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Numerical Solution to Singular Boundary Value Problems (SBVPS) using Modified Linear Multistep Formulas (LMF)

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Abstract

In this work, two block methods with characteristics of LMF are derived, analysed and numerically applied to solve second-order Singular Boundary Problems (SBVPs) of ordinary differential equations. The mathematical derivation of the proposed methods is based on the interpolation and collocation of the exact solution and its derivatives at some selected equidistant grid and off-grid points. The proposed strategy consists in a block method where the collocation at the initial point is avoided to circumvent the singularity at the starting end of the solution interval. The convergence analysis of the discrete solutions of the methods are examined. Finally, some second-order SVBPs of ordinary differential equations are numerically solved to demonstrate the efficiency and validity of the suggested technique, which is compared to various strategies available in the current literature. The result supports the good performance of the derived schemes.

Keywords: Convergence analysis, Order, Local Truncation error, Wavelet Newton approach, Chebyshev, Block method.
MSC2010: 65L04, 65L05, 65L06, 65L20.

1 Introduction

The solution of singular differential equations exhibits strange behaviour at the singular points; it is sometimes bounded, frequently unbounded, it may oscillate, or it is peculiar in some other way. SBVPs are fascinating to researchers because of their behaviour and presence in several science and engineering fields. Two-point singular boundary value problems:

$$u''(t) + \frac{\lambda}{t}u'(t) = k(t,u), \quad 0 < t \le t_N = 1 \tag{1.1}$$

is considered in this work together with any of the two-point boundary conditions: $u(0) = u_a$, $u(1) = u_b$ or $u(0) = u_a$, $u'(1) = u'_b$ or $u'(0) = u'_a$, $u(1) = u_b$ where λ, u_a, u_b, u'_a, and u'_b are known real values and $K(x,u)$ denotes a continuous real function, where we assume that the solution exists

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and is unique. [1] and [2] established the uniqueness and existence of the solution to the problem (1.1) subjected to any of the boundary conditions. According to [3] and [4], the mathematical expression of numerous problems arising in chemical kinetics, astrophysics, catalytic diffusion reactions, celestial mechanics, engineering, and various physical models resulted in second-order singular boundary value problems of ordinary differential equations of the type given in (1.1). [4], [5], [6], and [7] opined that SBVP has its application in the study of stellar structure, thermal explosion, and thermal distribution in the human head, rotationally symmetric shallow membrane caps, the problem of reactant concentration in a chemical reactor, reaction-diffusion processes inside a porous catalyst, the distribution of oxygen in a spherical shell, and many others, can be modelled by the system (1.1).

Significant effort has been expended in obtaining numerical solutions to the aforementioned unique problems. Different solutions for addressing (1.1) have been documented, where the main challenge occurs owing to the singularity at $x = 0$. Numerical techniques for solving the problem have been proposed by notable scholars in the field of numerical analysis (1.1). For example, finite difference methods (FDM) offered in [8] and [9], spline methods (SM) proposed in [10], and [11], the Jacobi-Gauss collocation method (JCM) described in [12] are examples of such techniques, the pseudospectral method (PM) proposed in [13], the quintic B-spline method in [14], the Bernstein basis polynomials in [15], the Pade approximation method (PAM) introduced in [16], [17]. Other papers on recently developed numerical or analytical strategies for tackling (1.1) are [3], [7], [18], [19], [20], [21] and [22]. In particular, [21] adopted the block method in combination with an ad-hoc formula to obtain a solution at the initial point before transferring the further solution to the primary method. One limitation of such a strategy is that the ad-hoc formulas will increase the computational cost. In this work, however, a class of numerical Methods have been derived for the solution of the Second-order singular boundary value problems. The method was achieved by omitting collocation at the initial points. This strategy allows the block method to produce approximation at this point. The methods are developed from continuous equations achieved by the techniques of collocation and interpolation. The derived methods are analyzed and numerically applied to solve second-order singular boundary value problems. The solutions were considered within intervals [0, 3] and [0, 4].

2 Formulation of the Suggested Block Method

Numerical solution of second order differential equation

$$u'' = f(t,\ u,\ u') \tag{2.1}$$

subject to any of the following boundary conditions: Dirichlet boundary conditions, $u(a) = u_a,\ u(b) = u_b$, Neumann boundary conditions, $u'(a) = u'_a,\ u'(b) = u'_b$, and Robin boundary conditions, $g_1(u(a), u'(a)) = v_a,\ g_2(u(b),\ u'(b)) = v_b$ are considered in this section. The methods are derived by considering

$$u(t) = \sum_{j=0}^{\infty} a_j t^j \tag{2.2}$$

as approximate solution to (2.1) whose partial sum is given as

$$u(t) = \sum_{j=0}^{k} a_j t^j, \tag{2.3}$$

where the a_r are unknown coefficients that will be determined. First and second derivatives of (2.3) are obtained as

$$u'(t) \quad = \quad \sum_{j=1}^{k} j\, a_j\, t^{j-1}, \tag{2.4}$$

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and

$$u''(t) \quad = \quad f(t,\, u,\, u') = \sum_{j=2}^{k} j\,(j-1)\,a_j\,t^{j-2}. \tag{2.5}$$

Interpolating and collocating equations (2.3), (2.4) and (2.5) at some carefully selected points and by imposing that

$$u(t_n) = u_n, \quad u'(t_n) = u'_n, \tag{2.6}$$

$$u''(t_{n+\frac{j}{2}}) = f_{n+\frac{j}{2}}, \quad j = 1,\ 2,\ \ldots, k \tag{2.7}$$

yields a system of $k-1$ equations with $k-1$ unknown coefficients. The system of algebraic equations (2.6) and (2.7) can be written in matrix form

$$\begin{pmatrix} 1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & \cdots & t_n^{k-1} \\ 0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & \cdots & D't_n^{k-2} \\ 0 & 1 & 2t_{n+\frac{1}{2}} & 3t_{n+\frac{1}{2}}^2 & 4t_{n+\frac{1}{2}}^3 & 5t_{n+\frac{1}{2}}^4 & \cdots & D't_{n+\frac{1}{2}}^{k-2} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 1 & 2t_{n+\frac{k}{2}} & 3t_{n+\frac{k}{2}}^2 & 4t_{n+\frac{k}{2}}^3 & 5t_{n+\frac{k}{2}}^4 & \cdots & D't_{n+\frac{k}{2}}^{k-2} \\ 0 & 0 & 2 & 6t_{n+\frac{1}{2}} & 12t_{n+\frac{1}{2}}^2 & 20t_{n+\frac{1}{2}}^3 & \cdots & D''t_{n+\frac{1}{2}}^{k-3} \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & 6t_{n+\frac{k}{2}} & 12t_{n+\frac{k}{2}}^2 & 20t_{n+\frac{k}{2}}^3 & \cdots & D''t_{n+\frac{k}{2}}^{k-3} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_4 \\ a_5 \\ \vdots \\ a_{k-1} \end{pmatrix} = \begin{pmatrix} u_n \\ f_n \\ f_{n+\frac{1}{2}} \\ \vdots \\ f_{n+\frac{k}{2}} \\ f_{n+\frac{1}{2}} \\ \vdots \\ f_{n+\frac{k}{2}} \end{pmatrix} \tag{2.8}$$

The algebraic equations represented in (2.8) are then solved for the unknown parameters, and substituting into the approximate solution (2.3), and simplified will yield a continuous scheme of the forms.

$$u(t) = \alpha_0(t)\,u_n + \alpha_1(t)\,h\,u'_n + h^2 \sum_{j=1}^{k} \beta_{\frac{j}{2}}(x)\,f_{n+\frac{j}{2}}, \tag{2.9}$$

where, $\alpha_j(s)$'s and $\beta_j(x)$'s are the coefficients that defined the methods and $x = \frac{t-t_{n+k-1}}{h}$. Evaluating (2.9) and it derivative at points $t_{n+j},\ j = 1,\, 2,\, \cdots,\, k$ yields the following main and additional formulas

$$u_{n+\frac{j}{2}} = \alpha_0\,u_n + \alpha_1\,h\,u'_n + h^2 \sum_{j=1}^{k} \beta_{\frac{j}{2}}\,f_{n+\frac{j}{2}}, \tag{2.10}$$

$$u'_{n+\frac{j}{2}} = \alpha_1\ u'_n + h \sum_{j=1}^{k} \beta_{\frac{j}{2}}\,f_{n+\frac{j}{2}}, \quad j = 1,\, 2, \ldots, k. \tag{2.11}$$

The combination of equations (2.10) and (2.11) is the required Direct Block Method (DBM). The theoretical procedure presented here is adopted to derive Hybrid Linear Multistep Formulas HLMF for the solution of problems of type (1.1) with the interval [0,3] and [0,4] yields the DBM which are termed Three Step Hybrid Direct Block Method (TSHDBM) and Four Step Hybrid Direct Block Method (FSHDBM) respectively. The coefficients of the TSHDBM and FSHDBM are as presented in tables 1 and 2.

3 Analysis of the method

We present in this section the analysis of the basic properties of the proposed methods.

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Table 1: The coefficients of the Hybrid Linear Multistep Formulas HLMF considered within $[0,3]$

	i	α_0	α_1	β_0	$\beta_{\frac{1}{2}}$	β_1	$\beta_{\frac{3}{2}}$	β_2	$\beta_{\frac{5}{2}}$	β_3	LTE
$u_{n+\frac{i}{2}}$	1	1	$\frac{1}{2}$	0	$\frac{19087}{40320}$	$-\frac{19987}{20160}$	$\frac{25561}{20160}$	$-\frac{9403}{10080}$	$\frac{14879}{40320}$	$-\frac{1231}{20160}$	$\frac{28549h^8y^{(8)}(x_n)}{30965760}+O\left(h^9\right)$
	2	1	1	0	$\frac{1609}{1260}$	$-\frac{1139}{504}$	$\frac{1843}{630}$	$-\frac{2713}{1260}$	$\frac{1073}{1260}$	$-\frac{71}{504}$	$\frac{1027h^8y^{(8)}(x_n)}{483840}+O\left(h^9\right)$
	3	1	$\frac{3}{2}$	0	$\frac{1881}{896}$	$-\frac{927}{280}$	$\frac{2055}{448}$	$-\frac{1881}{560}$	$\frac{5949}{4480}$	$-\frac{123}{560}$	$\frac{759h^8y^{(8)}(x_n)}{229376}+O\left(h^9\right)$
	4	1	2	0	$\frac{184}{63}$	$-\frac{1366}{315}$	$\frac{2032}{315}$	$-\frac{286}{63}$	$\frac{568}{315}$	$-\frac{94}{315}$	$\frac{17h^8y^{(8)}(x_n)}{3780}+O\left(h^9\right)$
	5	1	$\frac{5}{2}$	0	$\frac{30175}{8064}$	$-\frac{21625}{4032}$	$\frac{33625}{4032}$	$-\frac{11125}{2016}$	$\frac{18575}{8064}$	$-\frac{1525}{4032}$	$\frac{35225h^8y^{(8)}(x_n)}{6193152}+O\left(h^9\right)$
	6	1	3	0	$\frac{639}{140}$	$-\frac{1791}{280}$	$\frac{717}{70}$	$-\frac{909}{140}$	$\frac{423}{140}$	$-\frac{123}{280}$	$\frac{123h^8y^{(8)}(x_n)}{17920}+O\left(h^9\right)$
$u'_{n+\frac{i}{2}}$	1	0	1	0	$\frac{4277}{2880}$	$-\frac{2641}{960}$	$\frac{4991}{1440}$	$-\frac{3649}{1440}$	$\frac{959}{960}$	$-\frac{95}{576}$	$\frac{19087h^7y^{(8)}(x_n)}{7741440}+O\left(h^8\right)$
	2	0	1	0	$\frac{33}{20}$	$-\frac{203}{90}$	$\frac{287}{90}$	$-\frac{71}{90}$	$\frac{169}{180}$	$-\frac{7}{45}$	$\frac{1139h^7y^{(8)}x_n)}{483840}+O\left(h^8\right)$
	3	0	1	0	$\frac{105}{64}$	$-\frac{651}{320}$	$\frac{567}{160}$	$-\frac{393}{160}$	$\frac{309}{320}$	$-\frac{51}{320}$	$\frac{137h^7y^{(8)}(x_n)}{57344}+O\left(h^8\right)$
	4	0	1	0	$\frac{74}{45}$	$-\frac{31}{15}$	$\frac{172}{45}$	$-\frac{98}{45}$	$\frac{14}{15}$	$-\frac{7}{45}$	$\frac{143h^7y^{(8)}(x_n)}{60480}+O\left(h^8\right)$
	5	0	1	0	$\frac{105}{64}$	$-\frac{1175}{576}$	$\frac{1075}{288}$	$-\frac{175}{96}$	$\frac{665}{576}$	$-\frac{95}{576}$	$\frac{3715h^7y^{(8)}(x_n)}{1548288}+O\left(h^8\right)$
	6	0	1	0	$\frac{33}{20}$	$-\frac{21}{10}$	$\frac{39}{10}$	$-\frac{21}{10}$	$\frac{33}{20}$	0	$\frac{41h^7y^{(8)}(x_n)}{17920}+O\left(h^8\right)$

3.1 Local truncation error and order

Let $m(x)$ be a sufficiently differentiable function. Consider the following difference operator associated with the block method given in tables 1 and 2

$$\mathbf{L}\left[m(x);h\right]=\sum_j \alpha_j m(x_n+jh)-h\beta_j m'(x_n+jh)-h^2\gamma_j m''(x_n+jh),\quad j=0(1)k,\ k=3,\ 4. \tag{3.1}$$

where α_j and β_j are respectively given in the tables 1 and 2. The block methods in tables 1 and 2 and their associated difference operator is said to have order p if after expanding $m(x_n+jh)$, $m'(x_n+jh)$ and $m''(x_n+jh)$ in Taylor series about x_n, we have

$$\mathbf{L}\left[m(x);h\right]=c_0m(x_n)+c_1hm'(x_n)+c_3h^2m''(x_n)+\cdots+c_qh^qm^{(q)}(x_n)+\dots \tag{3.2}$$

with $c_0=c_1=\cdots=c_{p+1}=0$ and $c_{p+2}\neq 0$.

3.2 Consistency and convergence

In this section, the convergence of the proposed methods will be established. (see [4, 21, 23]) Let the $u(x)$ be the theoretical solution of (1.1), and $\{u_j\}_{j\in J}$ be the approximate solution at the grid points obtained by adopting proposed methods, that is, $u_j\simeq u(x_j)$. TSHBM is said to be of $p^{th}-$ theoretical order of convergence if for h sufficiently small, there exists a constant K independent of k such that

$$\max_{j\in J}||u(x_j)-u_j||\leq c\ h^p.$$

This implies that

$$\max_{0\leq j\leq N}||u(x_j)-u_j||\rightarrow 0 \text{ as } h\rightarrow\ 0.$$

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Table 2: The coefficients of the Hybrid Linear Multistep Formulas HLMF considered within [0, 4]

	i	α_0	α_1	β_0	$\beta_{\frac{1}{2}}$	β_1	$\beta_{\frac{3}{2}}$	β_2	$\beta_{\frac{5}{2}}$	β_3	$\beta_{\frac{7}{2}}$	β_4	LTE
$u_{n+\frac{i}{2}}$	1	1	$\frac{1}{2}$	0	$\frac{1070017}{1814400}$	$\frac{-12677941}{7257600}$	$\frac{4060853}{1209600}$	$\frac{-5980183}{1451520}$	$\frac{-118339}{36288}$	$\frac{-3917401}{2419200}$	$\frac{1670723}{3628800}$	$\frac{-416173}{7257600}$	$\frac{324901h^{10}y^{(10)}(x_n)}{1486356480}+O\left(h^{11}\right)$
	2	1	1	0	$\frac{12511}{8100}$	$\frac{-32377}{8100}$	$\frac{146089}{18900}$	$\frac{-215027}{22680}$	$\frac{85051}{11340}$	$\frac{-35177}{9450}$	$\frac{59989}{56700}$	$\frac{-14939}{113400}$	$\frac{58193h^{10}y^{(10)}(x_n)}{116121600}+O\left(h^{11}\right)$
	3	1	$\frac{3}{2}$	0	$\frac{112737}{44800}$	$\frac{-539283}{89600}$	$\frac{38643}{3200}$	$\frac{-52977}{3584}$	$\frac{104757}{8960}$	$\frac{-519909}{89600}$	$\frac{18471}{11200}$	$\frac{-18399}{89600}$	$\frac{71661h^{10}y^{(10)}(x_n)}{91750400}+O\left(h^{11}\right)$
	4	1	2	0	$\frac{49436}{14175}$	$\frac{-113642}{14175}$	$\frac{78572}{4725}$	$\frac{-8126}{405}$	$\frac{45044}{2835}$	$\frac{-37262}{4725}$	$\frac{31772}{14175}$	$\frac{-3956}{14175}$	$\frac{7703h^{10}y^{(10)}(x_n)}{7257600}+O\left(h^{11}\right)$
	5	1	$\frac{5}{2}$	0	$\frac{20225}{4536}$	$\frac{-2908025}{290304}$	$\frac{1026175}{48384}$	$\frac{-7300375}{290304}$	$\frac{208525}{10368}$	$\frac{-964925}{96768}$	$\frac{411325}{145152}$	$\frac{-102425}{290304}$	$\frac{56975h^{10}y^{(10)}(x_n)}{42467328}+O\left(h^{11}\right)$
	6	1	3	0	$\frac{543}{100}$	$\frac{-2103}{175}$	$\frac{18051}{700}$	$\frac{-8457}{280}$	$\frac{687}{28}$	$\frac{-1203}{100}$	$\frac{1203}{700}$	$\frac{-597}{1400}$	$\frac{93h^{10}y^{(10)}(x_n)}{57344}+O\left(h^{11}\right)$
	7	1	$\frac{7}{2}$	0	$\frac{3318329}{518400}$	$\frac{-14531881}{1036800}$	$\frac{81977}{2700}$	$\frac{-7310359}{207360}$	$\frac{3004337}{103680}$	$\frac{-4797541}{345600}$	$\frac{1046689}{259200}$	$\frac{-519253}{1036800}$	$\frac{2019731h^{10}y^{(10)}(x_n)}{1061683200}+O\left(h^{11}\right)$
	8	1	4	0	$\frac{104512}{14175}$	$\frac{-227104}{14175}$	$\frac{165184}{4725}$	$\frac{-22880}{567}$	$\frac{94912}{2835}$	$\frac{-74464}{4725}$	$\frac{69184}{14175}$	$\frac{-7912}{14175}$	$\frac{989h^{10}y^{(10)}(x_n)}{453600}+O\left(h^{11}\right)$
$u'_{n+\frac{i}{2}}$	1	0	1	0	$\frac{16083}{8960}$	$\frac{-1152169}{241920}$	$\frac{242653}{26880}$	$\frac{-296053}{26880}$	$\frac{2102243}{241920}$	$\frac{-115747}{26880}$	$\frac{32863}{26880}$	$\frac{-5257}{34560}$	$\frac{1070017h^{9}y^{(10)}(x_n)}{1857945600}+O\left(h^{10}\right)$
	2	0	1	0	$\frac{368}{189}$	$\frac{-703}{168}$	$\frac{179}{21}$	$\frac{-79417}{7560}$	$\frac{874}{105}$	$\frac{-3473}{840}$	$\frac{1111}{945}$	$\frac{-41}{280}$	$\frac{32377h^{9}y^{(10)}(x_n)}{58060800}+O\left(h^{10}\right)$
	3	0	1	0	$\frac{497}{256}$	$\frac{-35723}{8960}$	$\frac{80127}{8960}$	$\frac{-95783}{8960}$	$\frac{75577}{8960}$	$\frac{-37473}{8960}$	$\frac{2129}{1792}$	$\frac{-265}{1792}$	$\frac{12881h^{9}y^{(10)}(x_n)}{22937600}+O\left(h^{10}\right)$
	4	0	1	0	$\frac{68}{35}$	$\frac{-3784}{945}$	$\frac{964}{105}$	$\frac{-1087}{105}$	$\frac{7892}{945}$	$\frac{-436}{105}$	$\frac{124}{105}$	$\frac{-139}{945}$	$\frac{4063h^{9}y^{(10)}(x_n)}{7257600}+O\left(h^{10}\right)$
	5	0	1	0	$\frac{93965}{48384}$	$\frac{-21485}{5376}$	$\frac{49145}{5376}$	$\frac{-487225}{48384}$	$\frac{46415}{5376}$	$\frac{-22535}{5376}$	$\frac{57515}{48384}$	$\frac{-265}{1792}$	$\frac{41705h^{9}y^{(10)}(x_n)}{74317824}+O\left(h^{10}\right)$
	6	0	1	0	$\frac{68}{35}$	$\frac{-1121}{280}$	$\frac{321}{35}$	$\frac{-2843}{280}$	$\frac{314}{35}$	$\frac{-1107}{280}$	$\frac{41}{35}$	$\frac{-41}{280}$	$\frac{401h^{9}y^{(10)}(x_n)}{716800}+O\left(h^{10}\right)$
	7	0	1	0	$\frac{497}{256}$	$\frac{-27587}{6912}$	$\frac{7007}{768}$	$\frac{-38563}{3840}$	$\frac{303653}{34560}$	$\frac{-13573}{3840}$	$\frac{5257}{3840}$	$\frac{-5257}{34560}$	$\frac{149527h^{9}y^{(10)}(x_n)}{265420800}+O\left(h^{10}\right)$
	8	0	1	0	$\frac{368}{189}$	$\frac{-424}{105}$	$\frac{976}{105}$	$\frac{-9836}{945}$	$\frac{976}{105}$	$\frac{-424}{105}$	$\frac{368}{189}$	0	$\frac{989h^{9}y^{(10)}(x_n)}{1814400}+O\left(h^{10}\right)$

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The convergence shall be proof by first expressing the main and additional formulas in matrix form adopting the following notations. Let A represent the $12N \times 12N$ matrix define by

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1,2N} \\ \vdots & \vdots & \ddots & \vdots \\ A_{21} & A_{2N,2} & \dots & A_{2N,2N} \end{pmatrix},$$

where $A_{i,j}$ ara 6×6 submatrices, except $A_{i,N}$, $\;i = 1, \dots, 2N$ which have dimension 6×6 and $A_{i,2N}$, $\;i = 1, \dots, 2N$, which have 6×6. The submatrices are given below;

$$A_{N,N} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_{i,i-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad i = 2, \dots, N$$

$$A_{2N-1,2N} = h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; A_{i,i+1} = h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; i = (N+1), \dots, N$$

$$A_{2N,2N} = h \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; A_{i,i} = h \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; i = (N+1), \dots, (2N-1)$$

$$A_{N,2N} = h \begin{pmatrix} \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2} & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ \frac{-5}{2} & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; A_{i,N+i} = h \begin{pmatrix} \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3}{2} & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 1 & 0 \\ \frac{-5}{2} & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \; i = (N+1), \dots, (2N-1)$$

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and $A_{i,j} = I,\ i = 1, 2, \ldots, N-1$,where I is identity matrix. The rest of submatrices $A_{i,j}$ are null matrices. Also, let B be $12N \times (12N + 2)$ matrix defined by

$$A = \begin{pmatrix} B_{11} & B_{12} & \ldots & B_{1,2N} \\ \vdots & \vdots & \ddots & \vdots \\ B_{21} & B_{2N,2} & \ldots & B_{2N,2N} \end{pmatrix},$$

where the elements $B_{i,j}$ are 6×6 submatrices except the $B_{i,1},\ B_{i,N+1},\ i = 1, 2, \ldots, 2N$ which have dimension 6×6.

$$B_{1,1} = h^2 \begin{pmatrix} \frac{19087}{40320} & -\frac{19987}{20160} & \frac{25561}{20160} & -\frac{9403}{10080} & \frac{14879}{40320} & -\frac{1231}{20160} \\ \frac{1609}{1260} & -\frac{1139}{504} & \frac{1843}{630} & -\frac{2713}{1260} & \frac{1073}{1260} & -\frac{71}{504} \\ \frac{1881}{896} & -\frac{927}{280} & \frac{2055}{448} & -\frac{1881}{560} & \frac{5949}{4480} & -\frac{123}{560} \\ \frac{184}{63} & -\frac{1366}{315} & \frac{2032}{315} & -\frac{286}{63} & \frac{568}{315} & -\frac{94}{315} \\ \frac{30175}{8064} & -\frac{21625}{4032} & \frac{33625}{4032} & -\frac{11125}{2016} & \frac{18575}{8064} & -\frac{1525}{4032} \\ \frac{639}{140} & -\frac{1791}{280} & \frac{717}{70} & -\frac{909}{140} & \frac{423}{140} & -\frac{123}{280} \end{pmatrix},$$

$$B_{i,i} = h^2 \begin{pmatrix} \frac{19087}{40320} & -\frac{19987}{20160} & \frac{25561}{20160} & -\frac{9403}{10080} & \frac{14879}{40320} & -\frac{1231}{20160} \\ \frac{1609}{1260} & -\frac{1139}{504} & \frac{1843}{630} & -\frac{2713}{1260} & \frac{1073}{1260} & -\frac{71}{504} \\ \frac{1881}{896} & -\frac{927}{280} & \frac{2055}{448} & -\frac{1881}{560} & \frac{5949}{4480} & -\frac{123}{560} \\ \frac{184}{63} & -\frac{1366}{315} & \frac{2032}{315} & -\frac{286}{63} & \frac{568}{315} & -\frac{94}{315} \\ \frac{30175}{8064} & -\frac{21625}{4032} & \frac{33625}{4032} & -\frac{11125}{2016} & \frac{18575}{8064} & -\frac{1525}{4032} \\ \frac{639}{140} & -\frac{1791}{280} & \frac{717}{70} & -\frac{909}{140} & \frac{423}{140} & -\frac{123}{280} \end{pmatrix},\ i = 2, \ldots, 2N$$

$$B_{N+1,1} = h \begin{pmatrix} \frac{4277}{2880} & -\frac{2641}{960} & \frac{4991}{1440} & -\frac{3649}{1440} & \frac{959}{960} & -\frac{95}{576} \\ \frac{33}{20} & -\frac{203}{90} & \frac{287}{90} & -\frac{71}{90} & \frac{169}{180} & -\frac{7}{45} \\ \frac{105}{64} & -\frac{651}{320} & \frac{567}{160} & -\frac{393}{160} & \frac{309}{320} & -\frac{51}{320} \\ \frac{74}{45} & -\frac{31}{15} & \frac{172}{45} & -\frac{98}{45} & \frac{14}{15} & -\frac{7}{45} \\ \frac{105}{64} & -\frac{1175}{576} & \frac{1075}{288} & -\frac{175}{96} & \frac{665}{576} & -\frac{95}{576} \\ \frac{33}{20} & -\frac{21}{10} & \frac{39}{10} & -\frac{21}{10} & \frac{33}{20} & 0 \end{pmatrix},$$

$$B_{N+j,j} = h \begin{pmatrix} \frac{4277}{2880} & -\frac{2641}{960} & \frac{4991}{1440} & -\frac{3649}{1440} & \frac{959}{960} & -\frac{95}{576} \\ \frac{33}{20} & -\frac{203}{90} & \frac{287}{90} & -\frac{71}{90} & \frac{169}{180} & -\frac{7}{45} \\ \frac{105}{64} & -\frac{651}{320} & \frac{567}{160} & -\frac{393}{160} & \frac{309}{320} & -\frac{51}{320} \\ \frac{74}{45} & -\frac{31}{15} & \frac{172}{45} & -\frac{98}{45} & \frac{14}{15} & -\frac{7}{45} \\ \frac{105}{64} & -\frac{1175}{576} & \frac{1075}{288} & -\frac{175}{96} & \frac{665}{576} & -\frac{95}{576} \\ \frac{33}{20} & -\frac{21}{10} & \frac{39}{10} & -\frac{21}{10} & \frac{33}{20} & 0 \end{pmatrix},\ j = 2, \ldots, 2N$$

The rest of the submatrice $B_{i,j}$ that are not included are all Null matrices i.e $B_{i,j} = \mathbf{0}$. It should be noted that the entries of the submatrices $A_{i,j}$ and $B_{i,j}$ are the coefficients of the TSHDBM presented in table 1 for $n = 0, 1, \ldots, N-1$. We define the following vectors corresponding to the

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exact and function values.

$$\begin{aligned} U &= \left(u(t_{\frac{1}{2}}), u(t_1), u(t_{\frac{3}{2}}), u(t_2), \cdots, u(t_{N-1+\frac{1}{2}}),\ u'(t_0), u'(t_{\frac{1}{2}}), \cdots, u't_N)\right)^T \quad (3.3)\\ F &= \left(f(t_0, u(t_0), u'(t_0)), f(t_{\frac{1}{2}}, u(t_{\frac{1}{2}}), u'(t_{\frac{1}{2}})), \cdots, f(t_N, u(t_N), u'(t_N)))\right)^T. \quad (3.4)\end{aligned}$$

where u has $(6N-1)+(6N+1)=12N$ components and F has $(6N+1)+(6N+1)=12N+2$ components because due to the boundary condition in (1.1) $u(t_0)$ and $u(t_N)$ are known values, $y(x_0)=y_a$, $y(x_N)=y_b$. With this notation in mind, it is possible to write the exact form of the system that provided the approximate values of the problem at hand is given by

$$A_{12N\times 12N}U_{12N} + h^2 B_{12N\times(12N+2)}F_{(12N+2)} + \mathbf{C}_{12N} = \mathbb{L}(h)_{12N} \quad (3.5)$$

with

$$\mathbf{C}_{12N} = (-u_a, -u_a, -u_a - u_a, -u_a, -u_a, 0, \cdots, 0, -u_b, 0, \cdots, 0)^T,$$

while $\mathbb{L}(h)_{12N}$ represents the local truncation errors of the proposed formulas presented in table 1 which are given as

$$\mathbb{L}(h)_{12N} = (a_1, a_2, a_3, a_4, a_5, a_6, a_7 \ldots, a_{6N}, b_1, b_2, b_3, b_4, b_5, b_6, b_7 \ldots, b_6 N)^T \quad (3.6)$$

where

$$a_1 = \frac{28549 h^8 y^{(8)}(x_0)}{30965760} + O\left(h^9\right),\ a_2 = \frac{1027 h^8 y^{(8)}(x_0)}{483840} + O\left(h^9\right),\ a_3 = \frac{759 h^8 y^{(8)}(x_0)}{229376} + O\left(h^9\right)$$

$$a_4 = \frac{17 h^8 y^{(8)}(x_0)}{3780} + O\left(h^9\right),\ a_5 = \frac{35225 h^8 y^{(8)}(x_0)}{6193152} + O\left(h^9\right),\ a_6 = \frac{123 h^8 y^{(8)}(x_0)}{17920} + O\left(h^9\right)$$

$$a_7 = \frac{28549 h^8 y^{(8)}(x_1)}{30965760} + O\left(h^9\right),\ a_{6N} = \frac{123 h^8 y^{(8)}(x_{N-1})}{17920} + O\left(h^9\right)$$

$$b_1 = \frac{19087 h^7 y^{(8)}(x_0)}{7741440} + O\left(h^8\right),\ b_2 = \frac{1139 h^7 y^{(8)} x_0)}{483840} + O\left(h^8\right),\ b_3 = \frac{137 h^7 y^{(8)}(x_0)}{57344} + O\left(h^8\right)$$

$$b_4 = \frac{143 h^7 y^{(8)}(x_n)}{60480} + O\left(h^8\right),\ b_5 = \frac{3715 h^7 y^{(8)}(x_n)}{1548288} + O\left(h^8\right),\ b_6 = \frac{41 h^7 y^{(8)}(x_n)}{17920} + O\left(h^8\right)$$

$$b_7 = \frac{19087 h^7 y^{(8)}(x_1)}{7741440} + O\left(h^8\right),\ b_{6N} = \frac{41 h^7 y^{(8)}(x_{N-1})}{17920} + O\left(h^8\right)$$

Let defined the system that approximates (1.1) as

$$\bar{A}_{12N\times 12N}\bar{U}_{12N} + h^2 \bar{B}_{12N\times(12N+2)}\bar{F}_{(12N+2)} + \mathbf{C}_{12N} = 0 \quad (3.7)$$

where U_{12N} is approximated by vector $\bar{U}_{12N}$,

$$\bar{U}_{12N} = \left(u_{\frac{1}{2}}, u_1, u_{\frac{3}{2}}, u_2, u_{\frac{5}{2}}, u_3, \cdots, u_{N-1+\frac{1}{2}}, u'_0, u_{\frac{1}{2}}, \cdots, u'_N\right), \bar{F}_{12N+2} = \left(f_0, f_{\frac{1}{2}}, f_1, f_{\frac{3}{2}}, f_2 f_{\frac{3}{2}}, f_3 \cdots, f_N\right)$$

Subtracting equation (3.7) from (3.5) yields

$$\bar{A}_{12N\times 12N}\bar{E}_{12N} + h^2 \bar{B}_{12N\times(12N+2)}\left(F - \bar{F}\right) = \mathbb{L}(h)_{12N} \quad (3.8)$$

where

$$\bar{E}_{12N} = U_{12N} - \bar{U}_{12N} = (e_{\frac{1}{2}}, e_2, e_{\frac{3}{2}} \cdots, e_{N-1+\frac{1}{2}}, e'_0, e'_{\frac{1}{2}}, \cdots, e'_N)^T$$

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which are the errors associated with the solution and the derivatives. Since (1.1) is continuous in the close $[a,\ b]$ and differentiable in the open interval $(a\ b)$, it is true by Mean Value Theorem, for $i=0,\ 1,\ \cdots,N$ the following identities are valid.

$$f\big(t_i,u(t_i),u'(t_i)\big)-f\big(t_i,u_i,u'_i\big)=\big(u(t_i)-u_i\big)\frac{\partial f(c_i)}{\partial u}+\big(u'(t_i)-u'_i\big)\frac{\partial f(c_i)}{\partial u'}$$

where $c_i,\ \bar{c_i}$ are points on the line segment joining $\big(t_i,u(t_i),u'(t_i)\big)$ to $\big(t_i,u_i,u'_i\big)$.

Thus, we have

$$F-\bar{F}=\bar{J}_{12N\times(12N+2)}\bar{H}_{12N}$$

$$F-\bar{F}=\begin{pmatrix}\frac{\partial f(\epsilon_0)}{\partial u} & 0 & \dots & 0 & \frac{\partial f(\epsilon_0)}{\partial u'} & 0 & \dots & 0\\ 0 & \frac{\partial f(\epsilon_{\frac{1}{2}})}{\partial u} & \dots & 0 & 0 & \frac{\partial f(\epsilon_{\frac{1}{2}})}{\partial u'} & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \frac{\partial f(\epsilon_N)}{\partial u} & 0 & 0 & \dots & \frac{\partial f(\epsilon_N)}{\partial u'}\end{pmatrix}\begin{pmatrix}e_0\\ e_1\\ \vdots\\ e_N\\ e'_0\\ e'_1\\ \vdots\\ e'_N\end{pmatrix},$$

$$=\begin{pmatrix}0 & \dots & 0 & \frac{\partial f(\epsilon_0)}{\partial u'} & 0 & \dots & 0 & 0\\ \frac{\partial f(\epsilon_{\frac{1}{2}})}{\partial u} & \dots & 0 & 0 & \frac{\partial f(\epsilon_{\frac{1}{2}})}{\partial u'} & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & \frac{\partial f(\epsilon_{N-1})}{\partial u} & 0 & 0 & \dots & \frac{\partial f(\epsilon_{N-1})}{\partial u'} & 0\\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & \frac{\partial f(\epsilon_N)}{\partial u'}\end{pmatrix}\bar{E}_{12N}$$

where the second identity has been achieved through the fact that we know the exact boundary conditions, that is, $e_0=y(x_0)-y_0=0$, $e_N=y(t_N)-y_N=0$. Finally, using the above result, the equation (3.8) may be be written as

$$(\bar{A}_{12N\times 12N}+h^2\bar{B}_{12N\times(12N+2)}\bar{J}_{(12N+2)\times 12N})\bar{E}_{12N}=\mathbb{L}(h)_{12N} \tag{3.9}$$

and setting $\mathbb{Z}_{12N\times 12N}=\big(\bar{A}_{12N\times 12N}+h^2\bar{B}_{12N\times(12N+2)}\bar{J}_{(12N+2)\times 4N}\big)$, we have

$$\mathbb{Z}_{12N\times 12N}\bar{E}_{12N}=\mathbb{L}(h)_{12N}. \tag{3.10}$$

Dropping the dimension, we can write $\mathbb{Z}=\big(\bar{A}+h^2\bar{B}\bar{J}\big)$. For some selected values of $h>0$ matrix $\mathbb{Z}$ is invertible. Let $\mathbb{Z}_N=\mathbb{Z}_{12N\times 12N}$, for the matrix $\mathbb{Z}_N$ where the submatrices have many zeros entries, it is can be confirmed that for $N=1$, the determinant is $|\mathbb{Z}_1|=-h^{11}$. By mathematical induction, we know that $|\mathbb{Z}_N|=-Nh^{N+6}$ thus, $\mathbb{Z}_N$ is invertible as long as $h>0$. Hence,

$$\mathbb{Z}=\big(\bar{A}+h^2\bar{B}\bar{J}=(I-\mathbf{C})\bar{A}$$

where I is the identity matrix of order $12N$, and $\mathbf{C}=-h^2\bar{B}\bar{J}\mathbb{Z}^{-1}$. Therefore,

$$|\mathbb{Z}|=|I-\mathbf{C}||\bar{A}|$$

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As

$$|\lambda I - \mathbf{C}||\bar{A}| = \Pi_{i=1}^{12N}(\lambda - \lambda_i)$$

is the characteristic polynomial of $\mathbf{C}$ in order to have $|\lambda I - \mathbf{C}| \neq 0$ for $\lambda = 1$, it is sufficient to choose h such that $h^2 \in \{1/\lambda_i : \lambda_i \text{is an eigenvalue of } \mathbf{C}\}$. For such value of h the equation (3.10) can be written as

$$\bar{E} = (\mathbb{Z})^{-1}\mathbb{L}(h) \tag{3.11}$$

We consider the maximum norm in $\mathbb{R}^{12N}, ||\bar{E}|| = \max_i |e_i|$ in $\mathbb{R}^{12N}$ and the corresponding matrix induced norm in $\mathbb{R}^{12N\times 12N}$. By expanding series $(\mathbb{Z})^{-1}_{12N\times 12N}$ term by term around h, it can be shown after tedious calculations that $||(\mathbb{Z})^{-1}_{12N\times 12N}|| = \mathcal{O}(h^{-2})$. This is essentially related to the fact that the uniform norm of the inverse of $\mathbb{A}$ grows like h^{-2}, as one can verify rather simply . Consequently, from the equation in ((3.10)) and the form of the vector $\mathbb{L}(h)_{12N}$ in ((3.6)), assuming $u(x)$ has an

$$\begin{aligned} ||\bar{E}_{12N}|| &\leq ||(\mathbb{Z})^{-1}_{12N\times 12N}\mathbb{L}(h)_{12N}||. && (3.12)\\ &= \mathcal{O}(h^{-2})\mathcal{O}(h^{8}) && (3.13)\\ &= \langle(h^{6}) && (3.14)\end{aligned}$$

Thus, the method is 6-order convergent. In similar manner, it easy to verify that FSHDBM presented in table 2 is 8 order convergent.

4 Implementation

The derived formulas for methods within [0,3] and [0,4] presented in tables 1 and 2 are combined and applied in block form. The solutions are considered in the interval $[x_n, x_{n+k}]$, $n = 0,\ 1,\ 2,\ \ldots,\ N-k$ where N is the number of blocks. The formulas are written inform of $M(y) = 0$ with the following unknown values to be obtained

$$\bar{\mathbf{U}} = \left(u_0,\ u_{\frac{1}{2}},\ u_1, \ldots,\ u_{N-k},\ u'_0,\ u'_{\frac{1}{2}},\ u'_1, \ldots,\ u'_N\right)$$

The resulting system is solved using Newton's method given as

$$\bar{\mathbf{U}}^{i+1} = \bar{\mathbf{U}}^{i} - \frac{M^i}{\mathbf{J}^i}$$

where $\mathbf{J}$ is the Jacobian matrix of M. The following Taylor's approximations are considered as starting values to be used with the Newton's method.

$$\begin{aligned} u_{n+\frac{\bar{j}}{2}} &= u_n + \bar{j}\frac{h}{2}u'_n + \frac{1}{2}\left(\bar{j}\frac{h}{2}\right)^2 f_n,\\ u'_{n+\frac{\bar{j}}{2}} &= u'_n + \bar{j}\frac{h}{2}f_n. \end{aligned}$$

The algorithm adopted for the implementation suggested methods for solving (1.1) are as highlighted below:

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Algorithm 1 Algorithm for Implementing the suggested method

A **Input:** Supply the following:
1 N, the number of the main block
2 x_N, the end point of the integration interval
3 $f(x, u, u')$ and boundary conditions u_a, u_b.
B **Output:** Approximate (1.1) using the suggested method by
4 Define $h = \frac{x_N - x_0}{N-1}$, $\quad x = x_0, \quad n = 0$
5 **while** $x < x_N$ **do**
6 Obtain the values of $\left\{u_{\frac{i}{2}}, u'_{\frac{i}{2}}\right\}_{i=1,\ldots,k}$ set $x = x + h$, $n = n + 1$.
7 **end while**
8 **End**

5 Numerical application

Numerical examples are presented in this section to show the performance of the derived method. The errors are measured using $e_i = u(x_i, \tau) - \bar{u}(x_i, \tau)$ where $\bar{u}$ and u are approximate and exact solutions of the problems, respectively, and τ is an arbitrary time t in $[0, T]$. Also, the following error norms are defined for comparison:

$$L_2 - error = \left[\sum_{i=0}^{n} (e_i)^2\right]^{\frac{1}{2}}, \quad L_\infty - error = Max(e_i), \quad 0 \le i \le, \quad \text{and} \quad RMS - error = \left[\sum_{i=0}^{n} \frac{e_i^2}{n+1}\right]^{\frac{1}{2}}.$$

The following notation were adopted in the presentation of results: BNM → Block method in [26], ROC → Rate of convergence, NS → Number of steps, MAXAE → Maximum absolute error, ChWNA →Chebyshev wavelet Newton approach ([6]), GeWNA → Gegenbauer wavelet Newton approach ([6]) , HeWNA → Hermite wavelet Newton approach ([6]), LaWNA → Laguerre wavelet Newton Approach ([6]), WNA → wavelet Newton approach ([6]), and WQA → wavelet Quasilinearization approach ([6])

5.1 Example 1

Consider the nonlinear singular boundary value problem in physiology

$$u''(t) + \left(1 + \frac{r}{t}\right) u'(t) = \frac{5t^3 \left(5\, t^5 e^{u(t)} - t - r - 4\right)}{4 + t^5} \tag{5.1}$$

subject to $u'(0) = 0, \quad u(1) + 5u'(1) = ln\left(\frac{1}{5}\right) - 5$. The exact solution is given as $u(t) = ln(\frac{1}{4+t^5})$.

Table 3: Solution of example 5.1 using TSHDBM

t (grid value)	u-approx	u-Exact	u-Error
0.	-1.3862943729259658	-1.3862943611198906	1.1806×10^{-8}
0.1	-1.3862968729233076	-1.3862968611167654	1.1807×10^{-8}
0.2	-1.3863743697370740	-1.3863743579200614	1.1817×10^{-8}
0.3	-1.3869016885211218	-1.3869016766664655	1.1854×10^{-8}
0.8	-1.4650316132840980	-1.4650316016572746	1.1626×10^{-8}
0.9	-1.5239867831456570	-1.5239867721873073	1.0958×10^{-8}
1.	-1.6094379231199516	-1.6094379124341003	1.0685×10^{-8}

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Table 4: Solution of example 5.1 using FSHDBM

t (grid value)	u-approx	u-Exact	u-Error
0.	-1.3862943610889784	-1.3862943611198906	3.0912×10^{-11}
0.1	-1.3862968610858595	-1.3862968611167654	3.0906×10^{-11}
0.2	-1.3863743578891748	-1.3863743579200614	3.0886×10^{-11}
0.3	-1.3869016766356027	-1.3869016766664655	3.0862×10^{-11}
0.7	-1.4274530989039420	-1.4274530989357574	3.1815×10^{-11}
0.8	-1.4650316016256707	-1.4650316016572746	3.1604×10^{-11}
0.9	-1.5239867721569039	-1.5239867721873073	3.0403×10^{-11}
1.	-1.6094379124044592	-1.6094379124341003	2.9641×10^{-11}

The solution of Example 5.1 was considered within interval [0, 1] over ten (10) iterations. The results are shown in the tables 3 to 4. As shown in columns four of tables 3 to 4, the results of the method agree with the exact solution up to at least four decimal places.

Table 5: Comparison (MAXAE) and ROC of FSHDBM with that of using example 5.1

r=0.25				
NS	FSHDBM(MAXAE)	FSHDBM(ROC)	BNM(MAXAE)	BNM(ROC)
8	1.8930×10^{-5}	-	8.166×10^{-8}	6.24
16	7.5145×10^{-8}	7.98	1.231×10^{-9}	60.5
32	1.9588×10^{-10}	8.58	1.906×10^{-11}	6.01
64	7.2320×10^{-13}	8.08	2.967×10^{-13}	6.01
128	3.1086×10^{-15}	7.86	2.665×10^{-15}	6.80
256	4.4408×10^{-16}	2.80	1.554×10^{-15}	NA
512	8.3321×10^{-17}	2.41	8.882×10^{-16}	NA

Comparison of MAXAE of example (5.1) obtained using FSHDBM with those of BNM are presented in Tables 5 and 6 for $r = 0.25$ and $r = 1$. The efficiency curves using the MAXAE and NS in Tables 5 and 6 are displaced in Figures 5.1 and 5.1. It can be seen that FSHDBM compares well with BNM.

5.2 Example 2

Consider the nonlinear SBVP

$$u''(t) + \frac{2}{t}u'(t) + u^5(t) = 0, \tag{5.2}$$

Subject to $u'(0) = 0, \quad u(1) = \sqrt{\frac{3}{4}}$. The equation arise in the study of stellar structure. It exact solution is $u(t) = \sqrt{\frac{3}{3+t^2}}$. The example is solved within the interval [0, 1] over ten (10) iterations. The solution of example 5.2 was considered within the interval [0, 1] over ten (10) iterations. The results are presented in tables 7 to 8. As shown in columns four of tables 7 to 8, the results agreed

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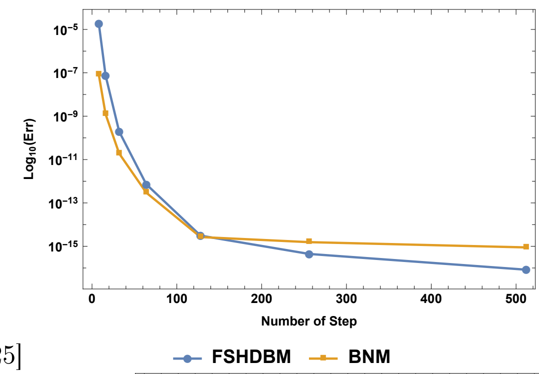
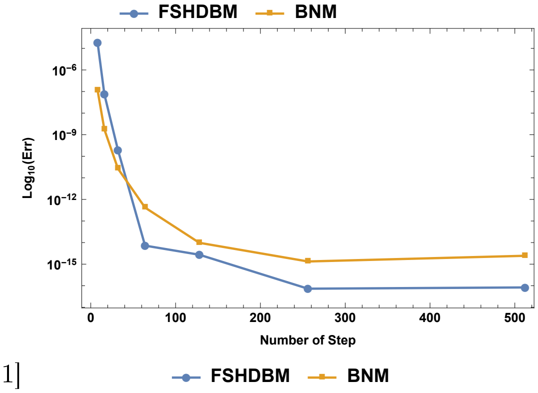
Table 6: Comparison of maximum absolute errors and ROC of FSHM with that of using example 5.1

r=1				
NS	FSHDBM(MAXAE)	FSHDBM(ROC)	BNM(MAXAE)	BNM(ROC)
8	1.8930×10^{-5}	-	1.145×10^{-7}	6.21
16	7.5145×10^{-8}	7.98	1.761×10^{-9}	6.02
32	1.9588×10^{-10}	8.58	2.682×10^{-11}	6.04
64	7.2289×10^{-15}	8.08	4.183×10^{-13}	6.00
128	2.7347×10^{-15}	7.86	9.769×10^{-15}	5.42
256	7.2953×10^{-17}	2.80	1.332×10^{-15}	NA
512	8.3321×10^{-17}	2.41	2.442×10^{-15}	NA



[For $r = 0.25$]

[For



$r = 1$]

Figure 1: [Comparison of errors obtained using FSHDBM and BNM on problem 5.1

with the exact solution up to at least six decimal places. Table 9 shows the comparison of errors (L_∞, L_2, RMS) of example 5.2 using TSHDBM and FSHDBM with those of WNA and WQA. Again, FSHDBM performed better than TSHDBM, WNA and WQA with just four iterations.

5.3 Example 3

$$u''(t) + \frac{1}{t}u'(t) + e^{u(t)} = 0 \tag{5.3}$$

subject to $u'(0) = 0, \quad u(1) = 0$. This problem was used to study thermal explosion in a cylindrical

Table 7: Solution of example 5.2 using TSHDBM

t (grid value)	u-approx	u-Exact	u-Error
0.	0.9999999999861606	1	$1.3839\ 10^{-11}$
0.1	0.9983374884457982	0.9983374884595826	$1.3784\ 10^{-11}$
0.2	0.9933992677849104	0.9933992677987828	1.387310^{-11}
0.3	0.9853292781505725	0.9853292781642932	$1.3721\ 10^{-11}$
0.7	0.9271455408157601	0.9271455408231195	$7.3595\ 10^{-12}$
0.8	0.9078412989984316	0.9078412990032037	$4.7721\ 10^{-12}$
0.9	0.8873565094138608	0.8873565094161139	$2.2531\ 10^{-12}$
1.	$\frac{\sqrt{3}}{2}$	0.8660254037844386	0

Table 8: Solution of example 5.2 using FSHDBM

t (grid value)	u-approx	u-Exact	u-Error
0.	1.0000000000000353	1.	6.5898×10^{-15}
0.1	0.9983374884596178	0.9983374884595826	6.5576×10^{-15}
0.2	0.9933992677988182	0.9933992677987828	6.5917×10^{-15}
0.3	0.9853292781643275	0.9853292781642932	6.4141×10^{-15}
0.7	0.9271455408231335	0.9271455408231195	2.6124×10^{-15}
0.8	0.9078412990032116	0.9078412990032037	1.5148×10^{-15}
0.9	0.8873565094161171	0.8873565094161139	6.3310×10^{-15}
1.	$\frac{\sqrt{3}}{2}$	0.8660254037844386	0

Table 9: Comparison L_∞, L_2 and RMS errors of example 5.2 obtained using TSHDBM, and FSHDBM with those of WNA and WQA [6]

Error	TSHDBM	FSHDBM	WNA	WQA
L_∞	9.5155×10^{-10}	9.384×10^{-12}	1.43×10^{-10}	1.43×10^{-10}
L_2	2.7840×10^{-9}	2.6023×10^{-11}	2.07069×10^{-10}	2.07069×10^{-10}
RMS	1.28743×10^{-9}	1.2373×10^{-11}	NA	NA

vessel. The exact solution of (5.3) is $u(t) = 2ln\left(\frac{4-2\sqrt{2}}{(3-2\sqrt{2})t^2+1}\right)$
Example 5.3 was considered within the interval [0, 1] over ten (10) iterations. The results are presented in tables 10 to 11. As shown in columns four of tables 10 to 11, the results of the method agreed with the exact solution up to at least twelve (12) decimal places.
Table 12 shows the comparison of errors (L_∞, L_2, RMS) of example 5.3 using TSHDBM and FSHDBM with those of WNA and WQA. FSTDDBM and FSHDBM performed better than WNA and WQA with just four iterations.

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Table 10: Solution of example 5.3 using TSHDBM within [0,1] over ten (10) iterations

t (grid value)	u-approx	u-Exact	u-Error
0.	0.31669436764085107	0.31669436764074954	1.0153×10^{-13}
0.1	0.3132658504981409	0.31326585049806327	7.7605×10^{-14}
0.2	0.3030154228322728	0.30301542283229976	2.6978×10^{-14}
0.3	0.2860472653046736	0.28604726530485386	1.8025×10^{-13}
0.7	0.15524810668220576	0.15524810668275627	5.5050×10^{-13}
0.8	0.10832276344401158	0.10832276344446458	4.5400×10^{-13}
0.9	0.05643860246897065	0.05643860246923624	2.6559×10^{-13}
1.	0	0.	0

Table 11: Solution of example 5.3 using FSHDBM within [0,1] over five (5) iterations

t (grid value)	u-approx	u-Exact	u-Error
0.	0.316694367640845589	0.31669436764074954	9.5712×10^{-14}
0.1	0.313265850498162312	0.31326585049806327	9.8782×10^{-14}
0.2	0.303015422832399008	0.30301542283229976	9.8606×10^{-14}
0.3	0.286047265304956432	0.28604726530485386	1.0223×10^{-13}
0.7	0.155248106682826018	0.15524810668275627	6.9448×10^{-14}
0.8	0.108322763444515644	0.10832276344446458	5.0841×10^{-14}
0.9	0.056438602469260776	0.05643860246923624	2.4403×10^{-14}
1	0	0	0

Table 12: Comparison L_∞, L_2 and RMS errors of example 5.3 obtained using TSHDBM, and FSHDBM with those of WNA and WQA [6]

Error	TSHDBM	WNA	WQA
L_∞	5.52899×10^{-11}	3.00502×10^{-10}	3.00502×10^{-10}
L_2	1.64448×10^{-10}	5.46469×10^{-10}	5.46469×10^{-10}
RMS	6.2147×10^{-11}	NA	NA

Error	FSHDBM	WNA	WQA
L_∞	6.85025×10^{-11}	3.00502×10^{-10}	3.00502×10^{-10}
L_2	2.2650×10^{-12}	5.46469×10^{-10}	5.46469×10^{-10}
RMS	9.7857×10^{-13}	NA	NA

5.4 Example 4

Consider the nonlinear SBVP

$$u''(t) + \frac{2}{t}u'(t) + e^{-y(t)} = 0,\ u'(0) = 0, \quad 2u(1) + u'(1) = 0. \tag{5.4}$$

The above nonlinear SBVP is discussed by [25] as a heat conduction model in the human head. However, an exact solution to this problem is yet to be found.

Table 13: Comparison of computed solutions of example 5.4 obtained using FSHDBM with those of Wavelet methods in [6]

t	FSHDBM	ChWNA	GeWNA	HeWNA	LaWNA
0	0.27002964789671	0.2699437790	0.269943779	0.269943779	0.269948773
0.1	0.26875690062969	0.2686763867	0.268676386	0.268676386	0.268676385
0.2	0.26493281753837	0.264853383	0.264853383	0.264853383	0.264853383
0.3	0.25853978938153	0.258462168	0.258462168	0.258462168	0.258462168
0.7	0.20649448302385	0.206431878	0.206431878	0.206431878	0.206431878
0.8	0.18655201416670	0.186495288	0.186495288	0.186495288	0.186495288
0.9	0.16365968158046	0.163609789	0.163609789	0.163609789	0.163609789
1	0.13769874661365	0.137656718	0.137656718	0.137656718	0.137656718

Table 13 compares solutions of example (5.4) obtained using FSTDDBM and FSHDBM with those obtained by ChWNA, GeWNA, HeWNA and LaWNA.

5.5 Example 5

Consider the nonlinear SBVP:

$$u''(t) + \frac{3}{t}u'(t) + \left(\frac{1}{8u^2(t)} - \frac{1}{2}\right) = 0, \tag{5.5}$$

subject to $u'(0) = 0, \quad u(0) = 1$. The above nonlinear SBVP is discussed in [0, 1] to study rotationally symmetric solutions of shallow membrane caps. The exact solution to this example is yet to be discovered. Table 14 compares the solutions of example (5.5) obtained using FSHDBM with those obtained by ChWNA, GeWNA, HeWNA and LaWNA.

6 Conclusion

A new strategy for developing block methods for solving singular boundary value problems has been introduced in this paper. The suggested strategy eliminates the use of ad-hoc formulas to obtain the approximation of the problem at the initial points. Some examples are considered for numerical experiments, and the results established the good performance of the technique.

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Table 14: Comparison of computed solutions of problem 5.5 obtained using FSHDBM with those of Wavelet methods in [6]

NS	FSHDBM	ChWNA	GeWNA	HeWNA	LaWNA
0.	0.954135307075277	0.954135307	0.954135307	0.954135307	0.954135307
0.1	0.954588728705079	0.954588729	0.954588729	0.954588729	0.954588729
0.2	0.955949645053102	0.955949645	0.955949645	0.955949645	0.955949645
0.3	0.958220004713764	0.958220005	0.958220005	0.958220005	0.958220005
0.7	0.976478969841676	0.976478970	0.976478970	0.976478970	0.976478970
0.8	0.983369348898582	0.983369349	0.983369349	0.983369349	0.983369349
0.9	0.991206375159024	0.991206375	0.991206375	0.991206375	0.991206375
1.	1	1	1	1	1

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