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Product of Quasi-Idempotents in Finite Semigroup of Partial Order-Preserving Transformations

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Abstract

Let X_n be the finite set $\{1, 2, \ldots, n\}$, and $\mathcal{PO}_n = O_n \cup \{\alpha : \mathrm{dom}(\alpha) \subset X_n(\forall x, y \in X_n), x \leq y \implies x\alpha \leq y\alpha\}$ be the semigroup of all partial order-preserving transformations from X_n to itself, where $\mathcal{O}_n = \{\alpha \in T_n : (\forall x, y \in X_n)x \leq y \implies x\alpha \leq y\alpha\}$ is the full order preserving transformation on X_n and $\mathcal{T}_n$ the semigroup of full transformations from X_n to itself. A transformation α in $\mathcal{PO}_n$ is called quasi-idempotent if $\alpha \neq \alpha^2 = \alpha^4$. In this article, we study quasi-idempotent elements in the semigroup of partial order-preserving transformations and show that semigroup $\mathcal{PO}_n$ is quasi-idempotent generated. Furthermore, an upper bound for quasi-idempotent rank of $\mathcal{PO}_n$ is obtained to be $\lceil \frac{5n-4}{2} \rceil$. Where $\lceil x \rceil$ denotes the least positive integer m such that $x \leq m \leq x + 1$.

Keywords: Partial order-preserving, Full order-preserving, Quasi-idempotent, generating set and rank.
MSC2010: 20M20.

1 INTRODUCTION

One of the most important generalizations of group theory and in fact, the leading area of research in modern algebra is the theory of semigroups. The theory become an interesting field in modern abstract algebra and the work done by Howie [1] in the full transformation semigroup $\mathcal{T}_n$ consisting of all mappings from a set X_n into itself, formed a major breakthrough and indeed the basis for further investigations into the area which has assumed an enviable place in the theory of semigroup. Since then, there have been many articles concerned with this idea in $\mathcal{T}_n$ (see for example, [2–5]). For any $\alpha \in \mathcal{PO}_n$(the semigroup of partial order-preserving transformations), if $\alpha = \alpha^2$ then α is called an idempotent; and if $\alpha \neq \alpha^2 = \alpha^4$ then α is called a quasi-idempotent. An element α of $\mathcal{PO}_n$, the semigroup of all partial order-preserving transformations of X_n, is said to have projection characteristic (r, s), or to belong to the set $[r, s]$, if $|\mathrm{dom}(\alpha)| = r$, $|\mathrm{im}(\alpha)| = s$, where $0 \leq s \leq r \leq n$.

Throughout this paper we shall use the notation $QE_1 = QE_1^n \cup QE_1^{n-1}$, where QE_1^n and QE_1^{n-1} to respectively mean the sets of quasi-idempotents in $[n, n-1]$ and $[n-1, n-1]$ in any subset QE of the semigroup $\mathcal{PO}_n$. For basic semigroup theory concepts see [6].

We now begin with a finite generating set. Let S be a semigroup and let $\emptyset \neq A \subseteq S$. The smallest subsemigroup of S containing A is called the subsemigroup generated by A and is denoted by $\langle A\rangle$. Clearly, $\langle A\rangle$ is the set of all finite products of elements of A.

If there exists a non-empty subset A of S such that $\langle A\rangle = S$, then A is called a generating set of S. Also, the *rank* of a finitely generated semigroup S is defined by

$$\operatorname{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A\rangle = S\}.$$

That is, the cardinality of a minimum generating set. If S is generated by the set E of idempotents, then the *idempotent rank* of S is defined by

$$\operatorname{idrank}(S) = \min\{|A| : A \subseteq E \text{ and } \langle A\rangle = S\}.$$

Generation of finite transformation subsemigroups includes the work of Gomes and Howie in [15] proved that both the rank and idempotent rank of $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$, where $\mathcal{S}_n$ is the symmetric group, are equal to $\frac{n(n-1)}{2}$. This was generalised by Howie and McFadden in [17] considered the semigroup $K(n,r) = \{\alpha \in \mathrm{Sing}_n : |\mathrm{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-1)$, and showed that both the rank and idempotent rank are equal to $S(n,r)$, the Sterling number of the second kind. Gomes and Howie in [13] investigated the rank of the semigroups $\mathcal{O}_n$ and $\mathcal{PO}_n$, (the semigroup of order-preserving full transformation and order-preserving partial transformations on X_n). It was shown that the rank of $\mathcal{O}_n$ is n and $\mathcal{PO}_n$ is $(2n-1)$, and the idempotent rank of $\mathcal{O}_n$ is $(2n-2)$ and $\mathcal{PO}_n$ is idempotent-generated and its idempotent rank is $(3n-2)$. Garba [14] in generalizing the work of [13] considered the semigroup $L(n,r) = \{\alpha \in \mathcal{O}_n : |\mathrm{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-2)$, and showed that both the rank and idempotent rank are equal to $\binom{n}{r}$ and the rank and idempotent rank of $M_{n,r} = \{\mathcal{PO}_n : |\mathrm{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-2)$ are both $\sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$.

In particular, if there exists a generating set A of S consisting of entirely quasi-idempotents, then A is called a quasi-idempotent generating set of S, and the *quasi-idempotent rank* of S is defined by

$$\operatorname{qrank}(S) = \min\{|A| : A \subseteq QE \text{ and } \langle \mathrm{A}\rangle = \mathrm{S}\}.$$

Umar [18] used quasi-idempotent elements to generate the semigroup

$$\mathcal{I}_n^- = \{\alpha \in \mathcal{I}_n : (\forall x \in \mathrm{dom}(\alpha)), x\alpha \leq x\},$$

of all partial one-to-one order-decreasing transformations in finite symmetric inverse semigroup and proved that the semigroup $\mathcal{I}_n^-$ is quasi-idempotent generated and its rank is equal to $\frac{n(n+1)}{2}$. Madu and Garba [7] showed that each element in the semigroup $\mathcal{IO}_n$ is expressible as a product of quasi-idempotents of defect one in $\mathcal{IO}_n$, and that the quasi-idempotent rank and depth of $\mathcal{IO}_n$ are $2(n-1)$ and $(n-1)$ respectively. Garba et al [8] proved that Sing_n is quasi-idempotent generated and that the quasi-idempotent rank of Sing_n is $\frac{n(n-1)}{2}$. Garba and Imam [9] proved that the semigroup $\mathcal{SI}_n$ (of all strictly partial one-to-one maps on X_n) is generated by quasi-idempotents of defect one and the best possible global lower bound for the number of quasi-idempotents (of defect and shift equal to one) required to generate $\mathcal{SI}_n$ is equal to $\lceil\frac{3(n-1)}{2}\rceil$. Bugay [10] proved among other results that for $n \geq 4$ the quasi-idempotent rank of $\mathcal{I}_n$ is 4. Bugay [11] proved that $\mathcal{I}(n,r) = \{\alpha \in \mathcal{I}_n : |\mathrm{im}(\alpha)| \leq r\}$ for $(1 \leq r \leq n-1)$ is quasi-idempotent generated and the quasi-idempotent rank of $\mathcal{I}(n,r)$ is $\binom{n}{2}$ if $r = 2$ and $\binom{n}{r} + 1$ if $r \geq 3$. Recently, Imam et al [12] showed that the semigroup $\mathcal{O}_n$ is generated by Quasi-idempotent of defect one and the upper bound for the quasi-idempotent rank is $\lceil\frac{3(n-2}{2}\rceil$.

In this article, we investigate the product of quasi-idempotents and obtain an upper bound for quasi-idempotents rank for $n \geq 4$ of $\mathcal{PO}_n$, that is the cardinality of a minimum quasi-idempotents generating set for $\mathcal{PO}_n$.

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2 PRELIMINARIES

Let $\mathcal{PO}_n$ be the Semigroup of partial order-preserving transformations on X_n defined by

$$\mathcal{PO}_n = \mathcal{O}_n \cup \{\alpha : \mathrm{dom}(\alpha) \subset X_n(\forall x, y \in X_n), x \leq y \implies x\alpha \leq y\alpha\},$$

and $\mathcal{O}_n$ the semigroups of full order-preserving transformations in Sing_n defined by

$$\{\mathcal{O}_n = \{\alpha \in \mathrm{Sing}_n : (\forall x, y \in X_n)x \leq y \implies x\alpha \leq y\alpha\},$$

where $\mathrm{Sing}_n = \mathcal{T}_n \setminus \mathcal{S}_n$ the semigroup of all singular transformations on X_n, that is

$$\mathrm{Sing}_n = \{\alpha \in \mathcal{T}_n : |\mathrm{im}(\alpha)| \leq n-1\}.$$

Hence, by Howie [6][Proposition 1.4.11], we have that

Proposition 2.1. *for any $\alpha, \beta \in \mathcal{PO}_n$*

$$\begin{aligned} \alpha\mathcal{L}\beta & \quad \text{if and only if} \quad \mathrm{im}(\alpha) = \mathrm{im}(\beta), \\ \alpha\mathcal{R}\beta & \quad \text{if and only if} \quad \ker(\alpha) = \ker(\beta), \\ \alpha\mathcal{J}\beta & \quad \text{if and only if} \quad |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)|. \end{aligned}$$

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R}.$$

These equivalences are known as Green's relations and the relation $\mathcal{D}$ is the composition of the two relations $\mathcal{L}$ and $\mathcal{R}$. The equivalences $\mathcal{D}$ and $\mathcal{J}$ coincide when the semigroup is finite. It is then evident from Proposition above that there are n J-classes in $\mathcal{PO}_n$. Thus, $\mathcal{PO}_n$ is a union of J-classes $J_0, J_1, \ldots, J_r, \ldots, J_{n-1}$, where

$$J_r = \{\alpha \in \mathcal{PO}_n : |\mathrm{im}(\alpha)| = r\}.$$

Gomes and Howie [13] showed that in $\mathcal{PO}_n$ and indeed in the larger semigroup $\mathcal{P}_n$ of all partial transformations of X_n. The $J-class$ $J_{n-1} = \{\alpha \in \mathcal{PO}_n : |\mathrm{im}(\alpha)| = n-1\}$ is the union of $[n, n-1]$ and $[n-1, n-1]$. Within $[n, n-1]$ there are $(n-1)$ $R-$ classes indexed by the equivalences $|1,2|, |2,3|, \ldots, |n-1, n|$ and within $[n-1, n-1]$, which consists of one-to-one partial order-preserving transformations, there are n R-classes, indexed by the domains $X_n \setminus \{1\}, X_n \setminus \{2\}, \ldots, X_n \setminus \{n\}$. So a generating set for $\mathcal{PO}_n$ covers the $R-classes$ in J_{n-1}. Thus $|J_{n-1}| = n(2n-1)$.

Consider a typical element β in $[n, n-1]$, with the kernel, $\ker(\beta) = |i, i+1|$ and $\mathrm{im}(\beta) = X_n \setminus \{j\}$, then β is always a decreasing element whenever $j > i$ and β can always be written as

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & i, i+1 & i+2 & \cdots & j & j+1 & \cdots & n \\ 1 & 2 & \cdots & i & i+1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix}.$$

Again β is increasing whenever $i > j$ and β can be written as

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & j-1, & j & \cdots & i, i+1 & i+2 & \cdots & n \\ 1 & 2 & \cdots & j-1 & j+1 & \cdots & i+1 & i+2 & \cdots & n \end{pmatrix}.$$

Similarly, for any element β in $[n-1, n-1]$, with $\mathrm{dom}\beta = X_n \setminus \{i\}$ and $\mathrm{im}\beta = X_n \setminus \{j\}$ then β is a decreasing element whenever $i < j$ and β can be written as

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & \cdots & j & j+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix}.$$

Similarly for $i > j$, the element β is a decreasing one and can be written as

$$\beta = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & \cdots & i-1 & i+1 & \cdots & n \\ 1 & 2 & \cdots & j-1 & j+1 & \cdots & i & i+1 & \cdots & n \end{pmatrix}.$$

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Lastly, for $i=j$ the element β in $[n-1,n-1]$ is the only identity maps and can be written as

$$\beta=\begin{pmatrix}1 & 2 & \cdots & i-1 & i+1 & \cdots & n\\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & n\end{pmatrix}.$$

Let X_n be the finite set $\{1,2,\ldots,n\}$ and $\mathcal{PO}_n$ the semigroups of partial order-preserving transformation on $\mathrm{X_n}$. We begin with the following definition of quasi-idempotent elements, stationery and non-stationery blocks of a transformation $\alpha\in\mathcal{PO}_n$.

Definition 2.2. *A transformation α in $\mathcal{PO}_n$ is called quasi-idempotent if α is not idempotent but α^2 is. That is, $\alpha\neq\alpha^2=\alpha^4$.*

3 THE MAIN RESULT

3.1 PRODUCTS OF QUASI-IDEMPOTENTS IN $\mathcal{PO}_n$

In this section, we consider the product of quasi-idempotents in the semigroup of partial order-preserving transformations. Let $QE_1=QE_1^n\cup QE_1^{n-1}$ where QE_1^n and QE_1^{n-1} are respectively, the set of quasi-idempotents in $[n,n-1]$ and $[n-1,n-1]$. Then

$$QE_I=\{\beta_1,\beta_2,\ldots,\beta_{n-1},\alpha_1,\alpha_2,\ldots,\alpha_{n-1},\mu_2,\mu_3,\ldots,\mu_{n-2},\delta_2,\delta_3,\ldots,\delta_{n-2}\}$$

where

$$\mu_i=\begin{pmatrix} i & i+1\\ i-1 & i\end{pmatrix}$$

for $i=2,\ldots,n-1$ and μ_i is decreasing and

$$\delta_i=\begin{pmatrix} i-1 & i\\ i & i+1\end{pmatrix}$$

for $i=2,\ldots,n-1$ which is increasing quasi-idempotents.

And also

$$\beta_i=\begin{pmatrix} i+1\\ i\end{pmatrix}$$

for $i=1,\ldots,n-1$ is a decreasing quasi-idempotents,

and

$$\alpha_i=\begin{pmatrix} i-1\\ i\end{pmatrix}$$

for $i=2,\ldots,n$ which is increasing.Thus, $\ker(\sigma_i)=|i,i+1|$, $\ker(\mu_i)=|i-1,i|$, $\mathrm{dom}(\beta_i)=X_n\setminus i$, $\mathrm{dom}(\alpha_i)=X_n\setminus i$, $\mathrm{im}(\sigma_i)=X_n\setminus i-1$, $\mathrm{im}(mu_i)=X_n\setminus i+1$, $\mathrm{im}(\beta_i)=X_n\setminus i+1$, and $\mathrm{im}(\alpha_i)=X_n\setminus i-1$. It is clear that the cardinality of quasi-idempotent in $\mathcal{J}_{n-1}$ of $\mathcal{PO}_n$ is $2(2n-3)$.

Lemma 3.1. *For $n\geq 4$, $[n-1,n-1]\subseteq\langle QE_1^{n-1}\rangle$*

Proof. Let $\alpha\in[n-1,n-1]\setminus QE_1^{n-1}$, with $\mathrm{dom}(\alpha)=X_n\setminus\{i\}$ and $\mathrm{im}(\alpha)=X_n\setminus\{j\}$. Then we consider three cases as follows:

Case I : If $i=j$, then

$$\alpha=\begin{cases}\begin{pmatrix}1 & 2 & \cdots & i-1 & i+1 & \cdots & n\\ 1 & 2 & \cdots & i-1 & i+1 & \cdots & n\end{pmatrix} & \text{for } i=1,2,\ldots,n-1;\\[2ex] \begin{pmatrix}1 & 2 & \cdots & n-1\\ 1 & 2 & \cdots & n-1\end{pmatrix} & \text{for } i=n.\end{cases}$$

And so, we have

$$\alpha = \begin{cases} \begin{pmatrix} i+1 \\ i \end{pmatrix} \begin{pmatrix} i \\ i+1 \end{pmatrix} = \beta_i \alpha_{i+1} & \text{for } i = 1, 2, \ldots, n-1 \\ \\ \begin{pmatrix} n-1 \\ n \end{pmatrix} \begin{pmatrix} n \\ n-1 \end{pmatrix} = \alpha_n \beta_{n-1} & \text{for } i = n. \end{cases}$$

Case II : If $i < j$, then

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & i-1 & i+1 & \cdots & j & j+1 & \cdots & n \\ 1 & 2 & \cdots & i-1 & i & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix}.$$

clearly, we have

$$\alpha = \begin{pmatrix} i+1 \\ i \end{pmatrix} \begin{pmatrix} i+2 \\ i+1 \end{pmatrix} \cdots \begin{pmatrix} j \\ j-1 \end{pmatrix} = \beta_i \beta_{i+1} \cdots \beta_{j-1}.$$

Case III : For $j < i$, then

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & \cdots & i-1 & i+1 & \cdots & n \\ 1 & 2 & \cdots & j-1 & j+1 & \cdots & i & i+1 & \cdots & n \end{pmatrix}.$$

And so, we have

$$\alpha = \begin{pmatrix} i-1 \\ i \end{pmatrix} \begin{pmatrix} i-2 \\ i-1 \end{pmatrix} \cdots \begin{pmatrix} j \\ j+1 \end{pmatrix} = \alpha_{i-1} \alpha_{i-2} \cdots \alpha_{j+1}.$$

Thus, in all cases α is a product of quasi-idempotents in QE_1^{n-1}. Hence $[n-1, n-1] \subseteq \langle QE_1^{n-1} \rangle$. □

The next lemma shows that the semigroup $\mathcal{O}_n$ is generated by its quasi-idempotents of defect 1 and can be found in [12].

Lemma 3.2. *For $n \geq 4$, $[n, n-1] \subseteq \langle QE_1^n \rangle$.*

The combined effect of lemmas 1 and 2 proved the following result.

Theorem 3.3. *For $n \geq 4$, the semigroup $\mathcal{PO}_n$ is quasi-idempotent generated. In particular, $\mathcal{PO}_n = \langle QE_1 \rangle$.*

$\mathcal{O}_n$ is generated by its quasi-idempotents of defect 1 and can be found in [12].

3.2 BOUND FOR QUASI-IDEMPOTENT RANK OF $\mathcal{PO}_n$

In this section, we obtain upper bound for the minimum cardinality of Quasi-idempotent generating set for the semigroup $\mathcal{PO}_n$. The quasi-idempotent rank of $\mathcal{PO}_n$ denoted by qrank($\mathcal{PO}_n$) is defined to be the minimum number of quasi-idempotents required to generate $\mathcal{PO}_n$, that is

$$\text{qrank}(\mathcal{PO}_n) = \min\{|Q| : Q \subseteq QE_1 \quad \text{and} \quad \langle Q \rangle = \mathcal{PO}_n\}.$$

We can finally state the main Theorem.

Theorem 3.4. *For $n \geq 4$, $2n-1 \leq qrank(\mathcal{PO}_n) \leq \lceil \frac{5n-4}{2} \rceil$.*

Proof. First, we note that, for any element $\alpha \in \mathcal{PO}_n$, of defect 1, with $\alpha = \epsilon_1 \epsilon_2 \cdots \epsilon_k$ where $\epsilon_i \in QE_1 (i = 1, 2, \ldots, k)$ we must have $\alpha \mathcal{R} \epsilon_1$ and $\alpha \mathcal{L} \epsilon_k$, that is, $\ker(\alpha) = \ker(\epsilon_1)$ and $\text{im}(\alpha) = \text{im}(\epsilon_k)$. Thus, it follows that any generating set of quasi-idempotents for $\mathcal{PO}_n$ must cover both the $\mathcal{R}$*-classes* and $\mathcal{L}$*-classes* in J_{n-1}. And so, since there are n $\mathcal{L}$*-classes* and $2n-1$ $\mathcal{R}$*-classes* in J_{n-1}, it is immediate that qrank($\mathcal{PO}_n$) $\geq n$ and qrank($\mathcal{PO}_n$) $\geq 2n-1$. But, since $2n-1 > n$, the inequality qrank($\mathcal{PO}_n$) $\geq n$ is superfluous. Hence, for all $n \geq 4$

$$2n - 1 \leq qrank(\mathcal{PO}_n). \tag{3.1}$$

Now, to show the other inequality, that is $\text{qrank}(\mathcal{PO}_n) \leq \lceil \frac{5n-4}{2} \rceil$, we consider the subset

$$A = QE_1 \setminus (\{\beta_i : i = 2, 3, \ldots, n-1\} \cup \{\delta_i : 4, 6, 8, \ldots, n-2\})$$

of QE_1 and observe that, for all $i = 2, 3, \ldots, n-1$,

$$\alpha_i \mu_i = \begin{pmatrix} i+1 \\ i \end{pmatrix} = \beta_i.$$

for $i = 4, 6, 8, \ldots, n-2$,

$$\mu_{i+1}\delta_{i+1}\delta_{i-1}\mu_{i-2} = \begin{pmatrix} i-1 & i \\ i & i+1 \end{pmatrix} = \delta_i.$$

This shows that $\mathcal{PO}_n = \langle A \rangle$. To obtain the required inequality we compute the cardinality of the set A. For this, we consider two case based on the parity of n.

Case I . If n is even, then
$|A| = |QE_1| - |\{\beta_j, \delta_i\}|$ $2 \leq j \leq n-1$ and $i = 4, 6, 8, \ldots, n-2$
$= 4n - 6 - (n-2) - (\frac{n}{2} - 2)$
$= \lceil \frac{5n-4}{2} \rceil$.

Case II . If n is odd, then
$|A| = |QE_1| - |\{\beta_j, \delta_i\}|$
$2 \leq j \leq n-1$ and $i = 4, 6, 8, \ldots, n-2$
$= 4n - 6 - (n-2) - (\frac{n-1}{2} - 2)$
$= \lceil \frac{5n-4}{2} \rceil$.

Thus, we have $\text{qrank}(\mathcal{PO}_n) \leq |A| = \lceil \frac{5n-4}{2} \rceil$.
This together with equation 3.1 give the required result. □

3.3 Conclusion

This article has been able to identify Quasi-idempotents as a new generating system of $\mathcal{PO}_n$ of a finite set of n-elements. Thus, it follows from this study that algebraic and combinatorial properties of $\mathcal{PO}_n$ can easily be deduced via studying the corresponding properties on the set of quasi-idempotents in this semigroup. Moreover, the upper bound obtained for the quasi-idempotents rank has reduced to some extent the size of the quasi-idempotents generating set. We suspect $\lceil \frac{5n-4}{2} \rceil$ to be the qrank, only that we are unable to establish the equality at the moment.

3.4 Competing interests

The authors declare that they have no competing interests.

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