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# Tropical Polynomial of Partial Contraction Transformation Semigroup

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**Abstract**

In this paper, we used tropical geometry on a partial transformation semigroup to create a tropical polynomial on a partial contraction mapping. Then we used the tropical polynomial to obtain a contraction mapping and plot tropical curves. Finally, we were able to find the roots of the tropical curve and determine their multiplicities.

**Keywords:** Multiplicity, Partial contraction, Root, Tropical geometry, Tropical polynomial, Partial transformation semigroup.
**MSC2010: 06F05.**

## 1 Introduction

A semigroup is an algebraic structure that consists of a set and an associative binary operation. It is a generalization of a group, but without the need for an identity element and inverses. This is why it's called a semigroup.
Let $N = \{1, 2, 3, \cdots, n\}$ be a finite chain a map $\alpha$ which has a domain and image both subset of $N$ is said to be partial. The collection of all partial transformation of $N$ is known as semigroup of partial transformation usually denoted by $P_n$
Let $CP_n = \{\alpha \in P_n :\mid x\alpha - y\alpha \mid\leq\mid x - y \mid \forall x, y \in Dom(\alpha)\}$ is known to be subsemigroup of $P_n$ (partial contraction transformation semigroup). The study of this semigroup and their respective subsemigroup was first initiated by Adeshola and Umar [1].

The study of semigroups is a relatively recent development, with researchers only beginning to explore them in the early 1900s. This was driven by the realization that it was important to analyze universal transformations, not just invertible ones. Early discoveries in the field include Cayley's theorem, which shows that any semigroup can be realized as a transformation semigroup, where functions of any kind replace the bijections of group theory. In this paper, we focus on the efforts of [2,3] to bring out the contraction element from transformation semigroups.

The name "tropical" was coined by French mathematician. Tropical geometry has established itself

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as an important new field bridging algebraic geometry whose techniques have been used to attack problems, these include enumerative geometry and arithmetic geometry. It builds on the older area of tropical mathematics more commonly known as max-plus algebra which arises in semigroup theory, computer science and optimization [4]. Tropical algebraic geometry is an intriguing new area of research in Mathematics that is focused on studying piecewise-linear functions that act like algebraic variety. The concept behind this area has been around for some time, with early ideas appearing in the works of [5–8]. However, it wasn't until the late 1900s that a concerted effort was made to solidify the basic definition of this theory. Interestingly, this effort was largely driven by the application of tropical algebraic geometry to enumerative algebraic geometry, as discovered by [9].

It's fascinating to see how tropical geometry has evolved into a distinct field of mathematics in such a short amount of time. What's even more impressive is the numerous connections that have been made to other areas of pure and applied mathematics. It's exciting to think about what new discoveries and applications will come from this field in the future.

## 2 Preliminary Notes

Contraction transformation semigroup has been studied by many researchers [4, 10–12]. Some of their results are given below:

**Theorem 2.1. [11]** Let $\alpha, \beta \in S$. Then

i $\alpha L^* \beta$ if and only if $Im\alpha = Im\beta$.

ii $\alpha R^* \beta$ if and only if $ker\alpha = ker\beta$.

iii $\alpha H^* \beta$ if and only if $Im\alpha = Im\beta and ker\alpha = ker\beta$.

iv $\alpha D^* \beta$ if and only if $|Im\alpha| = |Im\beta|$.

for $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_p \\ x_1 & x_2 & \cdots & x_p \end{pmatrix}$ and $\beta = \begin{pmatrix} B_1 & B_2 & \cdots & B_p \\ y_1 & y_2 & \cdots & y_p \end{pmatrix}$

**Corollary 2.1 [12]** The semigroup $DCT_n$ is not regular. Let $F(n,r) = |\{\alpha \in DCT_n : |im\alpha| = r\}|$. Then we have the following trivial results.

**Lemma 2.1 [12]** If $S = DCT_n$ then $F(n,r) = \begin{cases} 1 & if \quad r = 1 \\ 1 & if \quad r = n \end{cases}$

**Lemma 2.2 [10]** Let $\alpha \in OCP_n$. Then, for each $\alpha \in DOm(\alpha)$,

i if $\alpha < min(f(\alpha))$ , then $x\alpha > x$;

ii if $\alpha > max(f(\alpha))$ , then $x\alpha < x$;.

**Theorem 2.2 [10]** Let $\alpha \in OCP_n$. Then $\alpha$ can be decomposed as a product of three factors in $OCP_n$ as $\alpha = \alpha_1\alpha_2\alpha_3$, where $\alpha_1$ is an order-increasing partial map, $\alpha_2$ is a partial identity and $\alpha_3$ is an order-decreasing partial map.

**Theorem 2.3. [4]** Let $S = D\gamma_\eta, |\tau|$be the order of the roots of the tropical polynomial in $D\gamma_4$. Then, for all elements in $D\gamma_4$ atisfy $|\tau| < 4$ has unique multiplicity of height two.

However, in this paper we are going to focus on the root and multiplicity of partial contraction transformation semigroups of $P_2$ and $P_3$ by obtaining the tropical graph with the help of MATLAB R2019 V9.6.0.

**Definition 2.1: Semigroup [13]**: A semigroup in mathematics is an algebraic structure made up of a set and an associative binary operation. The most common multiplicative notation for the

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binary operation of a semigroup is $x.y$, or just $xy$, which represents the outcome of applying the semigroup operation to the ordered pair $(x, y)$. Formally, associativity is defined as $(xy)z = x(yz)$ for any $x, y$, and$z$ in the semigroup.

**Definition 2.2: Transformations: [13]** Let $X$ and $Y$ be the two non empty sets such that there is some rule $F$ which assigns to each element $y \in Y$, a unique element $x \in X$, then this rule is said to be a transformation or mapping.

**Definition 2.3 [14]** A point $x_0 \in T$ is a tropical root of order at least k of a tropical polynomial $P(x) =$ "$(x + x_0)^k q(x)$" for some k. The largest k for which this is possible is the multiplicity of the root $x_0$

**Definition 2.4 [14]** Let $P(x, y) =$ " $\sum_{i,j} a_{i,j} x^i y^j$" be a tropical polynomial. The tropical curve C defined by $P(x, y)$ is the set of points $(x_0, y_0)$in $R^2$ such that there exist pairs $(i, j) \neq (k, j)$ satisfying $P(x_0, y_0) = a_{i,j} + ix_0 + iy_0 = a_{k,l} + kx_0 + ly_0$.

# 3 Main Results

## Tropical Polynomial

A tropical monomial in $k$ variable is an expression which takes the form $x_1^{q_1}, x_2^{q_2}, \cdots x_k^{q_k}$. A tropical polynomial is the tropical linear addition of the tropical monomials i.e.

$$F(x) = \sum_k a_k x_k = \max_k \{a_k + x^k\}$$

## Partial Contraction Transformation Semigroup $P_n$

This can be denoted by $P_n$. It can be expressed by using the formula below to obtain different $P_n$.

$$P_n = (n + 1)^n$$

When $n = 1$

$$\begin{aligned} P_1 &= (1 + 1)^1 \\ &= 2^1 \\ &= 2 \end{aligned}$$

$P_1$ has 2 elements, then we form one mapping by setting the image of two elements as column and express it as tropical polynomial.

$$P_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ - \end{pmatrix}$$

When $n = 2$

$$\begin{aligned} P_2 &= (2 + 1)^2 \\ &= 3^2 \\ &= 9 \end{aligned}$$

$P_2$ has 9 elements, then we form one mapping by setting the image of two elements as column and express it as tropical polynomial.

$$\begin{aligned} P_2 = &\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \\ &\begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix} \\ &\begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix} \end{aligned}$$

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When $n = 3$

$$\begin{aligned} P_3 &= (3+1)^3 \\ &= 4^3 \\ &= 64 \end{aligned}$$

$P_3$ has 64 elements, then we form one mapping by setting the image of two elements as column and express it as tropical polynomial.

$$\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\1&3&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\3&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&3\end{pmatrix}$$

$$\begin{pmatrix}1&2&3\\3&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&3\end{pmatrix}\begin{pmatrix}1&2&3\\-&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\-&2&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\-&3&1\end{pmatrix}\begin{pmatrix}1&2&3\\-&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&3&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\1&-&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&-&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&-&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&-&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&-&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\3&-&3\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&1\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\1&1&-\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\1&3&-\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&-\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&-\end{pmatrix}\begin{pmatrix}1&2&3\\3&1&-\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&-\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\3&3&-\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\2&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\3&-&-\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&1&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&3&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&-\end{pmatrix}$$

Now using the elements to find the contraction

## Contraction transformation Semigroup in $P_2$

All the matrices above in $P_2$ are contraction.
Using $P_2$ to form a polynomial

Taking the image of the first and second matrices of $P_2$ to form tropical.

1. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{array}$
   $= x^2 + x + x + 2$
   $= x^2 + 2x + 2$
   $\max\{2x, 2 + x, 2\}$

2. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x^2 & x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \end{array}$
   $= x^3 + x^2 + 2x + x^2 + 2x + 1$
   $= x^3 + 2x^2 + 4x + 1$
   $\max\{3x, 2 + 2x, 4 + x, 1\}$

3. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x^3 & x^2 & x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix} \end{array}$
   $= x^4 + x^3 + 2x^2 + 2x + x^3 + 2x^2 + x + 2$
   $= x^4 + 2x^3 + 4x^2 + 3x + 2$
   $\max\{4x, 2 + 3x, 4 + 2x, 3 + x, 2\}$

4. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x^4 & x^3 & x^2 & x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 & 2 & 0 \\ 1 & 2 & 1 & 2 & 1 \end{pmatrix} \end{array}$
   $= x^5 + x^4 + 2x^3 + 2x^2 + 0 + x^4 + 2x^3 + x^2 + 2x + 1$
   $= x^5 + 2x^4 + 4x^3 + 3x^2 + 2x + 1$
   $\max\{5x, 2 + 4x, 4 + 3x, 3 + 2x, 2 + x, 1\}$

5. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x^5 & x^4 & x^3 & x^2 & x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 & 2 & 0 & 0 \\ 1 & 2 & 1 & 2 & 1 & 2 \end{pmatrix} \end{array}$
   $= x^6 + x^5 + 2x^4 + 2x^3 + x^5 + 2x^4 + x^3 + 2x^2 + x + 2$
   $= x^6 + 2x^5 + 4x^4 + 3x^3 + 2x^2 + x + 2$
   $= \max\{6x + 2 + 5x, 4 + 4x, 3 + 3x, 2 + 2x, x, 2\}$

6. $P_2(x) = \begin{array}{c} \\ x \\ 1 \end{array}\!\!\begin{array}{c} \begin{matrix} x^6 & x^5 & x^4 & x^3 & x^2 & x & 1 \end{matrix} \\ \begin{pmatrix} 1 & 1 & 2 & 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 0 \end{pmatrix} \end{array}$
   $= x^7 + x^6 + 2x^5 + 2x^4 + x + x^6 + 2x^5 + x^4 + 2x^3 + x^2 + 2x$
   $= x^7 + 2x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 + 3x$
   $\max\{7x, 2 + 6x, 4 + 5x, 3 + 4x, 2 + 3x, 2x, 3\}$

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$$\begin{pmatrix}1&2&3\\1&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\1&2&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&2&2\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\2&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\3&2&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&3\end{pmatrix}\begin{pmatrix}1&2&3\\3&3&2\end{pmatrix}\begin{pmatrix}1&2&3\\3&3&3\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&1&1\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&2\end{pmatrix}\begin{pmatrix}1&2&3\\1&-&1\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&1\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&-&2\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&3\end{pmatrix}\begin{pmatrix}1&2&3\\1&1&-\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&-\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\1&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\2&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\3&-&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&1&-\end{pmatrix}$$
$$\begin{pmatrix}1&2&3\\-&2&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&3&-\end{pmatrix}\begin{pmatrix}1&2&3\\-&-&-\end{pmatrix}$$

1. Using $P_3$ to form a tropical polynomial
   Taking the image of the first, second and third element of $P_3$ and for their tropical.

$$P_3 = \begin{array}{c} \\ x^2 \\ x \\ 1 \end{array}\begin{array}{c} \begin{matrix} x^2 & x & 1 \end{matrix} \\ \begin{pmatrix}2&3&3\\3&2&2\\3&1&2\end{pmatrix}\end{array}$$
$$= 2x^4 + 3x^3 + 3x^2 + 3x^3 + 2x^2 + 2x + 3x^2 + x + 2$$
$$= 2x^4 + 6x^3 + 8x^2 + 3x + 2$$
$$\max\{2 + 4x, 6 + 3x, 8 + 2x, 3 + x, 2\}$$

2. Forming the tropical by the images of some other element.

$$P_3 = \begin{array}{c} \\ x^2 \\ x \\ 1 \end{array}\begin{array}{c} \begin{matrix} x^2 & x & 1 \end{matrix} \\ \begin{pmatrix}1&2&2\\2&1&1\\2&1&2\end{pmatrix}\end{array}$$
$$= x^4 + 2x^3 + 2x^2 + 2x^3 + x^2 + x + 2x^2 + x + 2$$
$$= x^4 + 4x^3 + 5x^2 + 2x + 2$$
$$\max\{4x, 4 + 3x, 5 + 2x, 2 + x, 2\}$$

3. Forming the tropical by the images of some other elements.

$$P_3 = \begin{array}{c} \\ x^2 \\ x \\ 1 \end{array}\begin{array}{c} \begin{matrix} x^2 & x & 1 \end{matrix} \\ \begin{pmatrix}2&2&2\\2&2&3\\2&3&2\end{pmatrix}\end{array}$$
$$= 2x^4 + 2x^3 + 2x^2 + 2x^3 + 2x^2 + 3x + 2x^2 + 3x + 2$$
$$= 2x^4 + 4x^3 + 6x^2 + 6x + 2$$
$$\max\{2 + 4x, 4 + 3x, 6 + 2x, 6 + x, 2\}$$

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4. Using $P_3$ to form a tropical polynomial
   Taking the image of first, second and third elements of $P_3$ and form their tropical

$$P_3(x) = \begin{matrix} & x^2 & x & 1 \\ x^2 & 2 & 2 & 2 \\ x & 2 & 2 & 3 \\ 1 & 2 & 3 & 2 \end{matrix}$$

$$= 2x^4 + 2x^3 + 2x^2 + 2x^3 + 2x^2 + 3x + 2x^2 + 3x + 2$$

$$= 2x^4 + 4x^3 + 6x^2 + 6x + 2$$

$$\max\{2 + 4x, 4 + 3x, 6 + 2x, 6 + x, 2\}$$

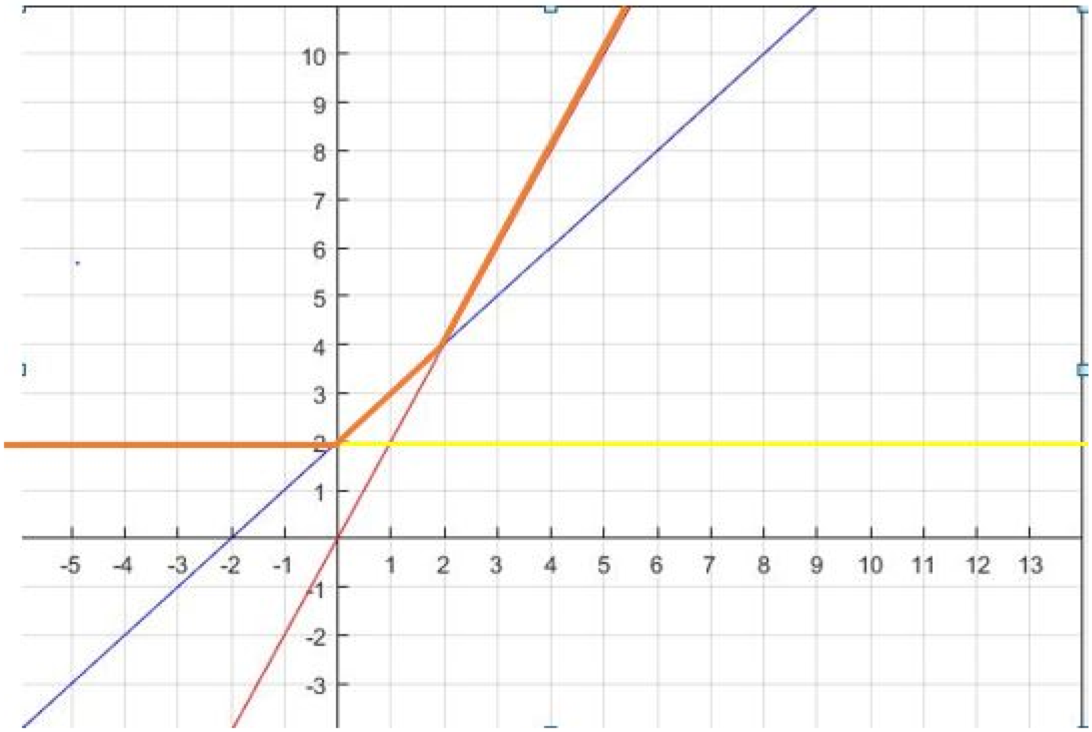
5. $P_3(x) = \begin{matrix} & x^2 & x & 1 \\ x^2 & 3 & 3 & 3 \\ x & 2 & 3 & 3 \\ 1 & 3 & 2 & 3 \end{matrix}$
   $= 3x^4 + 3x^3 + 3x^2 + 2x^3 + 3x^2 + 3x + 3x^2 + 2x + 3$
   $= 3x^4 + 5x^3 + 9x^2 + 5x + 3$
   $\max\{3 + 4x, 5 + 3x, 9 + 2x, 5 + x, 3\}$

### 3.1 Tropical graph and mulitiplicity of $P_2$

To sketch the tropical curve and find the multiplicity of each root, we will use some of the tropical polynomial obtained from $P_2$
Considering the tropical polynomial of $\max\{2x, 2 + x, 2\}$. we have the tropical curve as

**GRAPH I**

Figure 1: Max[2x, 2+x, 2]




We have from the above curve that the roots of the curve are $r_1 = 0$ and $r_2 = 2$. Hence,to obtain the multiplicity of the roots,we have

$$\begin{aligned} M(r_1 = 0) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 1 \\ &= |0 - 1| = |-1| = 1 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ when } m_2 = 1 \text{ and } m_3 = 2 \\ &= |1 - 2| = |-1| = 1 \end{aligned}$$

Hence,the multiplicity of the tropical polynomial of $\max\{2x, 2 + x, 2\}$ is (1,1).

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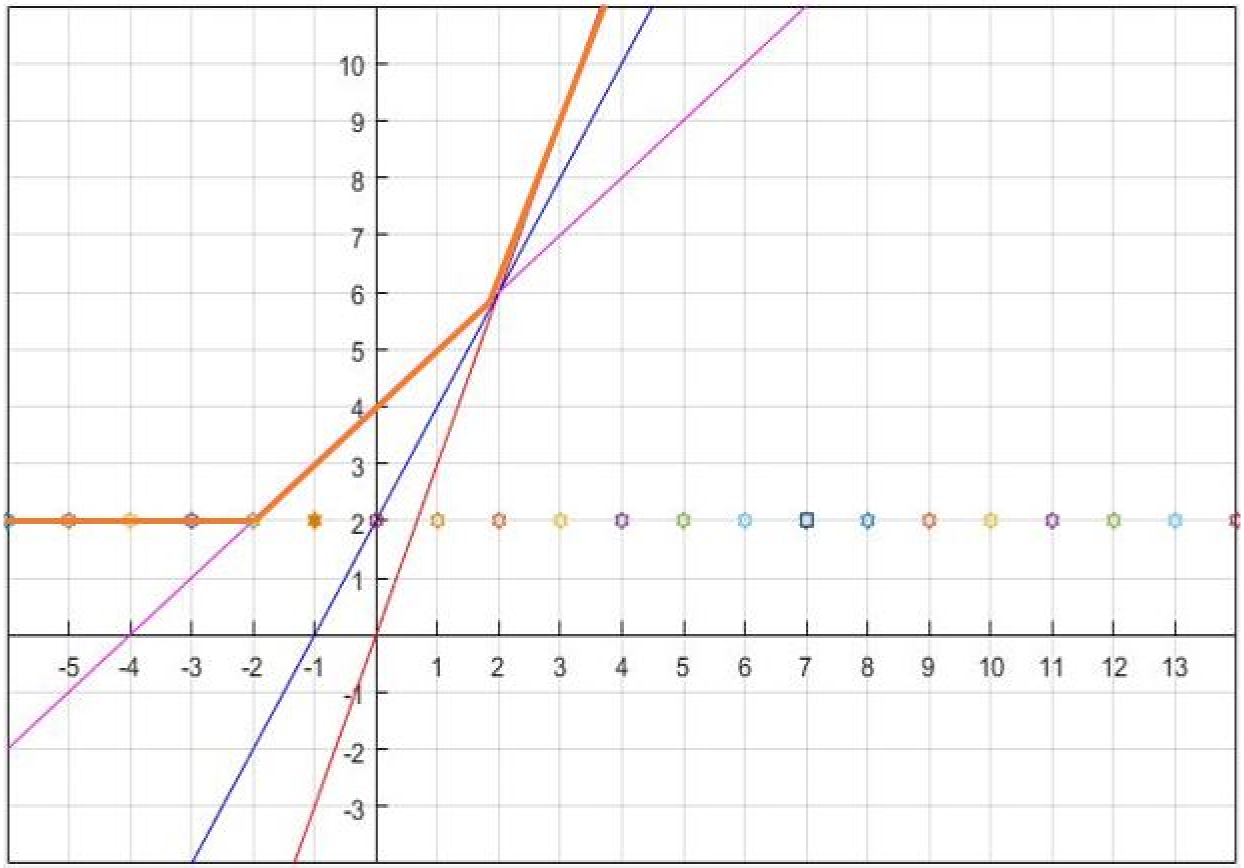
**GRAPH II**

Figure 2: $\max\{3x, 2+2x, 4+x, 2\}$




We have from the above curve that the roots of the curve are $r_1 = -1$ and $r_2 = 2$. Hence,to obtain the multiplicity of the roots,we have

$$\begin{aligned} M(r_1 = -1) &= |m_1 - m_2| \\ &= |0 - 1| = |-1| = 1 \\ M(r_2 = 2) &= |m_2 - m_3| \\ &= |1 - 3| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical polynomial of $\max\{3x, 2+2x, 4+x, 2\}$ is (1,2).

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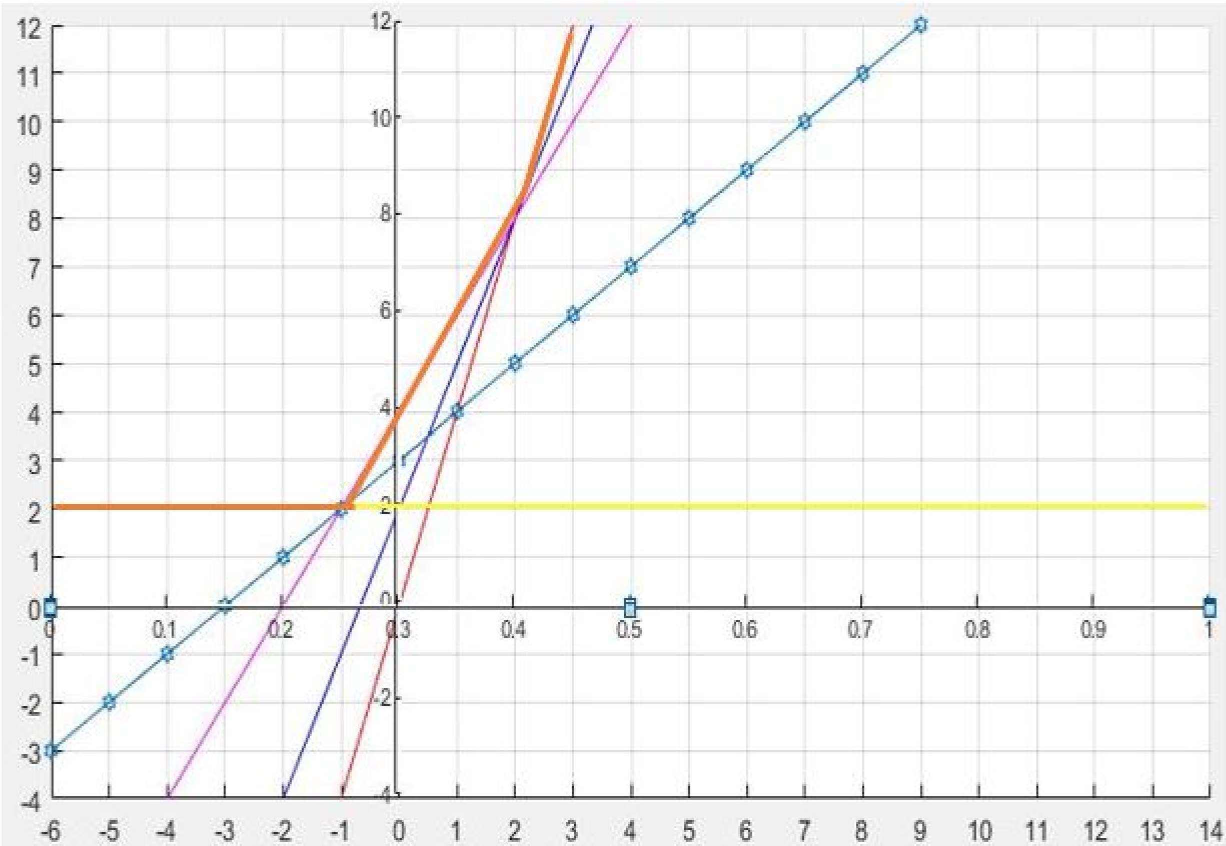
**GRAPH III**

Figure 3: $\max\{4x, 2+3x, 4+2x, 3+x, 2\}$



We have from the above curve that the roots of the curve are:$r_1 = -1$ and $r_2 = 2$.

$$\begin{aligned} M(r_1 = -1) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 2 \\ &= |0-2| = |-2| = 2 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ when } m_2 = 2 \text{ and } m_3 = 4 \\ &= |2-4| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical polynomial of $\max\{4x, 2+3x, 4+2x, 3+x, 2\}$ is (2,2).

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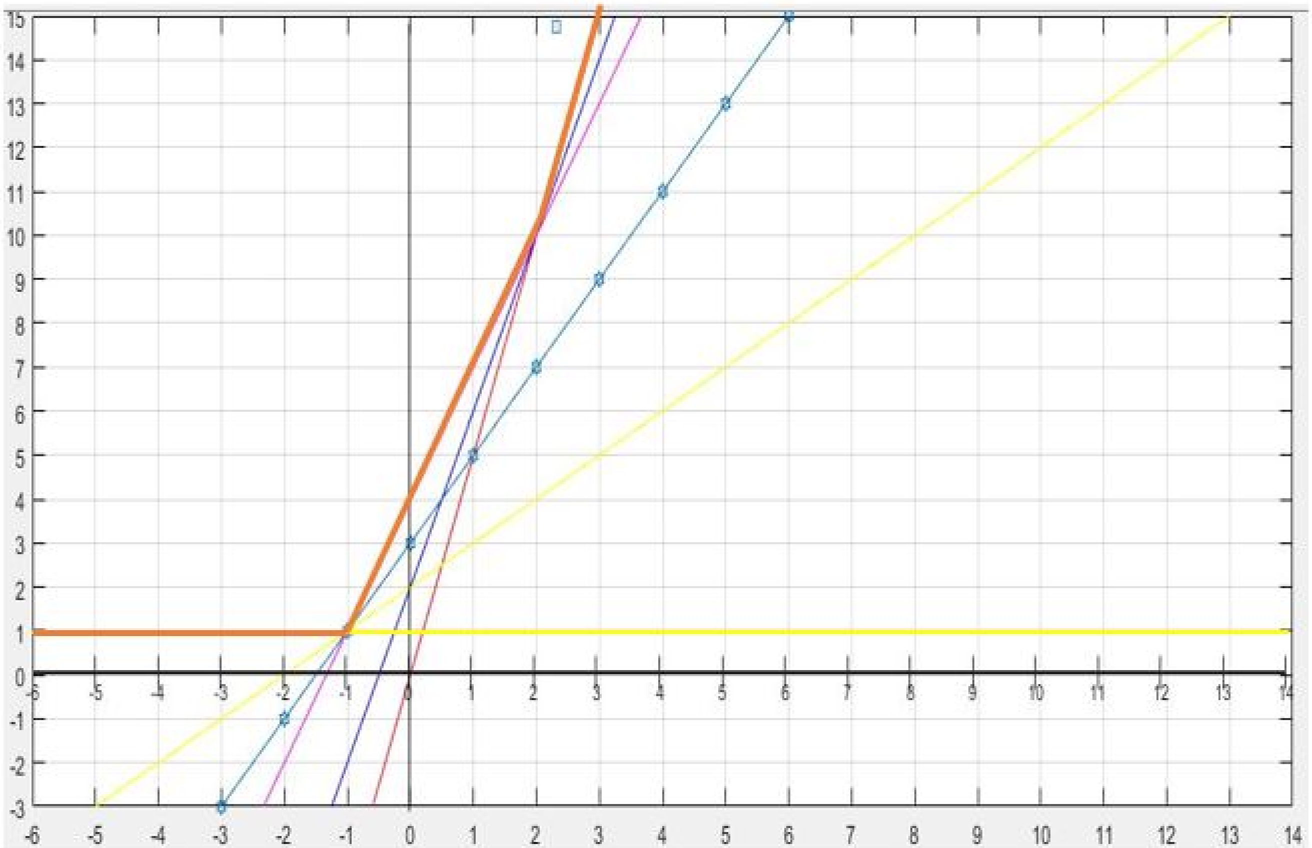
**GRAPH IV**

Figure 4: $\max\{5x, 2+4x, 4+3x, 3+2x, 2+x, 1\}$



We have from the above curve that the roots of the curve are:$r_1 = -1$ and $r_2 = 2$.

$$\begin{aligned} M(r_1 = -1) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 3 \\ &= |0-3| = |-3| = 3 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ when } m_2 = 3 \text{ and } m_3 = 5 \\ &= |3-5| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical polynomial of $\max\{5x, 2+4x, 4+3x, 3+2x, 2+x, 1\}$ is (3,2).

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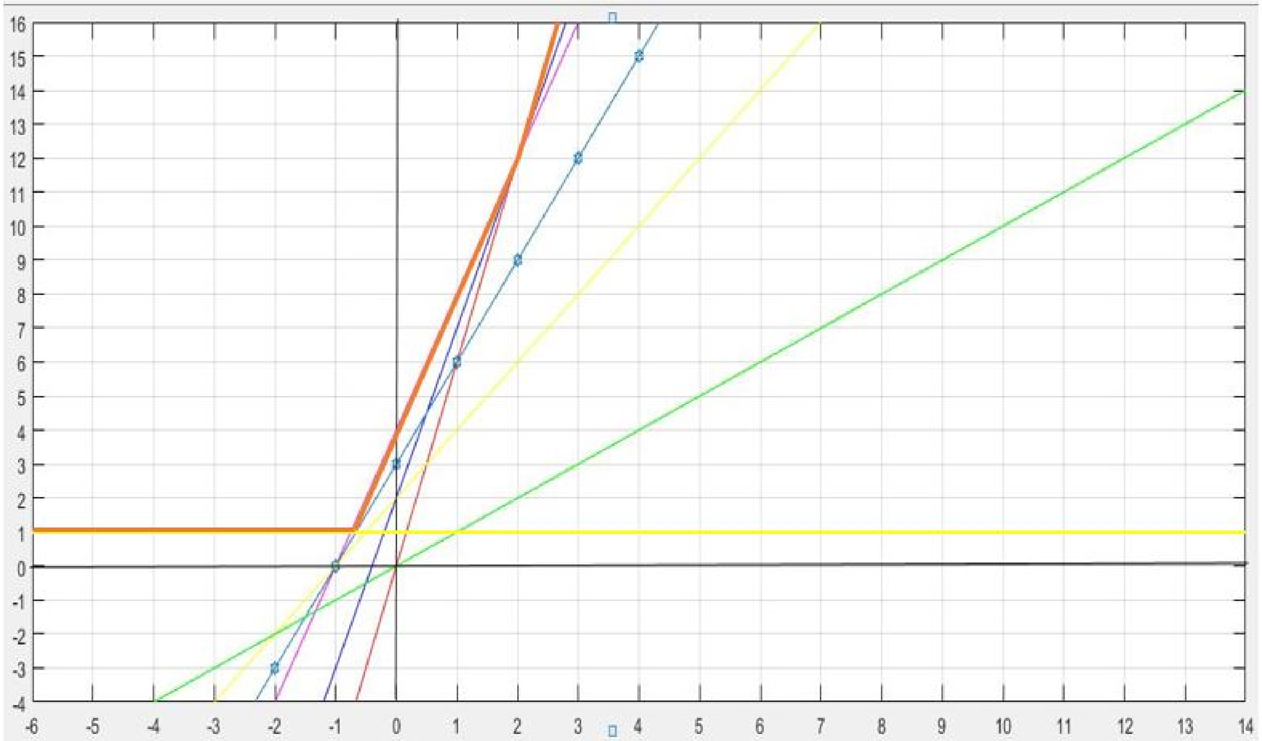
**GRAPH V**

Figure 5: $\max\{6x, 2+5x, 4+4x, 3+3x, 2+2x, x, 2\}$



We have from the above curve that the roots of the curve are $r_1 = -0.5$ and $r_2 = 2$. Hence,to obtain the multiplicity of the roots,we have

$$\begin{aligned} M(r_1 = -0.5) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 4 \\ &= |0 - 4| = |-4| = 4 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ when } m_2 = 4 \text{ and } m_3 = 6 \\ &= |4 - 6| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical polynomial of $\max\{6x, 2+5x, 4+4x, 3+3x, 2+2x, x, 2\}$ is (4,2).

## 3.2 Tropical graph and multiplicity of $P_3$

To sketch the tropical curve and find the multiplicity of each root, we will use some of the tropical polynomial obtained from $P_3$

Consider the tropical polynomial of $\max\{2+4x, 6+3x, 8+2x, 3+x, 2\}$. we have the tropical curve as follows:

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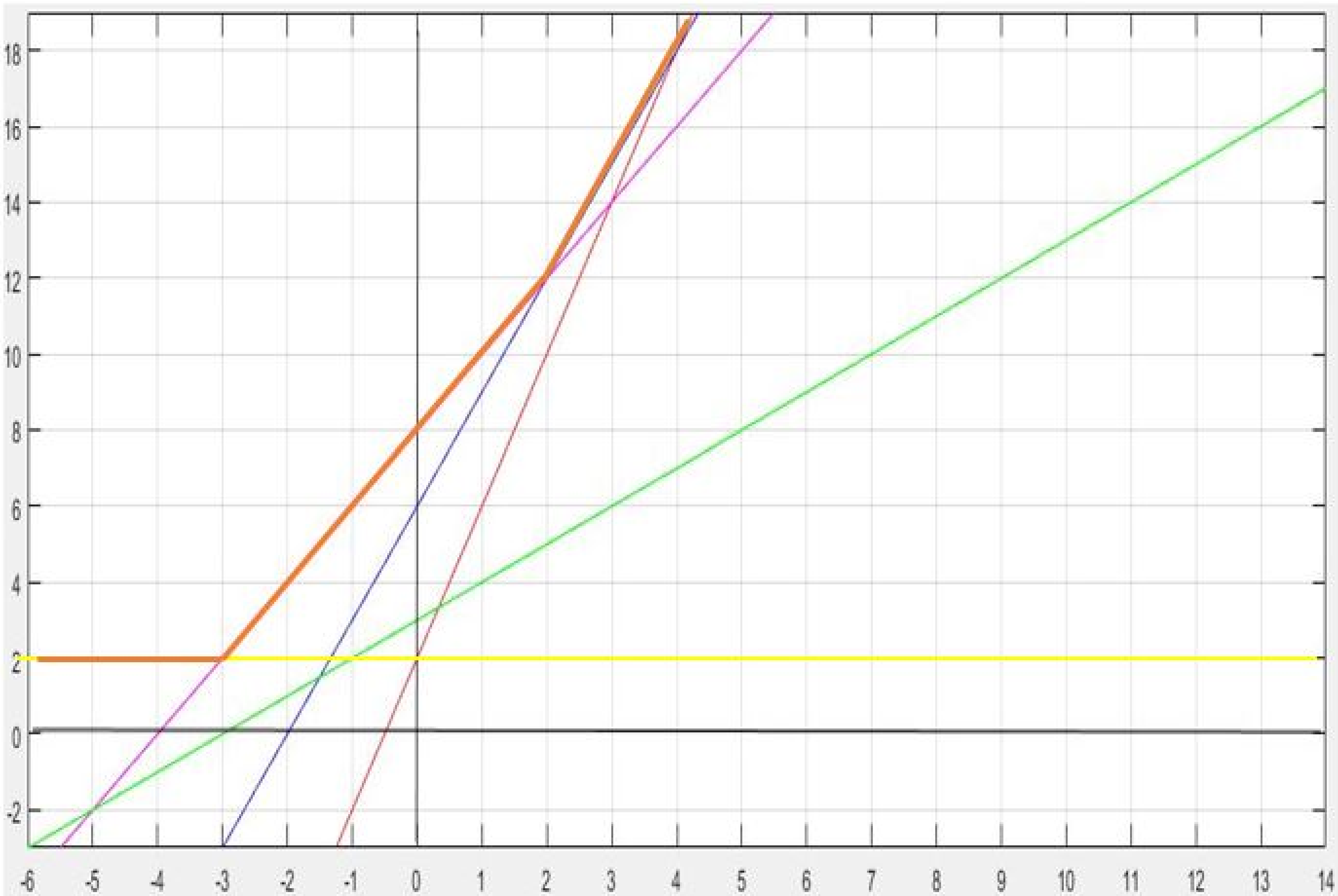
**GRAPH I**

Figure 6: $\max\{2+4x, 6+3x, 8+2x, 3+x, 2\}$




We have from the above curve that the roots of the curve are:$r_1 = -3, r_2 = 2$ and $r_3 = 4$.

$$\begin{aligned} M(r_1 = -3) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 2 \\ &= |0 - 2| = |-2| = 2 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ where } m_3 = 3 \text{ and } m_4 = 4 \\ &= |2 - 3 = |-1| = 1 \\ M(r_3 = 4) &= |m_3 - m_4| \\ &= |3 - 4| = |-1| = 1 \end{aligned}$$

Hence,the multiplicity of the tropical algebra $\max\{2+4x, 6+3x, 8+2x, 3+x, 2\}$ is (2,1,1).

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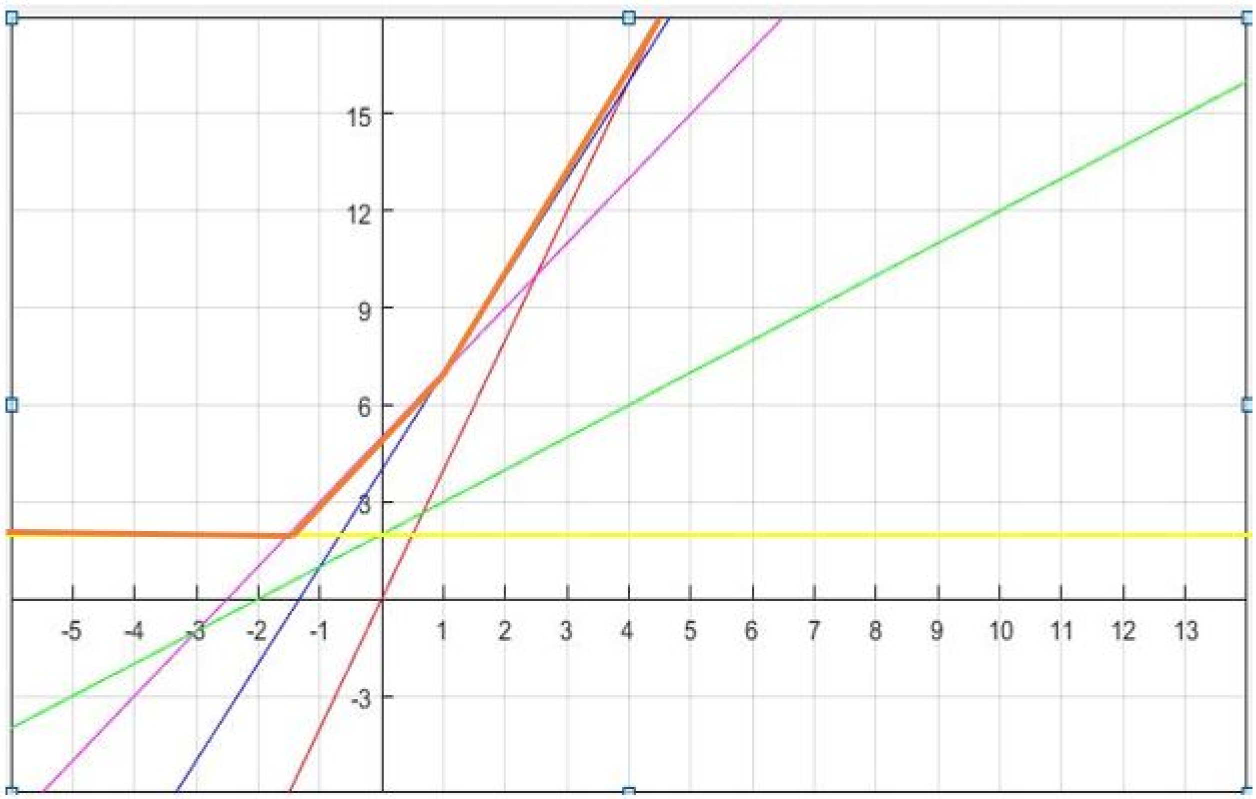
GRAPH II

Figure 7: $\max\{4x, 4+3x, 5+2x, 2+x, 2\}$




We have from the above curve that the roots of the curve are:$r_1 = -1.6, r_2 = 1$ and $r_3 = 4$.

$$\begin{aligned} M(r_1 = -1.6) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 2 \\ &= |0-2| = |-2| = 2 \\ M(r_2 = 1) &= |m_2 - m_3|, \text{ where } m_3 = 3 \text{ and } m_4 = 4 \\ &= |2-3 = |-1| = 1 \\ M(r_3 = 4) &= |m_3 - m_4| \\ &= |3-4| = |-1| = 1 \end{aligned}$$

Hence,the multiplicity of the tropical algebra $\max\{4x, 4+3x, 5+2x, 2+x, 2\}$ is (2,1,1).

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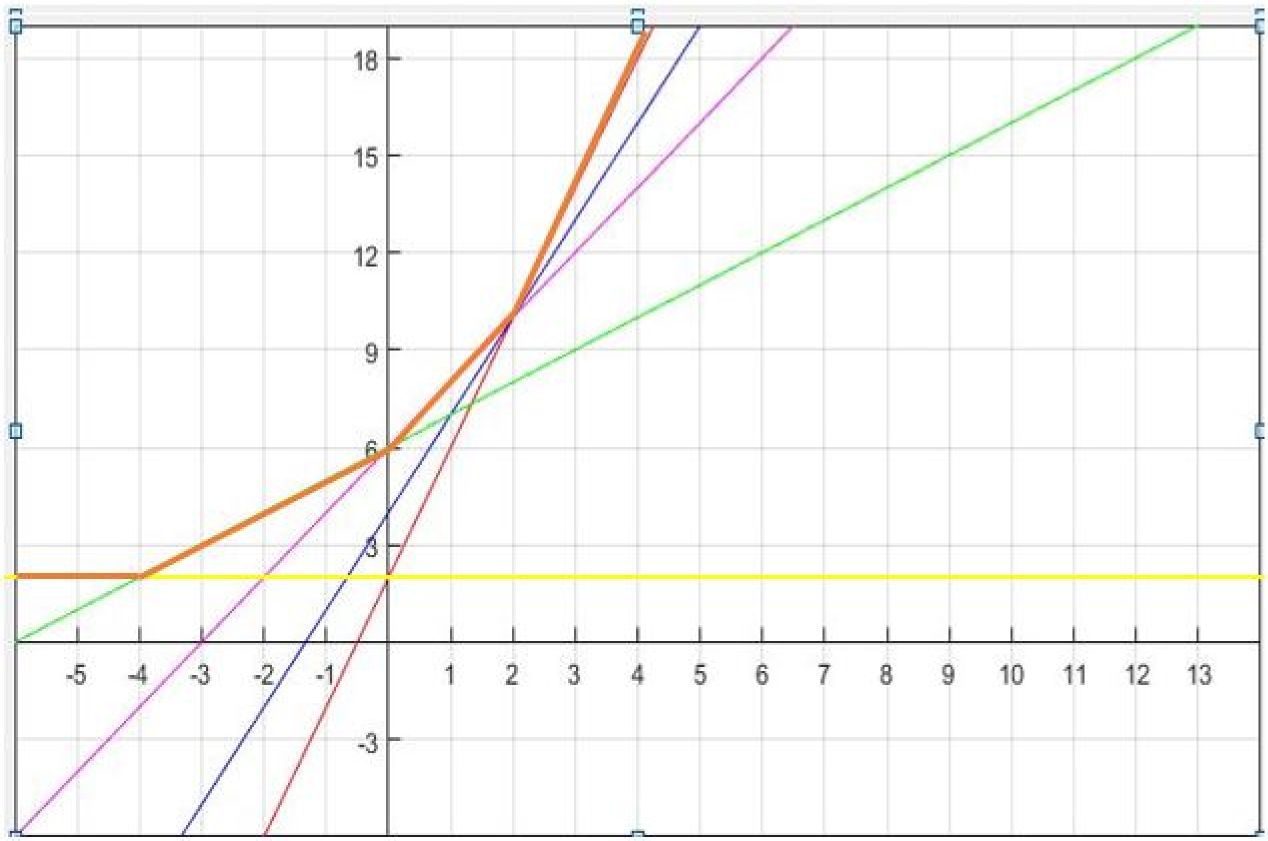
**GRAPH III**

Figure 8: $\max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$




We have from the above curve that the roots of the curve are: $r_1 = -4,$ and $r_2 = 2$.

$$\begin{aligned} M(r_1 = -2) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 2 \\ &= |0-4| = |-4| = 4 \\ M(r_2 = 2) &= |m_2 - m_3|, \text{ where } m_2 = 2 \text{ and } m_3 = 4 \\ &= |2-4| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical algebra $\max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$ is (4,2).

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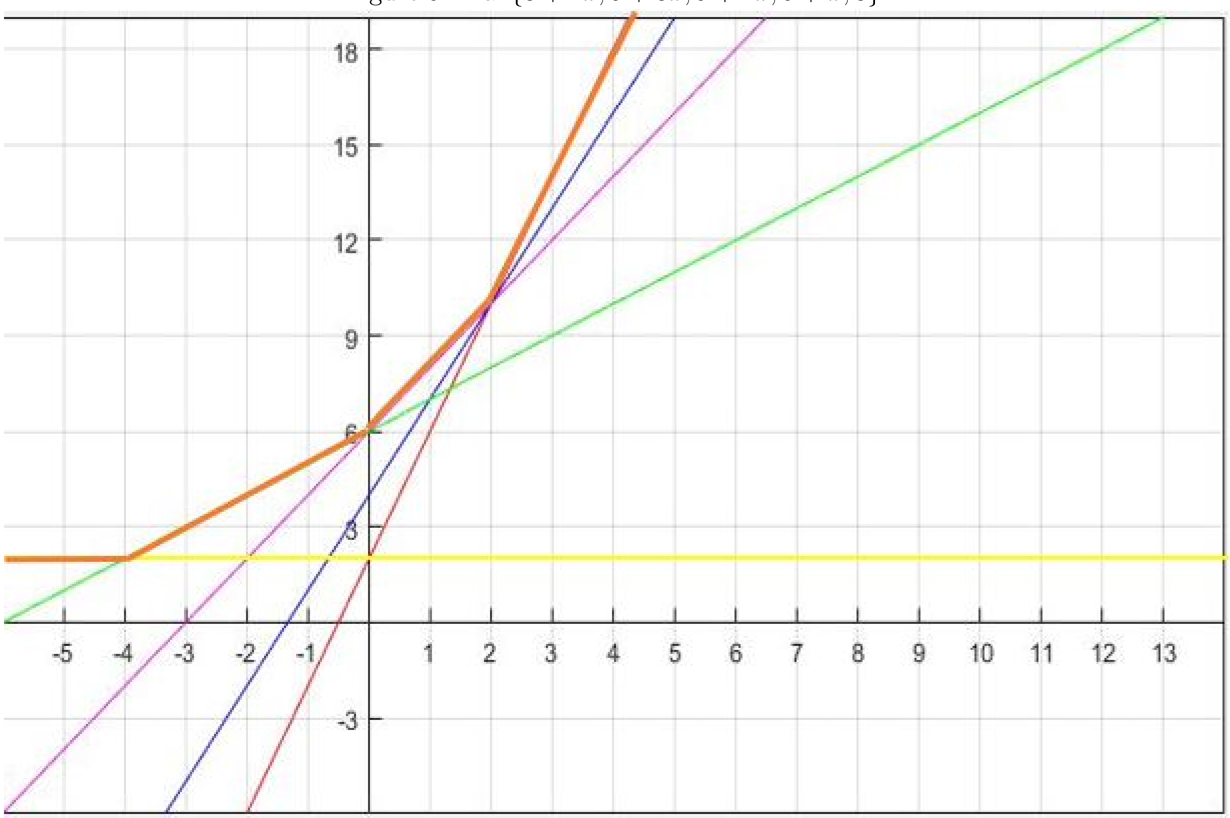
**GRAPH IV**

Figure 9: $\max\{3+4x, 5+3x, 9+2x, 5+x, 3\}$



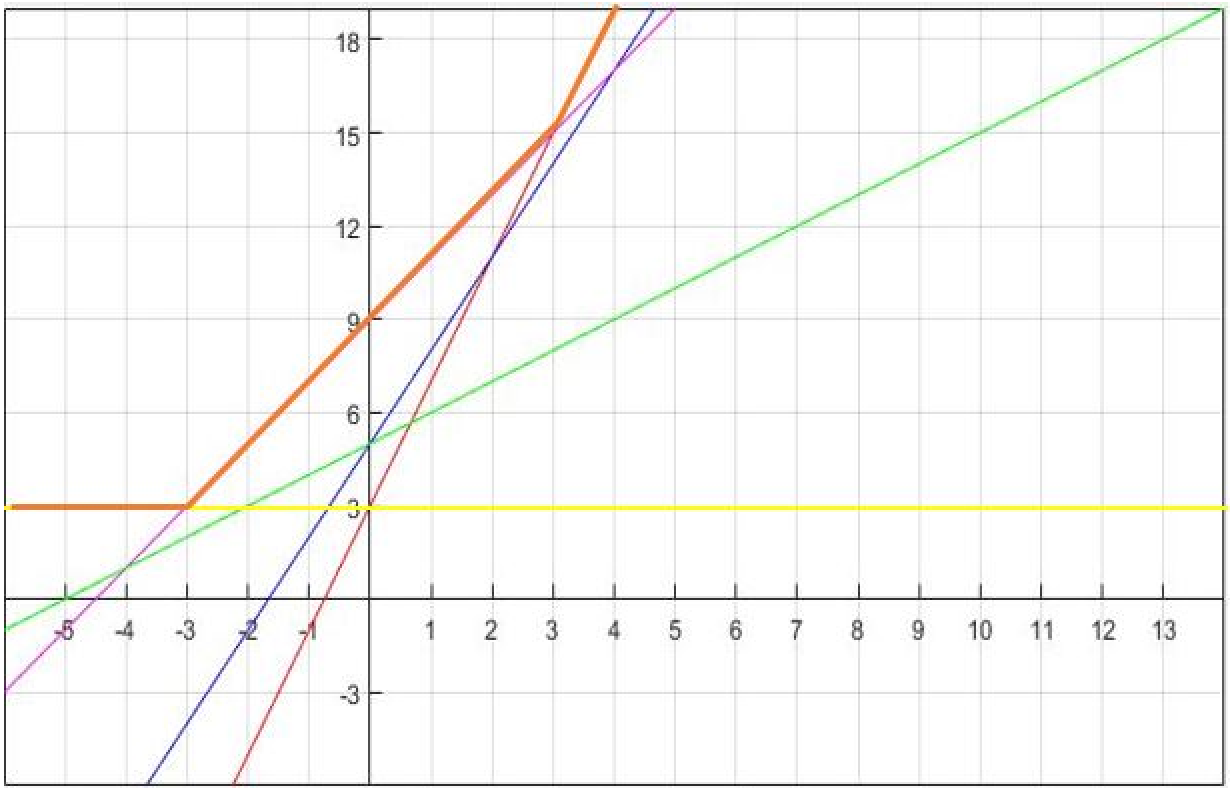

We have from the above curve that the roots of the curve are: $r_1 = -4, r_2 = 0$ and $r_3 = 2$.

$$\begin{aligned} M(r_1 = -4) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 1 \\ &= |0 - 1| = |-1| = 1 \\ M(r_2 = 0) &= |m_2 - m_3|, \text{ where } m_3 = 2 \text{ and } m_4 = 4 \\ &= |1 - 2| = |-1| = 1 \\ M(r_3 = 2) &= |m_3 - m_4| \\ &= |2 - 4| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical algebra $\max\{3+4x, 5+3x, 9+2x, 5+x, 3\}$ is (1,1,2).

**GRAPH V**

Figure 10: $\max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$




We have from the above curve that the roots of the curve are: $r_1 = -3$ and $r_2 = 3$.

$$\begin{aligned} M(r_1 = -3) &= |m_1 - m_2|, \text{ when } m_1 = 0 \text{ and } m_2 = 2 \\ &= |0-2| = |-2| = 2 \\ M(r_2 = 3) &= |m_2 - m_3|, \text{ where } m_2 = 2 \text{ and } m_3 = 4 \\ &= |2-4| = |-2| = 2 \end{aligned}$$

Hence,the multiplicity of the tropical algebra $\max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$ is (2,2).

**Lemma 3.1.** *Let C be a classical tropical curve of degree d then the sum of all points of tropical multiplicity of C is equal to d.*

*proof.* Let S be the sum of multiplicity of all point in C, consider $x^d + x^{d-1} + \cdots + x^{d-d}$ to be classical polynomial of degree d in C with multiplicity of $[a_1, a_2 \cdots a_n]$ then

$$S = a_1 + a_2 + \cdots + a_n = d \quad \forall \quad S, d \in C$$

# 4 Discussion and Conclusion

In this paper, tropical polynomials were formed on partial contraction transformation semigroup of $P_1, P_2 and P_3$(thesame method can be apply for $P_4, P_5$ etc), and tropical curves were also plotted using MATLAB 2019 V9.6.0. The root and multiplicity are obtained, lemma 3.1 shows that the sum of the multiplicity is equal to the number of the highest degree of the classical polynomial and below is a table showing the summary of the multiplicity.

Table 1: Order of Tropical Properties

| Classical | Tropical | Multiplicity |
|---|---|---|
| $x^2+2x+2$ | $max\{2x, 2+x, 2\}$ | [1, 1] |
| $x^2+2x+4x+1$ | $max\{3x, 2+2x, 4+x, 1\}$ | [1,2] |
| $x^4+2x^3+4x^2+3x+2$ | $max\{4x, 2+3x, 4+2x, 3+x, 2\}$ | [2,2] |
| $2x^4+6x^3+8x^2+3x+2$ | $max\{2+4x, 6+3x, 8+2x, 3+x, 2\}$ | [2,1,1] |
| $x^4+4x^3+5x^2+2x+2$ | $max\{4x, 4+3x, 5+2x, 2+x, 2\}$ | [2,1,1] |
| $2x^4+4x^3+6x^2+6x+2$ | $max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$ | [2,2] |
| $3x^4+5x^3+9x^2+5x+3$ | $max\{3+4x, 5+3x, 9+2x, 5+x, 3\}$ | [1,1,2] |
| $x^5+2x^4+4x^3+3x^2+2x+1$ | $max\{5x, 2+4x, 4+3x, 3+2x, 2+x, 1\}$ | [3,2] |
| $x^6+2x^5+4x^4+3x^3+2x^2+x+2$ | $max\{6x, 2+5x, 4+4x, 3+3x, 2+2x, x, 2\}$ | [4,2] |
| $2x^4+4x^3+6x^2+6x+2$ | $\max\{2+4x, 4+3x, 6+2x, 6+x, 2\}$ | [2,2] |

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## References

[1] Adeshola, A.D., and Umar, A. [2013], Combinatorial results for certain semigroups of order-preserving full contraction mapping of a finite chain; *Journal of Combinatorial Mathematics and Combinatorial Computing*, Vol. 1 pp. 1-11.

[2] Kehinde, R. [2012], Some algebraic and combinatorial properties of semigroup of injective partial contraction mapping and isometrics of a finite chain; *Ph.D. thesis, University of Ilorin, Nigeria*

[3] Akinwunmi, S.A., Mogbonju, M.M., Adeniji, A.O., Oyewola, D. O., Yakubu, G., Ibrahim, G. R., and Fati, M.O. [2021], Nildempotency structure of partial one-one contraction $CI_n$ transformation semigroups; *International journal of research and scientificinnovation (IJRSI)*, Vol. VIII, issue 1, pp. 230-233.

[4] Bakare, G.N., Ibrahim G.R., Usamot I.F., and Oyebo Y.T,. [2022], Result on order decreasing full transformation semigroup via tropical geometry; *Annals of Mathematics and Computer Science*, Vol 5 pp. 44-53.

[5] Sturmfels, B. [2007], A combinatorial introduction to tropical geometry. *Technical University of Berlin, Lecture notes series.*

[6] Brugalle, E, and Shaw, K. [2014], A Bit of Tropical Geometry; *The American Mathematical monthly, 121(7): PP. 563 – 589, retrieved from arXiv: 1311.2360v3 [Math. AG].*

[7] Ryan, H.D; [2015], An Introduction to Tropical geometry; $https : //sites.math.washington.edu/\ morrow/336_15/papers/ryan.pdf$.

[8] Protrka, I. [2017], An Invitation to Combinatorial Tropical Geometry; $https : //www.researchgate.net/publication/323529313$.

[9] Mikhalkin, G and Rau, J,[2018], Tropical Geometry; $https : //www.math.uni - tuebingen.de/user/jora/downloads/main.pdf$

[10] Balarabe, M., Iman, A.T., and Ade O., [2019], On semigroups of partial contractions of a finite chain; *Abacus (Mathematics Science Series)*, Vol. 44, No 1, pp 332-351.

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International Journal of Mathematical Sciences and Optimization: Theory and Applications
Vol. 9, No. 2, pp. 102 - 120
https://doi.org/10.5281/zenodo.10384737

[11] Umar, A. and Zubairu, M.M. [2021], On certain semigroups of contraction mappings of a finite chain; *Algebra and Discrete Mathematics* , Volume 32. Number 2, pp. 299-320.

[12] Zubairu, M. M., and Bashir, Ali., [2018], On certain combinatorial problems of the semigroup of partial and full contractions of a finite chain; *Bayero Journal of Pure and Applied Sciences*, 11(1): 377 - 380.

[13] A. Ibrahim, M. M. Mogbonju, A.O. Adeniji S. A. Akinwunmi,[2013], Angular loop model of 3 dimensional ALM transformation semigroup; *International Journal of Mathematical Science and Optimization: Theory and Application* Vol. 9, No. 2, pp. 13 -22.

[14] Ryan Hert Doenges [2015], An introduction to tropical Geometry ; $https : //sites.math.washington.edu/\ morrow/336_15/papers/ryan.pdf$.