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On the stability, boundedness and asymptotic behaviour of solutions of differential equations of third-order

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Abstract

In this paper, we construct a complete Lyapunov function which serves as a tool in providing sufficient conditions that ensure uniform asymptotic stability of the zero solution when $p(t,x,y,z) \equiv 0$; uniform ultimate boundedness and asymptotic behaviour of all solutions when $p(t,x,y,z) \neq 0$ of a certain third order non-linear ordinary differential equation. The results in this paper are new and in some ways generalize and improve on some existing results in literature. Finally, we provide an example to justify the correctness of the theorems.

Keywords: Third order; stability; boundedness; asymptotic behaviour; non-linear; Lyapunov function.
MSC2010: 34A34, 34C11, 34D20.

1 Introduction

The equation of interest in this paper, is the following third-order nonlinear differential equation

$$\dddot{x} + f(t,x,\dot{x},\ddot{x})\ddot{x} + \psi(t)g(x,\dot{x})\dot{x} + \phi(t)h(x,\dot{x},\ddot{x}) = p(t,x,\dot{x},\ddot{x}). \tag{1.1}$$

On setting $\dot{x} = y, \dot{y} = z$ in (1.1), we obtain the following system of first order differential equations

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f(t,x,y,z)z - \psi(t)g(x,y)y - \phi(t)h(x,y,z) + p(t,x,y,z), \tag{1.2}$$

where $h \in C(\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $g \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$; $f,p \in C\big([0,\infty)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R},\mathbb{R}\big)$. It is assumed that the partial derivatives $f_t(t,x,y,z)$, $f_x(t,x,y,z)$, $f_z(t,x,y,z)$, $g_x(x,y)$, $g_y(x,y)$, $h_x(x,y,z)$, $h_y(x,y,z)$, $h_z(x,y,z)$ and the derivatives $\dot{\psi}(t), \dot{\phi}(t)$ exist and are continuous with respect to their arguments. Similarly, we shall assume existence and uniqueness of solutions of (1.1) or (1.2).

In reality, differential equations have become tools for modelling many real life problems of concerns in biology, mathematical finance, engineering, medicine, just to mention but few [See, [1], [2], [3], [4]]. Hence, many notable researchers have devoted their effort to study the qualitative

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behaviour(boundedness, stability, convergence, periodicity, asymptotic behaviour) of solutions of differential equations without necessarily solving the differential equations analytically. In analyzing the qualitative properties of solutions of differential equations, Lyapunov function, which is a direct consequence of the second method (also called, the direct method) of Lyapunov, introduced by Lyapunov [5], has been considered to be an effective tool. Adams and Omeike [6], Ademola and Arawomo([7], [8]), Adeyanju and Adams([9], [10], [11]), Adeyanju [12], Adeyanju and Tunc [13], Ateş [14], Chukwu [15], Ezeilo [16], Nakashima [17], Olutimo [18], Olutimo et. al. ([3], [19]), Omeike([20], [21]), Qian [22], Swick [23], Tunc et.al. [24], Tunc([25]- [27]) and Yoshizawa [28] are some of the numerous authors who have employed the direct method of Lyapunov to study qualitative properties of various forms of third order differential equations. However, one major challenge of this method is the difficulty in constructing suitable Lyapunov functions, especially for nonlinear differential equations.

Specifically, Ezeilo [16] and Ogurtsov [29] investigated the stability property of the zero solution to the equation

$$\dddot{x} + \psi(x, \dot{x})\ddot{x} + \phi(\dot{x}) + g(x) = 0, \tag{1.3}$$

under certain conditions on $\psi(x, \dot{x}), \phi(\dot{x})$ and, $g(x)$. Earlier, the global stability of the zero solution of (1.3) when $\phi(\dot{x}) = b\dot{x}$ and $g(x) = cx$ (b and c are positive constants), was examined by [30]. Later, Omeike [20] and Tunc [26] proved some results on the boundedness and asymptotic behaviour of solutions to the equation

$$\dddot{x} + \psi(x, \dot{x})\ddot{x} + f(x, \dot{x}) = p(t, x, \dot{x}, \ddot{x}).$$

Furthermore, the duo of Ademola and Arawomo ([7], [8]) established some interesting results on the stability, boundedness, and asymptotic behaviour of solutions to the following differential equations using the direct method of Lyapunov

$$\dddot{x} + f(t, x, \dot{x}, \ddot{x})\ddot{x} + q(t)g(x, \dot{x}) + r(t)h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}),$$

and

$$\dddot{x} + \psi(t)f(x, \dot{x}, \ddot{x})\ddot{x} + \phi(t)g(x, \dot{x}) + \varphi(t)h(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}).$$

In 2018, Adeyanju and Adams [11] gave some theorems on the boundedness and asymptotic behaviour of solutions to the following nonlinear third-order differential equation

$$\dddot{x} + \psi(x, \dot{x}, \ddot{x})\ddot{x} + f(x, \dot{x}, \ddot{x}) = p(t, x, \dot{x}, \ddot{x}).$$

Being motivated by the works of Ademola and Arawomo([7], [8]) and other papers listed above, we now study the stability, boundedness and asymptotic behaviour of solutions to the equation (1.1) or its equivalent system (1.2) by using a suitable Lyapunov function and Yoshizawa [31] limit point argument.

Remark 1.1. *Equation (1.1) is more general than some other equations studied in the literature. In particular, (1.1) reduces to the third order differential equation considered in [7] when* $\psi(t)g(x, \dot{x})\dot{x}$ *appearing in (1.1) is replaced with* $q(t)g(x, \dot{x})$.

2 Main Results

Basic Assumptions

In addition to the assumptions imposed on functions f, g, h, p, ψ and ϕ appearing in (1.1) or system (1.2), suppose there exist the following positive constants a, b, σ_0, δ, $a_1, b_1, c_1, \sigma_1, \sigma_2$, and δ_1 such that for all $t \geq 0, x, y$ and z, we have:

(i) $a \leq f(t, x, y, z) \leq a_1, \ b \leq g(x, y) \leq b_1, \ h_x(x, 0, 0) \leq c_1;$

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(ii) $h(0,0,0) = 0, \delta \leq x^{-1}h(x,y,z) \leq \delta_1$ ($x \neq 0$), $\sigma_0 \leq \phi(t) \leq \psi(t)$, $\phi(t) \leq \sigma_1$, $\psi(t) \leq \sigma_2$, $\dot{\psi}(t) \leq \dot{\phi}(t) \leq 0$;

(iii) $f_t(t,x,y,z) \leq 0$, $yf_x(t,x,y,z) \leq 0$, $yg_x(x,y) \leq 0$;

(iv) $yf_z(t,x,y,z) \geq 0$, $h_y(x,y,0) \geq 0$, $h_z(x,0,z) \geq 0$;

(v) $ab - c_1 > 0$;

(vi) $\int_0^\infty |p(t,x,y,z)|dt < \infty$.

On the strength of the assumptions stated above, we have the followings as our main theorems.

Theorem 2.1. *Given that the conditions stated under the basic assumptions above are satisfied. Then all the solutions of* (1.2) *are uniformly ultimately bounded.*

Theorem 2.2. *If all the basic assumptions listed above hold, then every solution* $(x(t), y(t), z(t))$ *of system* (1.2) *is uniformly bounded and satisfies*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} y(t) = 0, \quad \lim_{t\to\infty} z(t) = 0. \tag{2.1}$$

Theorem 2.3. *If all the conditions of the basic assumptions are satisfied, any solution* $(x(t), y(t), z(t))$ *of system* (1.2) *passing through the initial condition*

$$x(0) = x_0, y(0) = y_0, z(0) = z_0, \tag{2.2}$$

must satisfy

$$|x(t)| \leq D, |y(t)| \leq D, |z(t)| \leq D, \tag{2.3}$$

for all $t \geq 0, x, y$ *and* z *where* D *is a positive constant.*

Theorem 2.4. *The zero solution of system* (1.2) *is uniformly asymptotically stable if the condition (i)-(v) listed in the basic assumptions are satisfied.*

Our Theorems 2.1 - 2.4 stated shall be proved using the following continuously differentiable scalar function $V(t) = V(t, x(t), y(t), z(t))$ given as

$$V(t) = e^{-p_1(t)}U(t,x,y,z), \tag{2.4}$$

where

$$p_1(t) = \int_0^t |p(s,x,y,z)|ds, \tag{2.5}$$

and $U(t,x,y,z) = U(t)$ is given by

$$\begin{aligned} 2U(t) &= \beta bc_1(ab-c_1)x^2 + 6a^2c_1\phi(t)\int_0^x h(\xi,0,0)d\xi \\ &+ 6a^2c_1\int_0^y \tau f(t,x,\tau,0)d\tau + 2a\psi(t)(2c_1+ab)\int_0^y \tau g(x,\tau)d\tau + \beta c_1^2y^2 \\ &+ (2ac_1+a^2b)z^2 + 2a(2c_1+ab)\phi(t)yh(x,0,0) + 2\beta ac_1(ab-c_1)xy \\ &+ 2\beta c_1(ab-c_1)xz + 6a^2c_1yz, \end{aligned} \tag{2.6}$$

where a, b, c_1 are as defined earlier and β is a constant satisfying

$$0 < \beta < \min\left\{ab(ab-c_1)^{-1}, a\sigma_0(ab-c_1)K_8^{-1}, 2a^2(ab-c_1)K_9^{-1}\right\}, \tag{2.7}$$

where

$$K_8 = \Big((\sigma_0\delta)^{-1}(ab-c_1)(\sigma_2 b_1 - b)^2 + a^2 b\Big),$$

and

$$K_9 = \Big(4c_1(\sigma_0\delta)^{-1}(ab-c_1)(f(t,x,y,z)-a)^2 + ab^2\Big).$$

The following lemma will be needed to establish the proofs of the theorems.

Lemma 2.1 *Suppose all the assumptions listed under the basic assumptions are satisfied. Then, there exist some positive constants* D_1, D_2 *and* D_3 *such that for all* $t \geq 0, x, y$ *and* z*, the function* $V(t)$ *defined in* (2.4) *and its derivative* $\dot{V}(t)$ *satisfy*

$$D_1\{x^2(t)+y^2(t)+z^2(t)\} \leq V(t) \leq D_2\{x^2(t)+y^2(t)+z^2(t)\}, \tag{2.8}$$

$$V(t) \to +\infty \text{ as } x^2(t)+y^2(t)+z^2(t) \to +\infty \tag{2.9}$$

and the derivative of $V(t)$ *along the solution path of* (1.2) *satisfies*

$$\dot{V}_{(1.2)}(t) \leq -D_3\{x^2(t)+y^2(t)+z^2(t)\}. \tag{2.10}$$

Proof.
Clearly, the function $V(t)$ defined in (2.4) becomes zero when $x=0, y=0$ and $z=0$. Next, we show that $V(t)$ is positive definite whenever x, y and z are not all zero. To do this, we have to show that $U(t)$ is positive definite, then, $V(t)$ is automatically positive definite. The function $U(t)$ defined in equation (2.6) can be written as

$$\begin{aligned}2U(t) =& 6a^2\phi(t)\int_0^x \big[c_1 - h_\xi(\xi,0,0)\big]h(\xi,0,0)d\xi + 2\phi(t)\big(c_1 y + ah(x,0,0)\big)^2\\ &+\beta c_1(ab-c_1)a^{-1}\Big(ab-\beta(ab-c_1)\Big)x^2 + 4c_1\phi(t)\int_0^y \Big(a\frac{\psi(t)}{\phi(t)}g(x,\tau)-c_1\Big)\tau d\tau\\ &+6a^2c_1\int_0^y \big(f(t,x,\tau,0)-a\big)\tau d\tau + 2ac_1(ay+z)^2 + a^2\phi(t)\big(by+h(x,0,0)\big)^2\\ &+ac_1\big(\beta(ab-c_1)a^{-1}x + ay + z\big)^2 + 2a^2b\phi(t)\int_0^y \Big(\frac{\psi(t)}{\phi(t)}g(x,\tau)-b\Big)\tau d\tau\\ &+a(ab-c_1)z^2 + \beta c_1^2 y^2.\end{aligned}$$

On applying some of the assumptions listed in (i) and (ii) of basic assumptions to $U(t)$, we have

$$\begin{aligned}2U(t) \geq& \beta c_1(ab-c_1)a^{-1}\Big(ab-\beta(ab-c_1)\Big)x^2 + 4c_1\phi(t)\int_0^y \Big(a\frac{\psi(t)}{\phi(t)}g(x,\tau)-c_1\Big)\tau d\tau + a(ab-c_1)z^2\\ \geq& \beta c_1(ab-c_1)a^{-1}(ab-\beta(ab-c_1))x^2 + 2c_1\sigma_0(ab-c_1)y^2 + a(ab-c_1)z^2.\end{aligned}$$

Thus, we can choose a positive constant

$$K_1 = \frac{1}{2}\min\left\{\beta c_1(ab-c_1)a^{-1}(ab-\beta(ab-c_1)), 2c_1\sigma_0(ab-c_1), a(ab-c_1)\right\},$$

such that

$$U(t) \geq K_1\{x^2(t)+y^2(t)+z^2(t)\}.$$

Also, it is clear from assumption (vi) and equation (2.5) that we can always find a positive constant α_0 such that for all $t > 0$, we have

$$0 \leq p_1(t) < \alpha_0. \tag{2.11}$$

Therefore, for some positive constant $K_2 = K_1 e^{-\alpha_0}$, we have

$$V(t) \geq K_2\{x^2(t)+y^2(t)+z^2(t)\}, \tag{2.12}$$

for all $t \geq 0, x, y$ and z. Inequality (2.12) shows that, $V(t) = 0$ if and only if $x^2(t)+y^2(t)+z^2(t) = 0$ and $V(t) > 0$ if and only if $x^2(t) + y^2(t) + z^2(t) \neq 0$. Therefore,

$$V(t) \to +\infty \text{ as } x^2(t) + y^2(t) + z^2(t) \to \infty. \tag{2.13}$$

Similarly, we obtain the upper bounds of functions $U(t)$ and $V(t)$. By assumptions (i) and (ii) under the basic assumptions and the fact that $2|xy| \leq x^2 + y^2$, we obtain

$$\begin{aligned} U(t) \leq & \frac{1}{2}\left[\beta bc_1(ab-c_1)x^2 + 3a^2c_1\sigma_1\delta_1x^2 + 3a^2a_1c_1y^2 + \sigma_2(ab_1c_1 + a^2bb_1)y^2 + \beta c_1^2y^2\right] \\ & + \frac{1}{2}\left[(2ac_1 + a^2b)z^2 + \delta_1\sigma_1(2ac_1 + a^2b)(x^2+y^2) + \beta ac_1(ab-c_1)(x^2+y^2)\right] \\ & + \frac{1}{2}\left[\beta c_1(ab-c_1)(x^2+z^2) + 3a^2c_1(y^2+z^2)\right] \\ \leq & \frac{1}{2}(\alpha_1x^2 + \alpha_2y^2 + \alpha_3z^2) \\ \leq & K_3(x^2(t) + y^2(t) + z^2(t)), \end{aligned}$$

where,

$$\begin{aligned} \alpha_1 &= c_1(ab-c_1)(a+b+1)\beta + a\sigma_1\delta_1[ab + c_1(2+3a)] > 0, \\ \alpha_2 &= c_1^2\beta + 3a^2c_1(a_1+1) + ab_1\sigma_2(c_1+ab) + ac_1(ab-c_1)\beta + a(2c_1+ab)\sigma_1\delta_1 > 0, \\ \alpha_3 &= 2ac_1 + a^2(b+3c_1) + c_1(ab-c_1)\beta > 0, \end{aligned}$$

and

$$K_3 = \frac{1}{2}\max\{\alpha_1, \alpha_2, \alpha_3\}.$$

Thus, from (2.11) we have

$$V(t) \leq K_4\{x^2(t) + y^2(t) + z^2(t)\}, \tag{2.14}$$

where $K_4 = K_3e^0 = K_3$. On combining (2.12) and (2.14) together, we obtain inequality (2.8) of Lemma 2.1.

Next, we proceed to obtain the time derivative of the function $V(t)$ along system (1.2).

$$\dot{V}_{(1.2)}(t) = -\dot{p}_1(t)e^{-p_1(t)}U(t) + e^{-p_1(t)}\dot{U}(t), \tag{2.15}$$

where,

$$-\dot{p}_1(t)e^{-p_1(t)}U(t) = -e^{-p_1(t)}U(t)|P(t,x,y,z)|, \tag{2.16}$$

and

$$\begin{aligned} \dot{U}_{(1.2)}(t) =& \beta bc_1(ab-c_1)xy + 3a^2c_1\dot{\phi}(t)\int_0^x h(\xi,0,0)d\xi + 3a^2c_1\phi(t)yh(x,0,0) \\ & + 3a^2c_1\Big[\int_0^y \tau f_t(t,x,\tau,0)d\tau + y\int_0^y \tau f_x(t,x,\tau,0)d\tau + f(t,x,y,0)yz\Big] \\ & + a\dot{\psi}(t)(2c_1+ab)\int_0^y \tau g(x,\tau)d\tau + a\psi(t)(2c_1+ab)y\int_0^y \tau g_x(x,\tau)d\tau \\ & + (2ac_1+a^2b)[-f(t,x,y,z)z^2 - \phi(t)h(x,y,z)z + zp(t,x,y,z)] \\ & + ac_1(ab-c_1)\beta xz + (2ac_1+a^2b)\dot{\phi}(t)yh(x,0,0) \\ & + (2ac_1+a^2b)\phi(t)zh(x,0,0) + (2ac_1+a^2b)\phi(t)y^2h_x(x,0,0) \\ & + abc_1\beta yz + ac_1(ab-c_1)\beta y^2 + 3a^2c_1z^2 \\ & + c_1(ab-c_1)\beta[-f(t,x,y,z)xz - \psi(t)g(x,y)xy - \phi(t)xh(x,y,z) \\ & + p(t,x,y,z)x] + 3a^2c_1[-f(t,x,y,z)yz - \psi(t)g(x,y)y^2 \\ & - \phi(t)h(x,y,z)y + yp(t,x,y,z)]. \end{aligned}$$

For ease of computations, we write $\dot{U}_{(1.2)}(t)$ as:

$$\dot{U}_{(1.2)}(t) = -U_1 - U_2 + U_3 + U_4 + U_5$$

where,

$$\begin{aligned}U_1 =& c_1(ab-c_1)\beta\phi(t)x^{-1}h(x,y,z)x^2 + ac_1\phi(t)\Big[a\frac{\psi(t)}{\phi(t)}g(x,y) - h_x(x,0,0)\Big]y^2\\ &+ a\phi(t)\Big[2ac_1\frac{\psi(t)}{\phi(t)}g(x,y) - (c_1+ab)h_x(x,0,0)\Big]y^2 + 2ac_1(f(t,x,y,z)-a)z^2 + a^2(bf(t,x,y,z)-c_1)z^2\\ &+ c_1(ab-c_1)\beta(\psi(t)g(x,y)-b)xy + c_1(ab-c_1)\beta(f(t,x,y,z)-a)xz - abc_1\beta yz - ac_1(ab-c_1)\beta y^2;\end{aligned}$$

$$U_2 = \phi(t)\big[3a^2c_1y + (2ac_1+a^2b)z\big]\big[h(x,y,z)-h(x,0,0)\big] + 3a^2c_1yz\big[f(t,x,y,z)-f(t,x,y,0)\big];$$

$$U_3 = 3a^2c_1\Big[\int_0^y \tau f_t(t,x,\tau,0)d\tau + y\int_0^y \tau f_x(t,x,\tau,0)d\tau\Big] + a\psi(t)(2c_1+ab)y\int_0^y \tau g_x(x,\tau)d\tau;$$

$$U_4 = 3a^2c_1\dot{\phi}(t)\int_0^x h(\xi,0,0)d\xi + a\dot{\psi}(t)(2c_1+ab)\int_0^y \tau g(x,\tau)d\tau + (2ac_1+a^2b)\dot{\phi}(t)yh(x,0,0);$$

and

$$U_5 = [c_1(ab-c_1)\beta x + 3a^2c_1y + (2ac_1+a^2b)z]p(t,x,y,z).$$

To establish inequality (2.10), we estimate the upper bound for each of $U_i, (i=1,2,3,4,5)$.
By applying assumptions stated in (i) and (ii) of basic assumptions, together with the obvious inequality $2|xy| \le (x^2+y^2)$ in U_1, we have

$$\begin{aligned}U_1 \ge& \frac{1}{2}c_1(ab-c_1)\beta\phi(t)x^{-1}h(x,y,z)x^2 + ac_1\phi(t)\Big[a\frac{\psi(t)}{\phi(t)}g(x,y) - h_x(x,0,0)\Big]y^2\\ &+ a\phi(t)\Big[2ac_1\frac{\psi(t)}{\phi(t)}g(x,y) - (c_1+ab)h_x(x,0,0)\Big]y^2 + 2ac_1(f(t,x,y,z)-a)z^2\\ &+ \frac{1}{4}c_1(ab-c_1)\beta\phi(t)x^{-1}h(x,y,z)\Big[x + 2\big(\phi(t)x^{-1}h(x,y,z)\big)^{-1}\big(\psi(t)g(x,y)-b\big)y\Big]^2\\ &- c_1(ab-c_1)\beta\big(\phi(t)x^{-1}h(x,y,z)\big)^{-1}\big(\psi(t)g(x,y)-b\big)^2y^2\\ &+ \frac{1}{4}c_1(ab-c_1)\beta\phi(t)x^{-1}h(x,y,z)\Big[x + 2\big(\phi(t)x^{-1}h(x,y,z)\big)^{-1}\big(f(t,x,y,z)-a\big)z\Big]^2\\ &- c_1(ab-c_1)\beta\big(\phi(t)x^{-1}h(x,y,z)\big)^{-1}\big(f(t,x,y,z)-a\big)^2z^2\\ &+ \beta a(c_1y - 2^{-1}bz)^2 - a^2bc_1\beta y^2 - \frac{1}{4}ab^2\beta z^2 + a^2(bf(t,x,y,z)-c_1)z^2\\ \ge& \frac{1}{2}c_1(ab-c_1)\beta\sigma_0\delta x^2 + ac_1\sigma_0(ab-c_1)y^2 + a^2(ab-c_1)z^2\\ &+ c_1\Big[a\sigma_0(ab-c_1) - \beta\Big(\big(\sigma_0\delta\big)^{-1}(ab-c_1)\big(\sigma_2b_1-b\big)^2 + a^2b\Big)\Big]y^2\\ &+ \frac{1}{4}\Big[2a^2(ab-c_1) - \beta\Big(4c_1\big(\sigma_0\delta\big)^{-1}(ab-c_1)\big(f(t,x,y,z)-a\big)^2 + ab^2\Big)\Big]z^2\\ \ge& \frac{1}{2}c_1(ab-c_1)\beta\sigma_0\delta x^2 + ac_1\sigma_0(ab-c_1)y^2 + a^2(ab-c_1)z^2\\ \ge& K_5\{x^2(t) + y^2(t) + z^2(t)\} \ge 0,\end{aligned}$$

where $K_5 = \frac{1}{2}\min\{c_1(ab-c_1)\beta\sigma_0\delta, 2ac_1\sigma_0(ab-c_1), 2a^2(ab-c_1)\}$.

Using mean value theorem with $0 \le \theta_i \le 1$ $(i = 1, 2, 3)$ and conditions given in (iv) of basic assumptions, we have the following estimate for U_2 for all $t \ge 0, x, y$ and z.

$$\begin{aligned} U_2 =& \phi(t)\big[3a^2c_1y + (2ac_1 + a^2b)z\big]\big[h(x,y,z) - h(x,0,0)\big] + 3a^2c_1yz\big[f(t,x,y,z) - f(t,x,y,0)\big] \\ \ge& \phi(t)\big[3a^2c_1h_y(x,\theta_1y,0)y^2 + (2ac_1 + a^2b)h_z(x,0,\theta_2z)z^2\big] + 3a^2c_1yf_z(t,x,y,\theta_3z)z^2 \\ \ge& 0. \end{aligned}$$

Also, by assumptions listed in (iii) of basic assumptions, we have

$$\begin{aligned} U_3 =& 3a^2c_1\big[\int_0^y \tau f_t(t,x,\tau,0)d\tau + y\int_0^y \tau f_x(t,x,\tau,0)d\tau\big] \\ &+ a\psi(t)(2c_1 + ab)y\int_0^y \tau g_x(x,\tau)d\tau \\ \le& 0. \end{aligned}$$

Similarly, by applying some of the assumptions in (i) and (ii) of basic assumptions to U_4, we have

$$\begin{aligned} U_4 =& \dot\psi(t)\Big[3a^2c_1\frac{\dot\phi(t)}{\dot\psi(t)}\int_0^x h(\xi,0,0)d\xi + a(2c_1 + ab)\int_0^y \tau g(x,\tau)d\tau + a(2c_1+ab)\frac{\dot\phi(t)}{\dot\psi(t)}yh(x,0,0)\Big] \\ \le& \frac{\dot\psi(t)}{2b}\Big[\frac{\dot\phi(t)}{\dot\psi(t)}\int_0^x \Big(6a^2bc_1 - 2a(2c_1+ab)\frac{\dot\phi(t)}{\dot\psi(t)}h_\xi(\xi,0,0)\Big)h(\xi,0,0)d\xi \\ &+ a(2c_1+ab)\Big(by + \frac{\dot\phi(t)}{\dot\psi(t)}x^{-1}h(x,0,0)x\Big)^2\Big] \\ \le& \frac{\dot\psi(t)}{2b}\Big[2ac_1(ab - c_1)\delta\frac{\dot\phi(t)}{\dot\psi(t)}x^2 + a(2c_1+ab)(\delta x + by)^2\Big] \\ \le& 0. \end{aligned}$$

Lastly, we have

$$\begin{aligned} U_5 &= [c_1(ab - c_1)\beta x + 3a^2c_1y + (2ac_1 + a^2b)z]p(t,x,y,z) \\ &\le K_6\{|x| + |y| + |z|\}|p(t,x,y,z)|, \end{aligned}$$

where $K_6 = \max\{c_1(ab - c_1)\beta, 3a^2c_1, (2ac_1 + a^2b)\}$.

On gathering the estimates for U_i, $(i = 1, 2, 3, 4, 5)$ into $\dot U(t)$ we obtain,

$$\dot U_{(1.2)}(t) \le -K_5\{x^2(t) + y^2(t) + z^2(t)\} + K_6\{|x| + |y| + |z|\}|p(t,x,y,z)|.$$

Similarly,

$$\begin{aligned} \dot V_{(1.2)}(t) \le& -e^{-p_1(t)}\Big(K_1\{x^2(t)+y^2(t)+z^2(t)\} - K_6\{|x|+|y|+|z|\}\Big)|p(t,x,y,z)| \\ &- K_5e^{-p_1(t)}\{x^2(t)+y^2(t)+z^2(t)\} \\ \le& -e^{-p_1(t)}\Big(K_1\{x^2(t)+y^2(t)+z^2(t)\} - K_63^{\frac12}\{X\}^{\frac12}\Big)|p(t,x,y,z)| \\ &- K_5e^{-p_1(t)}\{x^2(t)+y^2(t)+z^2(t)\}, \end{aligned} \tag{2.17}$$

for all $t \ge 0, x, y$ and z. If we choose $\{x^2(t) + y^2(t) + z^2(t)\}^{\frac12} \ge 3^{\frac12}K_1^{-1}K_6$ and taking into account assumption (vi) of the basic assumptions, we obtain for all $t \ge 0, x, y$ and z

$$\dot V_{(1.2)}(t) \le -K_7\{x^2(t) + y^2(t) + z^2(t)\}, \tag{2.18}$$

where $K_7 = K_5e^{-p_1(\infty)} > 0$. The proof of the lemma is now completes. □

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Proof of Theorem 2.1
Since the inequalities (2.12), (2.13), (2.14) and (2.18) satisfy all the conditions of [Theorem 35.1, [32]], then any solution $(x(t), y(t), z(t))$ of system (1.2) is uniform-ultimately bounded.

Proof of Theorem 2.2
By Lemma 2.1, the Lyapunov function defined in (2.4). has been shown to satisfy all the conditions of [Theorem 31.1, [32]]. Therefore, solution $(x(t), y(t), z(t))$ of system (1.2) is uniformly bounded. The conclusion of the proof of this theorem follows by employing the same reasoning as in Qian [[33], Theorem 2] using (2.8) and (2.18). Thus, any solution $(x(t), y(t), z(t))$ of system (1.2) is uniformly bounded.

Proof of Theorem 2.3
Suppose $(x(t), y(t), z(t))$ is any solution to the system (1.2). Then, under the assumptions of Theorem 2.3, estimates (2.12) and (2.17) of the proof of Lemma 2.1 still hold. If we make use of the fact that $|x| \leq 1 + x^2$, $|y| \leq 1 + y^2$, $|z| \leq 1 + z^2$ in (2.17), we have

$$\begin{aligned} \dot{V}_{(1.2)}(t) \leq & \; - \; e^{-p_1(t)}\Big(K_1\{x^2(t) + y^2(t) + z^2(t)\} - K_6\{|x| + |y| + |z|\}\Big)|p(t, x, y, z)| \\ \leq & \; e^{-p_1(t)}K_6\Big(3 + x^2(t) + y^2(t) + z^2(t)\Big)|p(t, x, y, z)|. \end{aligned}$$

By applying inequalities (2.11) and (2.12) in the above, we obtain

$$\dot{V}_{(1.2)}(t) - K_6K_2^{-1}|p(t, x, y, z)|V(t) \leq 3K_6|p(t, x, y, z)|,$$

for all $t \geq 0, x, y$ and z. On solving this differential equation, we have,

$$V(t) \leq \quad K_7,$$

where $K_7 = [V(t_0) + 3K_6p_0]\exp(K_6K_2^{-1}p_0) > 0$. From inequality (2.12), we get

$$\{x^2(t) + y^2(t) + z^2(t)\} \leq K_2^{-1}V(t) \leq K_2^{-1}K_7 = D^*. \tag{2.19}$$

Thus, inequality (2.3) of Theorem 2.3 follows from (2.19) by setting $D = \sqrt{D^*}$. This completes the proof of the theorem.

Proof of Theorem 2.4
To prove this theorem, we employ the usual limit point argument as indicated in [31] to show that if Lemma 2.1 holds, then the function $U(t) = U(t, x, y, z) \to 0$ as $t \to \infty$. On setting $p(t, x, y, z) = 0$ in (2.4), we have $V(t) = U(t)$. It then follows from estimates (2.12) of Lemma 2.1, that $V(t) = 0$ if and only if $x^2(t) + y^2(t) + z^2(t) = 0$, $V(t) > 0$ if and only if $x^2(t) + y^2(t) + z^2(t) > 0$ and $V(t) \to \infty$ if and only if $x^2(t) + y^2(t) + z^2(t) \to \infty$.

Suppose that $W = W(x, y, z)$ is any given solution of the system (1.2), and consider the function $V(t)$ corresponding to this solution. Then, we have from Lemma 2.1 that

$$\dot{V}(t) \leq -D_3\{x^2(t) + y^2(t) + z^2(t)\} \leq V(0), \forall t \geq 0.$$

In addition, $V(t)$ is non-negative, non-increasing and thereby tends to a non-negative limit, $V(\infty)$ as $t \to \infty$. Next, we show that $V(\infty) = 0$. But on the contrary, let's assume $V(\infty) > 0$ and consider the following set

$$Q = \{(x, y, z) : V(t, x, y, z) \leq V(0)\}.$$

It thus follows from the properties of the function $V(t)$ that the set Q is bounded and so also the set $W \subset Q$ is bounded. Likewise, the non-empty set containing all the limits points of W consists of the

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whole trajectories of the system (1.2) which is lying on the surface $V(t,x,y,z) = V(\infty)$. Suppose R is any given limit point of W, then there exists a half trajectory, W_R , issuing from R and lying on the surface $V(t,x,y,z) = V(\infty)$. Since for every point (x,y,z) on W_R , $V(t,x,y,z) \geq V(\infty)$, we can infer at once that $\dot{V}(t) = 0$ on W_R . It then follows from inequality (2.10) of Lemma 2.1 that, $x = 0, y = 0$ and $z = 0$. Therefore, the point $(0,0,0)$ must be lying on the surface $V(t,x,y,z) = V(\infty)$ which is a contradiction to our assumption that $V(t,x,y,z) \geq V(\infty)$. Hence $V(\infty) = 0$. Also, it is clear that all the conditions of Lemma 2.1 are met. Therefore, the zero solution of system (1.2) is uniformly asymptotically stable.

3 Example

In this section, we provide an example which is a special case of equation (1.1) or system (1.2).

Example 3.1. *In the system* (1.2)*, let*

$$f(t,x,y,z) = 7 + \frac{1}{2+t^4+|xy|\sin^2 y + e^{-|yz|}},\; g(x,y) = 3 - \frac{1}{1+e^{-|xy|}}, h(x,y,z) = 3x + \frac{x}{1+e^{(\frac{1}{1+|x|y^2z^2})}},$$
$$p(t,x,y,z) = \frac{1}{2+t^2+e^{(x+y+z)^2}}, \psi(t) = 1 + \frac{1}{2+t^4} \text{ and } \phi(t) = 1 + \frac{1}{4+t^4}.$$

We now proceed to show that functions f, g, h, p, ϕ and ψ satisfy all the assumptions placed on them under the basic assumptions.

Starting with function f, it is clear that

$$7 \leq f(t,x,y,z) = 7 + \frac{1}{2+t^4+|xy|\sin^2 y + e^{-|yz|}} \leq 7\frac{1}{2},\; \text{for all } t \geq 0, x, y, z,$$

which gives $a = 7$ and $a_1 = 7\frac{1}{2}$.
On differentiating f partially with respect to t, x, z one after the other, we have

$$f_t(t,x,y,z) = -\frac{4t^3}{(2+t^4+|xy|\sin^2 y + e^{-|yz|})^2} \leq 0 \text{ for all } t \geq 0, x, y, z.$$

$$f_x(t,x,y,z) = -\frac{|y|\sin^2 y}{(2+t^4+|xy|\sin^2 y + e^{-|yz|})^2},$$

and

$$yf_x(t,x,y,z) = -\frac{y^2\sin^2 y}{(2+t^4+|xy|\sin^2 y + e^{-|yz|})^2} \leq 0.$$

Similarly,

$$f_z(t,x,y,z) = \frac{e^{-|yz|}|y|}{(2+t^4+|xy|\sin^2 y + e^{-|yz|})^2},$$

and

$$yf_z(t,x,y,z) = \frac{e^{-|yz|}y^2}{(2+t^4+|xy|\sin^2 y + e^{-|yz|})^2} \geq 0,\; \text{for all } t \geq 0, x, y, z.$$

Next, we have from function g that

$$2 \leq g(x,y) = 3 - \frac{1}{1+e^{-|xy|}} \leq 3,\; \text{for all } x, y.$$

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Thus, $b = 2$ *and* $b_1 = 3$.

Differentiating g *partially with respect to* x *and multiply the result by* y *gives*

$$yg_x(x,y) = -\frac{e^{-|xy|}y^2}{1+e^{-|xy|}} \leq 0, \text{ for all } x,y.$$

Similarly, we have

$$h(x,y,z) = 3x + \frac{x}{1+e^{(\frac{1}{1+|x|y^2z^2})}}.$$

Obviously, $h(0,0,0) = 0$ *and*

$$3 \leq x^{-1}h(x,y,z) = 3 + \frac{1}{1+e^{(\frac{1}{1+|x|y^2z^2})}} \leq 4, \text{ for all } t \geq 0, x, y, z.$$

This gives $\delta = 3$ *and* $\delta_1 = 4$. *Differentiating function* h *partially with respect to* x, *we obtain*

$$h_x(x,y,z) = 3 + \frac{1}{1+e^{(\frac{1}{1+|x|y^2z^2})}} + \frac{xe^{(\frac{1}{1+|x|y^2z^2})}y^2z^2}{(1+e^{(\frac{1}{1+|x|y^2z^2})})^2(1+|x|y^2z^2)^2}, \text{ and } h_x(x,0,0) \leq 3 + \frac{1}{1+e} = c_1.$$

Thus, $ab - c_1 = 7 \times 2 - (3 + \frac{1}{1+e}) = 11 - \frac{1}{1+e} > 0.$
The derivative of h *with respect to* y *and* z *respectively are:*

$$h_y(x,y,z) = \frac{2x^2e^{(\frac{1}{1+|x|y^2z^2})}|y|z^2}{(1+e^{(\frac{1}{1+|x|y^2z^2})})^2(1+|x|y^2z^2)^2} \geq 0,$$

and

$$h_z(x,y,z) = \frac{2x^2e^{(\frac{1}{1+|x|y^2|z|})}y^2|z|}{(1+e^{(\frac{1}{1+|x|y^2z^2})})^2(1+|x|y^2z^2)^2} \geq 0.$$

Also, from functions $\phi(t)$ *and* $\psi(t)$, *we have*

$$\phi(t) = 1 + \frac{1}{4+t^4} \leq \psi(t) = 1 + \frac{1}{2+t^4}, \text{ for all } t \geq 0.$$

Clearly,

$$1 \leq \phi(t) \leq \psi(t), \text{ for all } t \geq 0 \text{ where } \sigma_0 = 1.$$

By differentiating both $\phi(t)$ *and* $\psi(t)$ *we obtain*

$$\dot{\phi}(t) = -\frac{4t^3}{(4+t^4)^2} \text{ and } \dot{\psi}(t) = -\frac{4t^3}{(2+t^4)^2}.$$

It is easy to check that

$$\dot{\psi}(t) \leq \dot{\phi}(t) \leq 0, \text{ for all } t \geq 0.$$

Finally, the integral of function $p(t,x,y,z)$ *with respect to* t *satisfies*

$$\begin{aligned}\int_0^\infty |p(t,x,y,z)|dt &= \int_0^\infty |\frac{1}{2+t^2+e^{(x+y+z)^2}}|dt \\ &= \frac{\pi}{2\sqrt{2+e^{(x+y+z)^2}}} \\ &\leq \frac{\pi}{2\sqrt{3}} < \infty,\end{aligned}$$

for all values of $t \geq 0, x, y, z$. *Therefore, all the conditions of the theorems are met by this example.*

4 Conclusion

With the aid of a new complete Lyapunov function, we have proved some theorems on the stability, boundedness and asymptotic behaviour of solutions to the equation (1.1) or system (1.2) under certain conditions placed on the nonlinear terms present in the equation considered.

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