tx - Ty\| \quad \leq \quad \delta\|x - y\| + 2\delta\|x - Tx\|, \ \forall x, y \in E. \tag{35}$$

T is called modified Zamfirescu contraction map ([29]), if

$$\begin{aligned} \|Tx - Ty\| \quad < \quad & \max\{\|x - y\|, \frac{1}{2}[\|x - Tx\| + \|y - Ty\|], \\ & \frac{1}{2}[\|x - Ty\| + \|y - Tx\|], \ \forall x, y \in E, \ x \neq y. \end{aligned} \tag{36}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

T is called Osilike contraction map ([31]), if there exists δ, L, satisfying $\delta \in [0, 1)$ and $L \geq 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|x - Tx\|, \ \forall x, y \in E. \tag{37}$$

Imoru and Olatinwo [33] proved stability theorems for Picard and Mann Iterative schemes using the contractive definition:
T is called Olatinwo contraction mapping ([33]), if there exists δ, satisfying $\delta \in [0, 1)$ and a monotone increasing function $\varphi : R^+ \to R^+$ with $\varphi(0) = 0$ such that

$$\|Tx - Ty\| \leq \delta\|x - y\| + \varphi(\|x - Tx\|), \ \forall x, y \in E. \tag{38}$$

A map $T : E \to E$, is called Hardy and Rogers contraction map ([34]), if there exist $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ satisfying $\delta_1 + \delta_2 + \delta_3 + \delta_4 + \delta_5 < 1$ such that

$$\begin{aligned} \|Tx - Ty\| &\leq \delta_1\|x - y\| + \delta_2\|x - Tx\| + \delta_3\|y - Ty\| + \delta_4\|x - Ty\| \\ &\quad +\delta_5\|y - Tx\|, \ \forall x, y \in E. \end{aligned} \tag{39}$$

T is called Ciric contraction map ([35]), if there exists h satisfying $h \in [0, 1)$ such that

$$\begin{aligned} \|Tx - Ty\| &\leq hmax\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \\ &\quad \|y - Tx\|\}, \ \forall x, y \in E. \end{aligned} \tag{40}$$

T is called modified Ciric contraction map ([35]), if

$$\begin{aligned} \|Tx - Ty\| &< hmax\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \\ &\quad \|y - Tx\|\}, \ \forall x, y \in E, \ x \neq y. \end{aligned} \tag{41}$$

Remark 10. It is important to note that in Banach contraction mapping, the operator $T : E \to E$ is continuous. While in Kannan contraction, it is not.

We present the generalizations of (1) – (21) for multivalued mappings in the framework of modular function spaces. These classes of multivalued mappings are versatile tools in the study of fixed point and possible relation to the solution of ordinary differential equations (ODE). Moreso, in this paper, we do comparison for various classes of $\rho-$contractive mappings in modular function spaces. Many fixed point results can be proved using our version of $\rho-$contractive definitions, thus, giving room for wider applications. Our method of proof can also be extended to various mappings that are non-expensive, pseudo-contractive and accretive in nature.

Let X_ρ be a modular space and $D \subset X_\rho$ be a $\rho-$proximinal set, for each $f \in X_\rho$ there exists an element $g \in D$ such that $\rho(f - g) = dist_\rho(f, D)$. Let $P_\rho(D)$ represent the family of nonempty $\rho-$bounded $\rho-$proximinal subsets of D, $C_\rho(D)$ represent the family of nonempty $\rho-$closed $\rho-$bounded subsets of D and $K_\rho(D)$ represent the family of $\rho-$compact subsets of D. Let $H_\rho(.,.)$ be the $\rho-$Hausdorff distance on $C_\rho(X_\rho)$, that is,
$H_\rho(A, B) = \max\{sup_{f\in A} dist_\rho(f, B), sup_{g\in B} dist_\rho(g, A)\}, \ A, B \in C_\rho(X_\rho)$.

A multivalued map $T : D \to C_\rho(X_\rho)$ is called:
Nadler contraction (or simply $\rho-$contraction) map, if there exists a constant δ, satisfying $\delta \in [0, 1)$, such that:

$$(I) \ H_\rho(Tf, Tg) \leq \delta\rho(f - g), \ \forall f, g \in D. \tag{42}$$

$\rho-$Rakotch contraction map, if there exists a monotone decreasing function $\phi : (0, \infty) \to [0, 1)$ such that

$$(II) \ H_\rho(Tf, Tg) \leq \phi(\rho(f - g)), \ \forall f, g \in D, \ f \neq g. \tag{43}$$

$\rho-$Edelstein contractive map, if

$$(III)\ H_\rho(Tf,Tg) \quad < \quad \rho(f-g),\ \forall f,g\in D,\ f\neq g. \tag{44}$$

$\rho-$ Kannan contraction map, if there exists δ_2 satisfying $\delta_2 \in [0,\frac{1}{2})$ such that

$$(IV) H_\rho(Tf,Tg) \leq \delta_2[\rho(f-u)+\rho(g-v)],\ \forall f,g\in D,\ \forall u\in Tf,\ \forall v\in Tg. \tag{45}$$

$\rho-$Bianchini contraction map, if there exists δ_3 satisfying $\delta_3\in[0,1)$ such that

$$\begin{aligned}(V)\ H_\rho(Tf,Tg) \quad &\leq \quad \delta_3 \max\{\rho(f-u),\rho(g-v)\}, \forall f,g\in D,\ \forall u\in Tf,\\ &\forall v\in Tg.\end{aligned} \tag{46}$$

modified $\rho-$Bianchini contraction map, if

$$\begin{aligned}(VI)\ H_\rho(Tf,Tg) \quad &< \quad \max\{\rho(f-u),\rho(g-v)\},\ \forall f,g\in D,\ f\neq g,\\ &\forall u\in Tf,\ \forall v\in Tg.\end{aligned} \tag{47}$$

$\rho-$ Reich contraction map, if there exists $\delta_1,\delta_2,\delta_3$ satisfying $\delta_1+\delta_2+\delta_3<1$ such that

$$\begin{aligned}(VII)\ H_\rho(Tf,Tg) \quad &\leq \quad \delta_1\rho(f-g)+\delta_2\rho(f-u)+\delta_3\rho(g-v),\ \forall f,g\in D,\\ &\forall u\in Tf,\ \forall v\in Tg.\end{aligned} \tag{48}$$

$\rho-$ Geraghty-Reich contraction map, if there exist monotonically decreasing functions ϕ_1,ϕ_2,ϕ_3 satisfying $\phi_1(t)+\phi_2(t)+\phi_3(t)<1$ such that

$$\begin{aligned}(VIII)\ H_\rho(Tf,Tg) \quad &\leq \quad \phi_1(\rho(f-g))\rho(f-g)+\phi_2(\rho(f-g))\rho(f-u)\\ &+\phi_3(\rho(f-g))\rho(g-v),\ \forall f,g\in D,\\ &\forall u\in Tf,\ \forall v\in Tg.\end{aligned} \tag{49}$$

$\rho-$Chatterjea contraction map, if there exists δ_3 satisfying $\delta_3\in[0,\frac{1}{2})$ such that

$$\begin{aligned}(IX)\ H_\rho(Tf,Tg) \quad &\leq \quad \delta_3[\rho(f-v)+\rho(g-u)],\ \forall f,g\in D,\ \forall u\in Tf,\\ &\forall v\in Tg.\end{aligned} \tag{50}$$

modified $\rho-$Chatterjea contraction map, if there exists h satisfying $h\in[0,1)$ such that

$$\begin{aligned}(X)\ H_\rho(Tf,Tg) \quad &\leq \quad h\ \max\{\rho(f-v),\rho(g-u)\},\ \forall f,g\in D,\ \forall u\in Tf,\\ &\forall v\in Tg.\end{aligned} \tag{51}$$

$\rho-$Chatterjea contractive-like map, if

$$\begin{aligned}(XI)\ H_\rho(Tf,Tg) \quad &< \quad \max\{\rho(f-v),\rho(g-u)\},\ \forall f,g\in D,\ \forall u\in Tf,\\ &\forall v\in Tg.\end{aligned} \tag{52}$$

extended $\rho-$Chatterjea contraction map, if there exist $\delta_1,\delta_2,\delta_3$ satisfying $\delta_1+\delta_2+\delta_3<1$ such that

$$\begin{aligned}(XII)\ H_\rho(Tf,Tg) \quad &\leq \quad \delta_1\rho(f-g)+\delta_2\rho(f-v)+\delta_3\rho(g-u),\ \forall f,g\in D,\\ &\forall u\in Tf,\ \forall v\in Tg.\end{aligned} \tag{53}$$

$\rho-$Sehgal contraction map, if

$$\begin{aligned}(XIII)\ H_\rho(Tf,Tg) \quad &< \quad \max\{\rho(f-g),\rho(f-v),\rho(g-u)\},\ \forall f,g\in D,\ f\neq g.\\ &\forall u\in Tf,\ \forall v\in Tg.\end{aligned} \tag{54}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

$\rho-$Zamfirescu contraction map, if there exists δ satisfying $\delta \in [0,1)$, such that

$$\begin{aligned}(XIV)\ H_\rho(Tf,Tg) \ &\leq\ \delta \max\{\rho(f-g), \frac{1}{2}[\rho(f-u)+\rho(g-v)],\\ &\frac{1}{2}[\rho(f-v)+\rho(g-u))]\},\ \forall f,g \in D,\\ &\forall u \in Tf,\ \forall v \in Tg. \qquad (55)\end{aligned}$$

modified $\rho-$Zamfirescu contraction map, if

$$\begin{aligned}(XV)\ H_\rho(Tf,Tg) \ &<\ max\{\rho(f-g), \frac{1}{2}[\rho(f-u)+\rho(g-v)],\\ &\frac{1}{2}[\rho(f-v)+\rho(g-u))]\},\ \forall f,g \in D,\ f \neq g,\\ &\forall u \in Tf,\ \forall v \in Tg. \qquad (56)\end{aligned}$$

$\rho-$quasi-contractive (or $\rho-$Osilike) map, if there exists δ and L satisfying $\delta \in [0,1)$ and $L \geq 0$, such that

$$\begin{aligned}(XVII)\ H_\rho(Tf,Tg) \ &\leq\ \delta\rho(f-g)+L\rho(u-f),\ \forall f,g \in D,\\ &\forall u \in Tf,\ L \geq 0. \qquad (57)\end{aligned}$$

$\rho-$quasi-contractive-like ($\rho-$Olatinwo) map, if there exists δ satisfying $\delta \in [0,1)$ and $\varphi : R^+ \to R^+$ a $\rho-$monotone increasing function with $\varphi(0) = 0$ such that

$$(XVIII)\ H_\rho(Tf,Tg) \leq \delta\rho(f-g)+\varphi(\rho(u-f)),\ \forall f,g \in D\ \ \forall u \in Tf. \qquad (58)$$

$\rho-$Hardy and Rogers contraction map, if there exist $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ satisfying $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5 < 1$ such that

$$\begin{aligned}(XIX)\ H_\rho(Tf,Tg) \ &\leq\ \delta_1\rho(f-g)+\delta_2\rho(f-u)+\delta_3\rho(g-v)+\delta_4\rho(f-v)\\ &+\delta_5\rho(g-u),\ \forall f,g \in D, \forall u \in Tf,\ \forall v \in Tg. \qquad (59)\end{aligned}$$

$\rho-$Ciric contraction map, if there exists h satisfying $h \in [0,1)$ such that

$$\begin{aligned}(XX)H_\rho(Tf,Tg) \ &\leq\ h\ \max\{\rho(f-g), \rho(f-u), \rho(g-v), \rho(f-v), \rho(g-u)\},\\ &\forall f,g \in D, \forall u \in Tf,\ \forall v \in Tg. \qquad (60)\end{aligned}$$

modified $\rho-$Ciric contraction map, if

$$\begin{aligned}(XXI)H_\rho(Tf,Tg) \ &<\ h\ \max\{\rho(f-g), \rho(f-h), \rho(g-v), \rho(f-v),\\ &\rho(g-u)\}, \forall f,g \in D, f \neq g, \forall u \in Tf, \forall v \in Tg. \quad (61)\end{aligned}$$

2 Main Results

2.1 Characterizations of Results in Modular Function Spaces

Theorem 1. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a:
(I) Nadler contraction (or $\rho-$ contracton) map, satisfying contractive condition (42);
(II) $\rho-$Rakotch contraction map, satisfying contractive condition (43);
(III) $\rho-$Edelstein contraction map, satisfying contractive condition (44);
(XIII) $\rho-$Sehgal contraction map, satisfying contractive condition (54). Then,
(a) T has a unique fixed point $f \in Tf$.
(b) Furthermore, $(I) \Rightarrow (II) \Rightarrow (III) \Rightarrow (XIII)$. That is, contractive condition (42) $\Rightarrow$ (43) $\Rightarrow$

(44) ⇒ (54).

Proof:
(a) First, we prove the existence of fixed point of the multivalued map T.
Since L_ρ is complete, and D is a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, then D is also complete.
Let f_1 be an arbitrary function in D.
Consider the sequence of functions $f_2 = u_1 \in P_\rho^T(f_1),\ f_3 = u_2 \in P_\rho^T(f_2),\ f_4 = u_3 \in P_\rho^T(f_3),\ \ldots,$ $f_{n+1} = u_n \in P_\rho^T(f_n)$. (62)
If this sequence converges to some $f = f_0 \in D$, then, we obtain the following:
$f = lim_{n\to\infty} f_n = lim_{n\to\infty} f_{n+1} = lim_{n\to\infty} u_n \in P_\rho^T(lim_{n\to\infty} f_n)$
$= lim_{n\to\infty} u_n \in lim_{n\to\infty}(P_\rho^T f_n) = Tf$.
Thus, $f \in Tf$, that is, f is the fixed point of the multivalued map T.

Next, we show that the sequence (62) $\rho-$converges to the unique fixed point $f \in F_\rho(T)$ using $\rho-$contractive condition (42). Also, if T satisfies (42) and (42) ⇒ (43) ⇒ (44) ⇒ (54), then, $\rho-$contractive conditions (43), (44) and (54) are satisfied. To do this, we use the completeness of D, that is, to show that the sequence is Cauchy and it converges in D.
Since T is a contraction, there exists a δ satisfying $\delta \in [0, 1)$ such that $\rho-$contractive condition (42) holds, that is

$$(I) H_\rho(Tf, Tg) \quad \leq \quad \delta\rho(f - g),\ \forall f, g \in D.$$

The following quantity $\rho(f_n - f_m)$ will be estimated for the sequence defined by (62) thus:

$$\begin{aligned} \rho(f - g) &\leq \rho(f - u) + H_\rho(Tf, Tg) + \rho(v - g) \\ &\leq \rho(f - u) + \delta\rho(f - g) + \rho(v - g),\ \forall f, g \in D, \\ &\quad \forall u \in Tf,\ \forall v \in Tg. \end{aligned} \tag{63}$$

That is,

$$\rho(f - g) \quad \leq \quad \frac{\rho(f - u) + \rho(g - v)}{1 - \delta},\ \forall f, g \in D,\ \forall u \in Tf,\ \forall v \in Tg. \tag{64}$$

Inequality (64) shows that f and g are both fixed points of the multivalued map T, thus $\rho(f-g) = 0$, or that $f = g$. This shows that the fixed point is unique.
Using (64), the Cauchy sequence $\{f_n\}$ is established as follows

$$\rho(f_n - g_m) \quad \leq \quad [\frac{\delta^n + \delta^m}{1 - \delta}]\rho(f_1 - u_2),\ \forall f, g \in D,\ \forall u_2 \in Tf_1, m > n. \tag{65}$$

Therefore, $\{f_n\}$ is a Cauchy sequence and it ρ-converges to $f \in F_\rho(T)$.

(b) Next, we show that $(I) \Rightarrow (II)$.
If there exists a constant δ, satisfying $\delta \in [0, 1)$ in $\rho-$contractive condition (42) and there exists a monotone decreasing function $\phi : (0, \infty) \to [0, 1)$ such that (43) holds then:

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) < \phi(\rho(f - g)),\ \forall f, g \in D,\ \ f \neq g. \tag{66}$$

(66) shows that if T satisfies contractive condition (42), then T also satisfies the condition (43). That is, $(I) \Rightarrow (II)$. This ends the proof.

$(II) \Rightarrow (III)$.
Let $\phi = I_d$, and $f \neq g$, then it follows that if T satisfies contractive condition (43), then T also satisfies the condition (44). This ends the proof.

$(III) \Rightarrow (XIII)$.
From contractive conditions (44) and (54), we see that

$$H_\rho(Tf, Tg) < \rho(f-g) < \max\{\rho(f-g), \rho(f-v), \rho(g-u)\}, \ \forall f, g \in D,$$

$\forall u \in Tf, \ \forall v \in Tg$.
Thus, $(III) \Rightarrow (XIII)$.

It follows that contractive condition (42) $\Rightarrow$ (43) $\Rightarrow$ (44) $\Rightarrow$ (54). This ends the proof.

Theorem 2. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a:
(IV) $\rho-$Kannan contracton map, satisfying contractive condition (45);
(V) $\rho-$Bianchini contraction map, satisfying contractive condition (46);
(VI) $\rho-$modified Bianchini contraction map, satisfying contractive condition (47);
(XIII) $\rho-$Sehgal contraction map, satisfying contractive condition (54). Then, (a) T has a unique fixed point $f \in Tf$.
(b) Furthermore, $(IV) \Rightarrow (V) \Rightarrow (VI) \Rightarrow (XIII)$. That is, contractive condition (45) $\Rightarrow$ (46) $\Rightarrow$ (47) $\Rightarrow$ (54).

Proof:
(a) First, we prove the existence of fixed point of the multivalued map T.
Since L_ρ is complete, and D is a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, then D is also complete.
Let f_1 be an arbitrary function in D.
Consider the sequence of functions $f_2 = u_1 \in P_\rho^T(f_1)$, $f_3 = u_2 \in P_\rho^T(f_2)$, $f_4 = u_3 \in P_\rho^T(f_3)$, $\ldots$,
$f_{n+1} = u_n \in P_\rho^T(f_n)$. (67)
If this sequence converges to some $f = f_0 \in D$, then, we obtain the following:
$f = lim_{n\to\infty} f_n = lim_{n\to\infty} f_{n+1} = lim_{n\to\infty} u_n \in P_\rho^T(lim_{n\to\infty} f_n)$
$= lim_{n\to\infty} u_n \in lim_{n\to\infty}(P_\rho^T f_n) = Tf$.
Thus, $f \in Tf$, that is, f is the fixed point of the multivalued map T.

Next, we show that the sequence (67) $\rho-$converges to the unique fixed point $f \in F_\rho(T)$ using Kannan $\rho-$contractive condition (45). Also, if T satisfies (45) and (45) $\Rightarrow$ (46) $\Rightarrow$ (47) $\Rightarrow$ (54), then, $\rho-$contractive conditions (46), (47) and (54) are satisfied. To do this, we use the completeness of D, that is, to show that the sequence is Cauchy and it converges in D.
Since in this case T is a Kannan contraction map, there exists a δ_2 satisfying $\delta_2 \in [0, \frac{1}{2})$ such that $\rho-$contractive condition (45) holds, that is

$$(IV) H_\rho(Tf, Tg) \leq \delta_2[\rho(f-u) + \rho(g-v)], \ \forall f, g \in D, \ \forall u \in Tf, \ \forall v \in Tg.$$

The following quantity $\rho(f_n - f_m)$ will be estimated for the sequence defined by (67) thus:

$$\begin{aligned} \rho(f-g) &\leq \rho(f-u) + H_\rho(Tf, Tg) + \rho(v-g) \\ &\leq \rho(f-u) + \delta_2[\rho(f-u) + \rho(g-v)] + \rho(v-g), \\ &\quad = (1+\delta_2)\rho(f-u) + (1+\delta_2)\rho(g-v), \ \forall f, g \in D, \\ &\quad \forall u \in Tf, \ \forall v \in Tg. \end{aligned} \tag{68}$$

Inequality (68) shows that f and g are both fixed points of the multivalued map T, thus $\rho(f-g) = 0$, or that $f = g$. This shows that the fixed point is unique.
Using (68), the Cauchy sequence $\{f_n\}$ is established as follows

$$\rho(f_n - g_m) \leq [2 + \delta^n + \delta^m]\rho(f_1 - u_2), \ \forall f, g \in D, \ \forall u_2 \in Tf_1, m > n. \tag{69}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

Therefore, $\{f_n\}$ is a Cauchy sequence and it ρ-converges to $f \in F_\rho(T)$.

(b) Next, we show that $(IV) \Rightarrow (V)$.
It is easy to see that: if T satisfies contractive condition (45), then T also satisfies the condition (46) from the following: If there exist δ_2, δ_3 satisfying $\delta_2 \in (0, \frac{1}{2})$, $\delta_3 \in [0,1)$ respectively, then

$$\begin{aligned} H_\rho(Tf, Tg) &\leq \delta_2[\rho(f-u)+\rho(g-v)] \leq \delta_3 max\{\rho(f-u), \rho(g-v)\}, \\ &\forall f, g \in D,\ \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{70}$$

Thus, $(IV) \Rightarrow (V)$. This ends the proof.

Next, we show that $(V) \Rightarrow (VI)$.
If there exists δ_3, satisfying $\delta_3 \in [0,1)$ in (46), then (46) and (47) are related as follows:

$$\begin{aligned} H_\rho(Tf, Tg) &\leq \delta_3 \max\{\rho(f-u), \rho(g-v)\} < \max\{\rho(f-u), \rho(g-v)\}, \\ &\forall f, g \in D,\ f \neq g,\ \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{71}$$

Thus, $(V) \Rightarrow (VI)$. This ends the proof.

Next, we show that $(VI) \Rightarrow (XIII)$. From contractive conditions (44) and (54), we see that

$$\begin{aligned} H_\rho(Tf, Tg) &< \max\{\rho(f-u), \rho(g-v)\} < \max\{\rho(f-g), \rho(f-v), \rho(g-u)\}, \\ &\forall f, g \in D,\ f \neq g,\ \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{72}$$

Thus, $(VI) \Rightarrow (XIII)$.

It follows that contractive condition (45) $\Rightarrow$ (46) $\Rightarrow$ (47) $\Rightarrow$ (54). The proof is complete.

Theorem 3. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a:
(IV) $\rho-$Kannan contracton map, satisfying contractive condition (45);
(VII) $\rho-$Reich contraction map, satisfying contractive condition (48);
(VIII) $\rho-$modified Reich contraction map, satisfying contractive condition (49);
(XIII) $\rho-$Sehgal contraction map, satisfying contractive condition (54). Then,
(a) T has a unique fixed point $f \in Tf$.
(b) Furthermore, $(IV) \Rightarrow (VII) \Rightarrow (VIII) \Rightarrow (XIII)$. That is, contractive condition (45) $\Rightarrow$ (48) $\Rightarrow$ (49) $\Rightarrow$ (54).

Proof:
(a) First, we prove the existence of fixed point of the multivalued map T.
Since L_ρ is complete, and D is a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, then D is also complete.
Let f_1 be an arbitrary function in D.
Consider the sequence of functions $f_2 = u_1 \in P_\rho^T(f_1)$, $f_3 = u_2 \in P_\rho^T(f_2)$, $f_4 = u_3 \in P_\rho^T(f_3)$, ..., $f_{n+1} = u_n \in P_\rho^T(f_n)$. (73)
If this sequence converges to some $f = f_0 \in D$, then, we obtain the following:
$f = lim_{n\to\infty} f_n = lim_{n\to\infty} f_{n+1} = lim_{n\to\infty} u_n \in P_\rho^T(lim_{n\to\infty} f_n)$
$= lim_{n\to\infty} u_n \in lim_{n\to\infty}(P_\rho^T f_n) = Tf$.
Thus, $f \in Tf$, that is, f is the fixed point of the multivalued map T.

Next, we show that the sequence (73) $\rho-$converges to the unique fixed point $f \in F_\rho(T)$ using Kannan $\rho-$contractive condition (45). Also, if T satisfies (45) and (45) $\Rightarrow$ (48) $\Rightarrow$ (49) $\Rightarrow$ (54), then, $\rho-$contractive conditions (48), (49) and (54) are satisfied. To do this, we use the completeness of D, that is, to show that the sequence is Cauchy and it converges in D.

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

Since in this case T is a Kannan contraction mapping, there exists a δ_2 satisfying $\delta_2 \in [0, \frac{1}{2})$ such that $\rho-$contractive condition (45) holds, that is

$$(IV) H_\rho(Tf, Tg) \leq \delta_2[\rho(f-u) + \rho(g-v)],\ \forall f, g \in D,\ \forall u \in Tf,\ \forall v \in Tg.$$

The following quantity $\rho(f_n - f_m)$ will be estimated for the sequence defined by (73) thus:

$$\begin{aligned} \rho(f-g) &\leq \rho(f-u) + H_\rho(Tf, Tg) + \rho(v-g) \\ &\leq \rho(f-u) + \delta_2[\rho(f-u) + \rho(g-v)] + \rho(v-g), \\ &= (1+\delta_2)\rho(f-u) + (1+\delta_2)\rho(g-v),\ \forall f, g \in D, \\ &\quad \forall u \in Tf,\ \forall v \in Tg. \end{aligned} \tag{74}$$

Inequality (74) shows that f and g are both fixed points of the multivalued map T, thus $\rho(f-g) = 0$, or that $f = g$. This shows that the fixed point is unique.
Using (74), the Cauchy sequence $\{f_n\}$ is established as follows

$$\rho(f_n - g_m) \leq [2 + \delta^n + \delta^m]\rho(f_1 - u_2),\ \forall f, g \in D,\ \forall u_2 \in Tf_1, m > n. \tag{75}$$

Therefore, $\{f_n\}$ is a Cauchy sequence and it ρ-converges to $f \in F_\rho(T)$.

(b) Next, we show that $(IV) \Rightarrow (VII)$.
if T satisfies contractive condition (45), then T also satisfies the condition (48) from the following:
If there exist δ_2, satisfying $\delta_2 \in [0, \frac{1}{2})$ in (45) and $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ in (48), then we have

$$\begin{aligned} H_\rho(Tf, Tg) &\leq \delta_2[\rho(f-u) + \rho(g-v)] \leq \delta_1\rho(f-g) + \delta_2\rho(f-u) \\ &\quad + \delta_3\rho(g-v)\}, \forall f, g \in D,\ \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{76}$$

Thus, $(IV) \Rightarrow (VII)$. This ends the proof.

Next, we show that $(VII) \Rightarrow (VIII)$.
If there exist $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ in (48), and there exist monotonically decreasing functions ϕ_1, ϕ_2, ϕ_3 satisfying $\phi_1(t) + \phi_2(t) + \phi_3(t) < 1$ in (49), then (48) and (49) are related as follows:

$$\begin{aligned} H_\rho(Tf, Tg) &\leq \delta_1\rho(f-g) + \delta_2\rho(f-u) + \delta_3\rho(g-v)\} \\ &\quad \leq \phi_1(\rho(f-g))\rho(f-g) + \phi_2(\rho(f-g))\rho(f-u) \\ &\quad + \phi_3(\rho(f-g))\rho(g-v), \forall f, g \in D, \\ &\quad \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{77}$$

Thus, $(VII) \Rightarrow (VIII)$. This ends the proof.

Next, we show that $(VIII) \Rightarrow (XIII)$.
If there exist monotonically decreasing functions ϕ_1, ϕ_2, ϕ_3 satisfying $\phi_1(t) + \phi_2(t) + \phi_3(t) < 1$ in (49), then (49) and (54) are related as follows:

$$\begin{aligned} H_\rho(Tf, Tg) &\leq \phi_1(\rho(f-g))\rho(f-g) + \phi_2(\rho(f-g))\rho(f-u) \\ &\quad + \phi_3(\rho(f-g))\rho(g-v) \\ &\quad < \max\{\rho(f-g), \rho(f-v), \rho(g-u)\}, \forall f, g \in D, f \neq g, \\ &\quad \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{78}$$

Thus, $(VIII) \Rightarrow (XIII)$. This ends the proof.

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

It follows that contractive condition (45) $\Rightarrow$ (48) $\Rightarrow$ (49) $\Rightarrow$ (54). The proof is complete.

Theorem 4. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P^T_ρ is a:
(IX) $\rho-$Chatterjea contracton map, satisfying contractive condition (50);
(XII) extended $\rho-$Chatterjea contraction map, satisfying contractive condition (53);
(XIII) $\rho-$Sehgal contraction map, satisfying contractive condition (54). Then,
(a) T has a unique fixed point $f \in Tf$.
(b) Furthermore, $(IX) \Rightarrow (XII) \Rightarrow (XIII)$. That is, contractive condition (50) $\Rightarrow$ (53) $\Rightarrow$ (54).

Proof:
(a) First, we prove the existence of fixed point of the multivalued map T.
Since L_ρ is complete, and D is a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, then D is also complete.
Let f_1 be an arbitrary function in D.
Consider the sequence of functions $f_2 = u_1 \in P^T_\rho(f_1)$, $f_3 = u_2 \in P^T_\rho(f_2)$, $f_4 = u_3 \in P^T_\rho(f_3)$, $\ldots$, $f_{n+1} = u_n \in P^T_\rho(f_n)$.
If this sequence converges to some $f = f_0 \in D$, then, we obtain the following:
$f = lim_{n\to\infty} f_n = lim_{n\to\infty} f_{n+1} = lim_{n\to\infty} u_n \in P^T_\rho(lim_{n\to\infty} f_n)$
$= lim_{n\to\infty} u_n \in lim_{n\to\infty}(P^T_\rho f_n) = Tf$.
Thus, $f \in Tf$, that is, f is the fixed point of the multivalued map T.

Next, we show that the sequence (79) $\rho-$converges to the unique fixed point $f \in F_\rho(T)$ using Chatterjea $\rho-$contractive condition (50). Also, if T satisfies (50) and (50) $\Rightarrow$ (53) $\Rightarrow$ (54), then, $\rho-$contractive conditions (53) and (54) are satisfied. To do this, we use the completeness of D, that is, to show that the sequence is Cauchy and it converges in D. Since in this case T is a $\rho-$Chatterjea contraction map, there exists a δ_3 satisfying $\delta_2 \in [0, \frac{1}{2})$ such that $\rho-$contractive condition (50) holds, that is

$$(IV) H_\rho(Tf, Tg) \leq \delta_3[\rho(f - v) + \rho(g - u)],\ \forall f, g \in D,\ \forall u \in Tf,\ \forall v \in Tg.$$

The following quantity $\rho(f_n - f_m)$ will be estimated for the sequence defined by (79) thus:

$$\begin{aligned}
\rho(f - g) &\leq \rho(f - u) + H_\rho(Tf, Tg) + \rho(v - g)\\
&\leq \rho(f - u) + \delta_3[\rho(f - v) + \rho(g - u)] + \rho(v - g),\\
&\leq \rho(f - u) + \frac{\delta_3}{1 - \delta_3}\rho(f - g) + \frac{2\delta_3}{1 - \delta_3}\rho(f - u) + \rho(v - g)\\
&= \frac{\delta_3}{1 - \delta_3}\rho(f - g) + \frac{1 + \delta_3}{1 - \delta_3}\rho(f - u) + \rho(v - g)\\
&\leq \frac{1 + \delta_3}{1 - 2\delta_3}\rho(f - u) + \frac{1 - \delta_3}{1 - 2\delta_3}\rho(v - g),\ \forall f, g \in D,\\
&\quad \forall u \in Tf,\ \forall v \in Tg. \qquad (79)
\end{aligned}$$

Inequality (79) shows that f and g are both fixed points of the multivalued map T, thus $\rho(f - g) = 0$, or that $f = g$. This shows that the fixed point is unique. Using (79), the Cauchy sequence $\{f_n\}$ is established as follows

$$\rho(f_n - g_m) \leq [\frac{2 + \delta_3^n - \delta_3^m}{1 - 2\delta_3}]\rho(f_1 - u_2),\ \forall f, g \in D,\ \forall u_2 \in Tf_1. \qquad (80)$$

Therefore, $\{f_n\}$ is a Cauchy sequence and it ρ-converges to $f \in F_\rho(T)$.

(b) Next, we show that $(XI) \Rightarrow (XII)$.
If T satisfies contractive condition (50), then T also satisfies the condition (53) from the following:

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

If there exist δ_3, satisfying $\delta_3 \in (0, \frac{1}{2})$ in (50) and $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ in (53), then (50) and (53) are related as follows:

$$\begin{aligned} H_\rho(Tf, Tg) \quad &\leq \quad \delta_3[\rho(f-v) + \rho(g-u)] < \delta_1\rho(f-g) + \delta_2\rho(f-v) \\ &\quad +\delta_3\rho(g-u)\}, \forall f, g \in D, \ \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{81}$$

Thus, $(IX) \Rightarrow (XII)$. This ends the proof.

Next, we show that $(XII) \Rightarrow (XIII)$.
If there exist $\delta_1, \delta_2, \delta_3$ satisfying $\delta_1 + \delta_2 + \delta_3 < 1$ in (53), then (53) and (54) are related as follows:

$$\begin{aligned} H_\rho(Tf, Tg) \quad &\leq \quad \delta_1\rho(f-g) + \delta_2\rho(f-v) + \delta_3\rho(g-u)\} \\ &\quad < \max\{\rho(f-g), \rho(f-v), \rho(g-u)\}, \forall f, g \in D, f \neq g, \\ &\quad \forall u \in Tf, \forall v \in Tg. \end{aligned} \tag{82}$$

Thus, $(XII) \Rightarrow (XIII)$. This ends the proof.

It follows that contractive condition (50) $\Rightarrow$ (53) $\Rightarrow$ (54). The proof is complete.

Theorem 5. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a:
(I) Nadler contracton mapping, satisfying contractive condition (42);
(IV) extended $\rho-$Kannan contraction mapping, satisfying contractive condition (45);
(XII) extended $\rho-$Chatterjea contraction mapping, satisfying contractive condition (53);
(XIX) $\rho-$Hardy and Rogers contraction mapping, satisfying contractive condition (59). Then, T has a unique fixed point $f \in Tf$. Furthermore, $(I) \Rightarrow (XIX)$, $(IV) \Rightarrow (XIX)$ and $(XII) \Rightarrow (XIX)$. That is, contractive condition (42) $\Rightarrow$ (59), (45) $\Rightarrow$ (59) and (53) $\Rightarrow$ (59).

Proof:
(a) First, we prove the existence of fixed point of the multivalued map T.
Since L_ρ is complete, and D is a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, then D is also complete.
Let f_1 be an arbitrary function in D.
Consider the sequence of functions $f_2 = u_1 \in P_\rho^T(f_1)$, $f_3 = u_2 \in P_\rho^T(f_2)$, $f_4 = u_3 \in P_\rho^T(f_3)$, $\ldots$, $f_{n+1} = u_n \in P_\rho^T(f_n)$. (83)
If this sequence converges to some $f = f_0 \in D$, then, we obtain the following:
$f = lim_{n\to\infty} f_n = lim_{n\to\infty} f_{n+1} = lim_{n\to\infty} u_n \in P_\rho^T(lim_{n\to\infty} f_n)$
$= lim_{n\to\infty} u_n \in lim_{n\to\infty}(P_\rho^T f_n) = Tf$.
Thus, $f \in Tf$, that is, f is the fixed point of the multivalued map T.

Next, we show that the sequence (83) $\rho-$converges to the unique fixed point $f \in F_\rho(T)$ using $\rho-$contractive condition (42). Also, if T satisfies (42) and (42) $\Rightarrow$ (59), (45) $\Rightarrow$ (59), (53) $\Rightarrow$ (59), then, $\rho-$contractive conditions (45), (53) and (59) are satisfied. To do this, we use the completeness of D, that is, to show that the sequence is Cauchy and it converges in D. Since T is a contraction, there exists a δ satisfying $\delta \in [0, 1)$ such that $\rho-$contractive condition (42) holds, that is

$$(I) H_\rho(Tf, Tg) \quad \leq \quad \delta\rho(f-g), \ \forall f, g \in D.$$

The following quantity $\rho(f_n - f_m)$ will be estimated for the sequence defined by (83) thus:

$$\begin{aligned} \rho(f-g) \quad &\leq \quad \rho(f-u) + H_\rho(Tf, Tg) + \rho(v-g) \\ &\leq \quad \rho(f-u) + \delta\rho(f-g) + \rho(v-g), \ \forall f, g \in D, \\ &\quad \forall u \in Tf, \ \forall v \in Tg. \end{aligned} \tag{84}$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

That is,

$$\rho(f-g) \quad \leq \quad \frac{\rho(f-u)+\rho(g-v)}{1-\delta}, \ \forall f,g \in D, \ \forall u \in Tf, \ \forall v \in Tg. \qquad (85)$$

Inequality (85) shows that f and g are both fixed points of the multivalued map T, thus $\rho(f-g)=0$, or that $f=g$. This shows that the fixed point is unique.
Using (85), the Cauchy sequence $\{f_n\}$ is established as follows

$$\rho(f_n-g_m) \quad \leq \quad [\frac{\delta^n+\delta^m}{1-\delta}]\rho(f_1-u_2), \ \forall f,g \in D, \ \forall u_2 \in Tf_1, m>n. \qquad (86)$$

Therefore, $\{f_n\}$ is a Cauchy sequence and it ρ-converges to $f \in F_\rho(T)$.

(b) Next, we show that $(I) \Rightarrow (II)$.
If there exists a constant δ, satisfying $\delta \in [0,1)$ in $\rho-$contractive condition (42) and there exist $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5$ satisfying $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5<1$ in (59), then (42) and (59) are related as follows

$$\begin{aligned} H_\rho(Tf,Tg) \quad < \quad & \delta\rho(f-g), \\ & \leq \delta_1\rho(f-g)+\delta_2\rho(f-u)+\delta_3\rho(g-v)+\delta_4\rho(f-v) \\ & +\delta_5\rho(g-u), \ \forall f,g \in D, \forall u \in Tf, \ \forall v \in Tg. \qquad (87) \end{aligned}$$

(87) shows that if T satisfies contractive condition (42), then T also satisfies the condition (59). That is, $(I) \Rightarrow (XIX)$. This ends the proof.

Next, we show that $(IV) \Rightarrow (XIX)$.
If there exist δ_2, satisfying $\delta_2 \in [0,\frac{1}{2})$ in (45) and $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5$ satisfying $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5<1$ in (59), then (45) and (59) are related as follows:

$$\begin{aligned} H_\rho(Tf,Tg) \quad \leq \quad & \delta_2[\rho(f-u)+\rho(g-v)] \\ & \leq \delta_1\rho(f-g)+\delta_2\rho(f-u)+\delta_3\rho(g-v)+\delta_4\rho(f-v) \\ & +\delta_5\rho(g-u), \forall f,g \in D, \ \forall u \in Tf, \forall v \in Tg. \qquad (88) \end{aligned}$$

Thus, $(IV) \Rightarrow (XIX)$. This ends the proof.

Next, we show that $(XII) \Rightarrow (XIX)$.
If there exist $\delta_1,\delta_2,\delta_3$ satisfying $\delta_1+\delta_2+\delta_3<1$ in (53) and $\delta_1,\delta_2,\delta_3,\delta_4,\delta_5$ satisfying $\delta_1+\delta_2+\delta_3+\delta_4+\delta_5<1$ in (59), then (53) and (59) are related as follows:

$$\begin{aligned} H_\rho(Tf,Tg) \quad \leq \quad & \leq \delta_1\rho(f-g)+\delta_2\rho(f-v)+\delta_3\rho(g-u) \\ & \leq \delta_1\rho(f-g)+\delta_2\rho(f-u)+\delta_3\rho(g-v)+\delta_4\rho(f-v) \\ & +\delta_5\rho(g-u), \forall f,g \in D, f \neq g, \ \forall u \in Tf, \forall v \in Tg. \qquad (89) \end{aligned}$$

Thus, $(XII) \Rightarrow (XIX)$. This ends the proof.

It follows that contractive condition (42) $\Rightarrow$ (45), (45) $\Rightarrow$ (53), and (53) $\Rightarrow$ (59). The proof is complete.

Theorem 6. Let D be a $\rho-$closed, $\rho-$bounded and convex subset of a $\rho-$complete modular space L_ρ, and $T: D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a:
(XIV) $\rho-$Zamfirescu contracton mapping, satisfying contractive condition (55);
(XV) modified $\rho-$Zamfirescu contraction mapping, satisfying contractive condition (56);
(XVI) $\rho-$ quasi-contractive mapping, satisfying contractive condition (57);
(XVII) $\rho-$ quasi-contractive-like mapping, satisfying contractive condition (58). Then,

IJMSO
International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 23 - 43
https://doi.org/10.5281/zenodo.8404255

(a) T has a unique fixed point $f \in Tf$.
(b) Furthermore, $(XIV) \Rightarrow (XV)$ and $(XIV) \Rightarrow (XVI) \Rightarrow (XVII)$. That is, contractive condition (55) $\Rightarrow$ (56) and (55) $\Rightarrow$ (57) $\Rightarrow$ (58).

Proof:
The proof of the first part of Theorem 6 follows from that of Theorem 1.
Next, we show that $(XIV) \Rightarrow (XV)$ and $(XIV) \Rightarrow (XVI) \Rightarrow (XVII)$.
$(XIV) \Rightarrow (XV)$: If there exists a δ, satisfying $\delta \in [0,1)$, in (55), then (55) and (56) are related as follows:

$$\begin{aligned}(XIV)\ H_\rho(Tf,Tg) \ \leq \delta \quad & max\{\rho(f-g), \frac{1}{2}[\rho(f-u)+\rho(g-v)],\\ & \frac{1}{2}[\rho(f-v)+\rho(g-u))]\}\\ & < max\{\rho(f-g), \frac{1}{2}[\rho(f-u)+\rho(g-v)],\\ & \frac{1}{2}[\rho(f-v)+\rho(g-u))]\}\ \forall f,g \in D,\ f \neq g,\\ & \forall u \in Tf,\ \forall v \in Tg. \end{aligned} \tag{90}$$

$(XIV) \Rightarrow (XVI)$.
If there exists a δ, satisfying $\delta \in [0,1)$, in (55) and (57) respectively, then (55) and (57) are related as follows: (55) can be simplified as:

$$\begin{aligned}(XIV)\ H_\rho(Tf,Tg) \ \leq \delta \quad & \rho(f-g)+2\delta\rho(f-u),\ \forall f,g \in D,\ f \neq g,\\ & \forall u \in Tf. \end{aligned} \tag{91}$$

Let $L = 2\delta$ in (57), we get (91), thus

$$\begin{aligned}(XIV)\ H_\rho(Tf,Tg) \ \leq \delta \quad & max\{\rho(f-g), \frac{1}{2}[\rho(f-u)+\rho(g-v)],\\ & \frac{1}{2}[\rho(f-v)+\rho(g-u))]\}\\ & \leq \delta\rho(f-g)+L\rho(u-f)\ \forall f,g \in D,\ f \neq g,\\ & \forall u \in Tf,\ \forall v \in Tg. \end{aligned} \tag{92}$$

$(XVI) \Rightarrow (XVII)$.
If there exists a δ, L satisfying $\delta \in [0,1)$, $L \geq 0$ in (57) and $\delta \in [0,1)$ and $\varphi : R^+ \to R^+$ a $\rho-$monotone increasing function in (58), then (57) and (58) are related as follows: If $\varphi(t) = Lt$, then (58) becomes (57), thus

$$\begin{aligned}(XIV)\ H_\rho(Tf,Tg) \ \leq \delta \quad & \rho(f-g)+L\rho(u-f)\\ & \leq \delta\rho(f-g)+\varphi(\rho(u-f))\ \forall f,g \in D,\\ & \forall u \in Tf,\ \forall v \in Tg. \end{aligned} \tag{93}$$

Thus, it follows that contractive condition (55) $\Rightarrow$ (56) and (55) $\Rightarrow$ (57) $\Rightarrow$ (58). The proof is complete.

Example 7. Let X the real number system, define ρ on X by $\rho(f) = |f|^k$, $k \geq 1$. Let $D = \{f \in L_\rho : 0 \leq f(x) \leq 1\}$. Let $T : D \to P_\rho(D)$ be the multivalued mapping such that P_ρ^T is $\rho-$quasi-contractive satisfying $T(f) = \{\delta f\}$, where $\delta = \frac{1}{8}$.
Clearly, D is non-empty $\rho-$ compact, $\rho-$bounded convex subset of $L_\rho = X$ which satisfies (UC) condition. $F_\rho(T) = \{f_0\}$, where $f_0 = 0$, and P_ρ^T is a $\rho-$contraction with $P_\rho^T(f) = \{Tf\}\ \ \forall\ f \in D$. Notice that if $k = 2$, then $\rho(f) = |f|^2$ is NOT normed.

Remark 8. The existence results in a modular function space in **Theorems 1-6** above are new results and **Example 7** illustrates a multivalued $\rho-$quasi-contractive map in a modular function space which do not belong to a normed linear spsce.

3 Conclusion

In conclusion, this research extends known contractive and contraction maps in metric and normed linear spaces to multivalued types in the framework of modular function spaces. The maps were compared through well constructed fixed point theorems with proofs. The comparison of the various ρ- contractive and ρ-contraction mappings in modular function spaces reveals several important insights and considerations. The choice of ρ-contraction mapping plays a significant role in determining the behaviour and convergence properties of iterative schemes in these spaces. An example is given to show a multivalued $\rho-$quasi-contractive map in a modular function space which do not belong to other particular spsces. The classes of multivalued maps considered in this study have several applications that have good potentials for further research. In practical applications, a thorough understanding of the properties of various contraction mappings and their interactions with modular function spaces is essential to come up with effective numerical schemes. Researchers should continue to explore and refine the comparison of these mappings, seeking to uncover new insights and methodologies that can enhance the accuracy, efficiency, and applicability of iterative schemes within modular function spaces.

Funding. No funds, grants or other support was received.

Competing interest. The authors declare that they have no competing interest.

Author contributions. The first two authors wrote the main paper while the third author constructed example 7. All the authors proofread and approved the final manuscript.

Acknowledgment. The authors are grateful to Professor J. O. Olaleru for supervising their Ph.D. Theses. They wish to also appreciate the anomymous referees for their valuable criticism and contributions leading to the improvement of this paper.

References

[1] Nakano, H., Modular Semi-Ordered Spaces. Maruzen, Tokyo, (1950).

[2] Musielak, J. and Orlicz, W., On modular spaces. *Stud. Math.*, **18,** 591-597 (1959).

[3] Khamsi, M. A., Kozlowski, W. M., Fixed Point Theory in Modular Function Spaces, Springer International Publishing, Switzerland, 2015.

[4] Chistyakov, V. V., Metric modular spaces, I basic concepts, *Nonlinear Anal Theor Meth Appl.*; **72**: 1-14 (2010).

[5] Chistyakov, V. V., Metric modular spaces, II Applications to superposition operators, *Nonlinear Anal Theor Meth Appl.*; **72**: 15-30 (2010).

[6] Okeke, G. A., Khan, S. H, Approximation of fixed point of multivalued $\rho-$quasi-contractive mappings in modular function spaces, *Arab J. Math. Sci.,* **26 (1/2)**, 75-93 (2019).

[7] Okeke, G. A., Francis, D. and Gibali A., On fixed point theorems for a class of $\alpha - \nu$ - Meir – Keeler – type contraction mapping in modular extended b-metric spaces, *The Journal of Analysis,* **30(3)**, pp. 1257-1282 (2022).

[8] Okeke, G. A. and Francis, D., Some fixed-point theorems for a general class of mappings in modular G-metric spaces, *Arab Journal of Mathematical Sciences,* **Vol. 28 No. 2,** pp. 203-216 (2022).

[9] Okeke, G. A., Francis D. and Abbas, M., Common fixed point theorems in modular metric spaces with applications to nonlinear integral equation of Urysohn type, *The Journal of Analysis,* **30**, 1069-1114 (2022).

[10] Akewe, H. and Olaoluwa, H., Multistep-type construction of fixed point for multivalued $\rho-$quasi-contractive-like maps in modular function spaces, *Arab Journal of Mathematical Sciences*, **27 (2)**, 189-213 (2021).

[11] Khamsi, M. A., Kozlowski, W. M. and Reich, S., Fixed point theory in modular function spaces. *Nonlinear Anal.*, **14,** 935-953 (1990).

[12] Khamsi, M. A., A convexity property in modular function spaces. *Math. Japonica.*, **44 (2),** 269-279 (1996).

[13] Khan, S. H., Abbas, M., Approximating fixed points of multivalued $\rho-$nonexpansive mappings in modular function spaces, *Fixed Point Theory Appl.*, **34,** 9 pages (2014).

[14] Khan, S. H., Abbas, M. and Ali, S., Fixed point approximation of multivalued $\rho-$quasi-nonexpansive mappings in modular function spaces, *J. Nonlinear Sci. Appl.*, **10**, 3168–3179 (2017).

[15] Khan, S. H., Approximating fixed points of $(\lambda, \rho)-$firmly nonexpansive mappings in modular function spaces, *Arab. J. Math.*, **7,** doi:10.1007/s40065-018-0204-x (2018).

[16] Kutbi, M. A., Latif A., Fixed points of multivalued mappings in modular function spaces, *Fixed Point Theory Appl.*, **(2009)**, 786357, 12 pages (2009).

[17] Rakotch, E., A note on contractivem appings *Proc. Amer. Math. Soc.*, **13**, 459-465 (1962).

[18] Edelstein, M., An extension of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **12**, 7-10 (1960).

[19] Edelstein, M., On fixed and periodic points under contractive mappings, *J. London Math. Soc.*, **37**, 74-79 (1962).

[20] Kannan, R., Some results on fixed points, *Bulletin Calcutta Mathematical Society,* **10**, 71-76 (1968).

[21] Kannan, R., Some results on fixed points. I I, *Amer. M ath. Monthly* **76** 405-408 (1969).

[22] Bianchini, R. M. T., Su un problema di S. Reich aguardante la teoría dei punti fissi, *Boll. Un. Mat. Ital.* **5**, 103-108 (1972).

[23] Reich, S., Some remarks concerning contraction mappings, *Oanad. Math. Bull.* **14**, 121-124 (1971).

[24] Reich, S., Kannan's fixed point theorem, *Boll. Un. Mat. Ital. (4)* **4**, 1 -11 (1971).

[25] Reich, S., Remarks on fixed points, *Atti. Accad. Naz. Lincei Rend. CI. Sei. Fis. Mat Natur. (8),* **52**, , 689-697 (1972).

[26] Chatterjea, S. K., Fixed point theorems, *Comptes Rendus de l' Academie Bulgare des Sciences,* 25(6), 727-730 (1972).

[27] Chatterjea, S. K., Fixed point theorems for a sequence of mappings with contractive iterates, *Publ. Inst Math. (Beograd)* **14 (28)**,15-18 (1972).

[28] Sehgal, V. M., On fixed and periodic points for a class of mappings, *J. London Math. Soc. (2),* **5**, 571-576 (1972).

[29] Zamfirescu, T., Fixed point theorems in metric spaces, *Archiv der Mathematik (Basel),* **23**, 292-298 (1972).

[30] Zamfirescu, T., A theorem on fixed points, *Atti Acad. Naz. Lincei Rend. Cl. Sei. Fis. Mat. Natur. (8),* **52**, 832 - 834 (1973).

[31] Osilike, M. O., Stability results for Ishikawa fixed point iteration procedure, *Indian Journal of Pure and Applied Mathematics,* **26** (10), 937-941 (1995/96).

[32] Osilike, M. O. and Udomene, A., *Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings,* Indian Journal of Pure and Applied Mathematics, **30**, 1229-1234 (1999).

[33] Imoru, C. O., Olatinwo, M. O., On the stability of Picard and Mann iteration. *Carpath. J. Math.*, **19**, 155-160 (2003).

[34] Hardy, G. E. and Rogers, T. D., A generalization of a fixed point theorem of Reich, *Canad. Math. Bull.*, **16**, 201-206 (1973).

[35] Ciric, L. B., Generalized contractions and fixed-point theorems, *Publ. Inst. Math. (Beograd) (N.S.)* **12 (26)**, 19-26 (1971).

[36] Ciric, L. B., A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.,* **45**, 267-273 (1974).

[37] Akewe, H., *Approximation of fixed and common fixed points of generalized contractive-like operators.* University of Lagos, Lagos, Nigeria, Ph.D. Thesis, 112 pages, (2010).

[38] Berinde V., Picard iteration converges faster than the Mann iteration in the class of quasicontractive operators,*Fixed Point Theory Appl.,* **2004**, 97-105 (2004).

[39] Berinde, V., On the stability of some fixed point procedures, *Buletinul Stiintific al Universitatii din Baia Mare. Seria B. Fascicola Mathematica-Informatica,* **XVIII** (1), 7–14 (2002).

[40] Berinde, V., Iterative approximation of fixed points, *Baia Mare, Editura Efemeride* (2002).

[41] Ishikawa, S., Fixed points by a new iteration method, *Proceedings of the American Mathematical Society,* 44, 147-150 (1974).

[42] Mann, W. R., Mean value methods in iterations, *Proceedings of the American Mathematical Society,* 44, 506-510 (1953).

[43] Noor, M. A., New approximation schemes for general variational inequalities, *Journal of Mathematical Analysis and Applications,* **251**, 217-229 (2000).

[44] Picard E., Sur l'application des methodes d'approximations successives a l'etude de certaines equations differentielles ordinaires, *Journal de Mathematicas,* **9**, 217-271 (1893).

[45] Rhoades, B. E., Comment on two fixed point iteration method, *Journal of Online Mathematics and Its Applications,* **56**, 741-750 (1976).

[46] Rhoades, B. E., A comparison of various definition of contractive mapping, *Transactions of the American Mathematical Society*, **226**, 257-290 (1977).

[47] Rhoades, B. E., Fixed point theorems and stability results for fixed point iteration procedures, *Indian Journal of Pure and Applied Mathematics*, **21,** 1-9 (1990).

[48] Rhoades, B. E., Fixed point theorems and stability results for fixed point iteration procedures II, *Indian Journal of Pure and Applied Mathematics,* **24 (11)**, 691-703 (1993).

[49] Okeke, G. A., Olaleru, J. O. and Olatinwo, M. O., Existence and approximation of fixed point of a nonlinear mapping satisfying rational type contractive inequality condition in complex-valued Banach spaces, *International Journal of Mathematical Analysis and Optimization: Theory and Applications*, **Vol. 2020, No. 1,** pp. 707 - 717 (2020).

[50] Okeke, G. A., $\rho - T$-Stability of some fixed point iterations with respect to operators satisfying contractive conditions of integral type in modular function spaces, *International Journal of Mathematical Analysis and Optimization: Theory and Applications*, **Vol. 2019, No. 1,** pp. 412 - 426 (2019).