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One-step fourth derivative block integrator for the numerical solution of singularly perturbed third-order boundary value problems

O. O. Olaiya [1*]

1. National Mathematical Centre, Abuja, Nigeria.
* Corresponding authors: olaiyaolumide.o@gmail.com

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Abstract

Third-order singularly perturbed problems are common models that are prevalent in applied mathematics and engineering. They model physical phenomenon in fluid dynamics, optimal control, reaction-diffusion processes and many other fields. A one-step fourth derivative block integrator is derived in this work to numerically solve general singularly perturbed third-order problems in ordinary differential equations with prescribed initial or boundary conditions. The derivation of this block of integrators was achieved via the collocation technique where a shifted Chebyshev polynomial of the first kind is used as the trial solution. The characteristics of the method are shown and the numerical examples shows an excellent performance in terms of accuracy of this block method as compared to existing methods already in literature.

Keywords: Block method, Boundary value problem, Hybrid method, Linear Multistep method, Initial Value Problem.
MSC2010: 65L05, 65L06, 65L10.

1 Introduction

In this work, the numerical solution of the singularly perturbed third-order problem of the form

$$\begin{aligned} &\epsilon^n y'''(x) + a(x)y(x) + f(x), \quad x \in [a,b] \\ &y(a) = \alpha_1, \quad y(b) = \beta_1, \quad y'(a) = \gamma_1 \end{aligned} \tag{1.1}$$

is considered. Here α, β, γ are real constants, $f(x)$ is a smooth function and $a \leq x \leq b$ and the parameter ϵ is the perturbation parameter, n is a positive integer. We note here that, other boundary conditions like $y(a) = \alpha_2$, $y(b) = \beta_2$, $y''(a) = \gamma_2$ may be considered.

Singular perturbation problems (SPPs) models several physical occurrences in different branches of applied sciences. Such file of application include fluid mechanics or dynamics, chemical reactor theory, optimal control theory and host of many others. The perturbation parameter is usually a small constant which is the coefficient of the highest derivative in a given differential equation,

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see [1]. Due to the presence of perturbation parameter, the solution varies swiftly. With this, the main concern with such problems is the swift growth or deterioration of their solutions in one or more narrow boundary layer region(s). As a result, not only determining analytical solutions to such problems is difficult but also the convergence analysis, see [2]. The prevalence of this small constant coefficient, numerical approaches to the solution of this kind of differential equation remains the only viable means of obtaining a solution, especially for higher order differential equation. Many researchers on SPPs have mostly restricted their search for numerical solution to second-order problems, see [3–6]. Hence, the search for an efficient numerical scheme is the motivation and goal in this paper.

Some methods for solving singular perturbation problems include Local discontinuous Galerkin (LDG) method used in [7, 8], initial-value technqiue proposed in [9], exponentially fitted mesh method in [10], C^0-continuous interior penalty method [11], cubic splines in [12] and quartic splines in [13]. Akram in [14] used quartic splines while Saini and Mishra in [15] used quartic B-splines to solve third-order self-adjoint SPBVPs. Other methods can be found in [16–18], just to mention few. The major drawback in obtaining the solution of problems of this nature is that the solution depends on the perturbation parameter, see [2, 19]. Hence, a more efficient and robust method would be required for the solution of this problem, see [20, 21].

In this work, linear multistep method is derived to form a block method which is rich in handling pieces of solution in a given domain. The advantage of this technique helps in dealing with the perturbation parameter, even as it decreases [22–24]. Other advantages includes producing smaller global errors (at the end of the range of integration).
The technique employed is the collocation method where the trial solution $y(x)$ of (1.1) is given as $q(t) = \sum_{j=0}^{m} \rho_j T^*(t)_j$. Here, $T^*(t)$ is the shifted Chebyshev polynomial of the first kind. The collocation/interpolation technique affords us the derivation of the main and additional methods, which are unified to form a block integrator. These integrators are single step hybrid method called one-step fourth derivative block integrator (OSFDBI). This technique has several interesting advantages among which are; they overcome the intersections of pieces of solutions, they are self starting which implies that they do not require any starting values from eternal methods and solutions can be obtained at intra-points other than the usual grid-points, among others. Thus, the drawback mentioned earlier can be circumvented with the use of this method, see [24].

2 Derivation of the method

In this section, the derivation of a continuous implicit three intra-step hybrid block method is described, for the solution of (1.1) over the integration interval $[a, b]$,

$$\pi_N \equiv \{a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b\}$$

with h the constant step-size, $h = t_i - t_{i-1}$, $i = 1, 2, \ldots, N$.

Three off-step points u, v, w considered are such that $0 < u < v < w < 1$ on the interval $[t_n, t_{n+1}]$, so that $t_n < t_{n+u} < t_{n+v} < t_{n+w} < t_{n+1}$, where $t_{n+u} = t_n + uh$, $t_{n+v} = t_n + vh$, $t_{n+w} = t_n + wh$. Thus, we consider the approximation $q(t)$ of the solution $y(t)$ given by the polynomial

$$y(t) \simeq q(t) = \sum_{j=0}^{9} \rho_j T_j^* \,. \tag{2.1}$$

with the d^{th} derivatives given as

$$y^{(d)}(t) \simeq q^{(d)}(t) = \sum_{j=0}^{9} \rho_j T_j^{*(d)}, ; \quad d = 1(1)4 \tag{2.2}$$

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where ρ_j are coefficients to be determined and T^* is the shifted Chebyshev polynomial of degree j, $t \in [a, b]$. The shifted Chebychev polynomial on the interval $[a, b]$, is given as

$$T_n^*(t) = T_n(x), \quad x = \frac{2}{b-a}\left(t - \frac{a+b}{2}\right) \tag{2.3}$$

with leading coefficient $2^{n-1}\left(\frac{2}{b-a}\right)^n$. Evaluating (2.1), (2.2) for $d = 1, 2$ each at the point $t = t_n$, then evaluating the third derivative in (2.2) $(d = 3)$ at the points t_{n+j} for $j = 0, u, v, w, 1$, and finally, evaluating the fourth derivative in (2.2) $(d = 4)$ at the points t_{n+j} $j = 0, 1$ so that the following system of equations are obtained $q(t_n) = y_n$, $q'(t_n) = y'_n$, $q''(t_n) = y''_n$, $q'''(t_n) = f_n$, $q'''(t_{n+u}) = f_{n+u}$, $q'''(t_{n+v}) = f_{n+v}$, $q'''(t_{n+w}) = f_{n+w}$, $q'''(t_{n+1}) = f_{n+1}$, $q^{(4)}(t_n) = g_n$, $q^{(4)}(t_{n+1}) = g_{n+1}$. Put in matrix form with ten real unknown constants ρ_j $j = 0(1)9$ to be determined.

$$\begin{pmatrix}
T_0^*(t_n) & T_1^*(t_0) & T_2^*(t_0) & T_3^*(t_0) & T_4^*(t_0) & \cdots & T_9^*(t_0) \\
T_0^{*\prime}(t_n) & T_1^{*\prime}(t_n) & T_2^{*\prime}(t_n) & T_3^{*\prime}(t_n) & T_4^{*\prime}(t_n) & \cdots & T_9^{*\prime}(t_n) \\
T_0^{*\prime\prime}(t_n) & T_1^{*\prime\prime}(t_n) & T_2^{*\prime\prime}(t_n) & T_3^{*\prime\prime}(t_n) & T_4^{*\prime\prime}(t_n) & \cdots & T_9^{*\prime\prime}(t_n) \\
T_0^{*\prime\prime\prime}(t_n) & T_1^{*\prime\prime\prime}(t_n) & T_2^{*\prime\prime\prime}(t_n) & T_3^{*\prime\prime\prime}(t_n) & T_4^{*\prime\prime\prime}(t_n) & \cdots & T_9^{*\prime\prime\prime}(t_n) \\
T_0^{*\prime\prime\prime}(t_{n+u}) & T_1^{*\prime\prime\prime}(t_{n+u}) & T_2^{*\prime\prime\prime}(t_{n+u}) & T_3^{*\prime\prime\prime}(t_{n+u}) & T_4^{*\prime\prime\prime}(t_{n+u}) & \cdots & T_9^{*\prime\prime\prime}(t_{n+u}) \\
T_0^{*\prime\prime\prime}(t_{n+v}) & T_1^{*\prime\prime\prime}(t_{n+v}) & T_2^{*\prime\prime\prime}(t_{n+v}) & T_3^{*\prime\prime\prime}(t_{n+v}) & T_4^{*\prime\prime\prime}(t_{n+v}) & \cdots & T_9^{*\prime\prime\prime}(t_{n+v}) \\
T_0^{*\prime\prime\prime}(t_{n+w}) & T_1^{*\prime\prime\prime}(t_{n+w}) & T_2^{*\prime\prime\prime}(t_{n+w}) & T_3^{*\prime\prime\prime}(t_{n+w}) & T_4^{*\prime\prime\prime}(t_{n+w}) & \cdots & T_9^{*\prime\prime\prime}(t_{n+w}) \\
T_0^{*\prime\prime\prime}(t_{n+1}) & T_1^{*\prime\prime\prime}(t_{n+1}) & T_2^{*\prime\prime\prime}(t_{n+1}) & T_3^{*\prime\prime\prime}(t_{n+1}) & T_4^{*\prime\prime\prime}(t_{n+1}) & \cdots & T_9^{*\prime\prime\prime}(t_{n+1}) \\
T_0^{*(4)}(t_n) & T_1^{*(4)}(t_n) & T_2^{*(4)}(t_n) & T_3^{*(4)}(t_n) & T_4^{*(4)}(t_n) & \cdots & T_9^{*(4)}(t_n) \\
T_0^{*(4)}(t_{n+1}) & T_1^{*(4)}(t_{n+1}) & T_2^{*(4)}(t_{n+1}) & T_3^{*(4)}(t_{n+1}) & T_4^{*(4)}(t_{n+1}) & \cdots & T_9^{*(4)}(t_{n+1})
\end{pmatrix}
\begin{pmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \\ \rho_3 \\ \rho_4 \\ \rho_5 \\ \rho_6 \\ \rho_7 \\ \rho_8 \\ \rho_9 \end{pmatrix}
=
\begin{pmatrix} y_n \\ y'_n \\ y''_n \\ f_n \\ f_{n+u} \\ f_{n+v} \\ f_{n+w} \\ f_{n+1} \\ g_n \\ g_{n+1} \end{pmatrix}$$

where $y'_n \simeq y'(x_n)$, $f_n = f(x_n, y_n, y'_n, y''_n)$, and $g_n = g(x_n, y, y', y'', y''')$. $T_0^*(t) = 0$, $T_1^*(t) = 2t - 1$, $T_{n+1}^*(t) = 2(2t-1)T_n^*(t) - T_{n-1}^*(t)$; $\quad n \geq 1$. Solving for the unknown ρ in the above matrix using the computer algebraic system in Mathematica, the coefficients ρ_j, $j = 0, 1, \ldots, 9$ are obtained but not included here as they are cumbersome. Substituting the values of ρ_j into (2.1) and after some simplifications, the following approximate form is obtained

$$\begin{aligned} q(t + jh) = \quad & \alpha_0 y_0 + \alpha_1 h y'_0 + \alpha_2 h^2 y''_0 \\ & + h^3 \sum_{i=0}^{4} \beta_i f_{n+s_i} + h^4 \sum_{i=0}^{1} \gamma_i g_{n+i} \Big|_{\{s_0=0, s_1=u, s_2=v, s_3=w, s_4=1\}} \end{aligned} \tag{2.4}$$

Evaluating the formula in (2.4) and its derivative up to the 2nd at the point $t = t_{n+1}$, the following one step formulae are obtained

$$\begin{aligned} y_{n+1} = \quad & y_n + h y'_n + \frac{h^2 y''_n}{2} + h^3 \left(\frac{A_0}{5040 u^2 v^2 w^2} f_n + \frac{(3v(8w-3)-9w+4)}{5040(u-1)^2 u^2 (u-v)(u-w)} f_{n+u} \right. \\ & \left. + \frac{(u(9-24w)+9w-4)}{5040(v-1)^2 v^2 (u-v)(v-w)} f_{n+v} + \frac{(u(9-24v)+9v-4)}{5040(w-1)^2 w^2 (u-w) + \frac{A_1}{5040(u-1)^2(v-1)^2(w-1)^2} f_{n+1}(w-v)} f_{n+w} \right) \\ & + h^4 \left(\frac{(3u(4v(7w-2)-8w+3)+v(9-24w)+9w-4)}{5040 uvw} g_n + \frac{(3u(v(6-14w)+6w-3)+9v(2w-1)-9w+5)}{5040(u-1)(v-1)(w-1)} g_{n+1} \right) \end{aligned}$$

$$\begin{aligned} A_0 \quad = \quad & \left(3u^2\left(8v^2\left(28w^2 - 2w - 1\right) + v(3 - 2w(8w+1)) + (3 - 8w)w\right) - u(2vw + v + w)(3v(8w-3) - 9w + 4)\right. \\ & \left. + vw(v(9 - 24w) + 9w - 4)\right) \end{aligned}$$

(2.6)

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$$\begin{aligned}A_1 \;=\;& 3u^2\left(8v^2(w(7w-12)+4)+v(6(27-16w)w-55)+w(32w-55)+20\right)+u\left(-3v^2(6w(16w-27)+55)\right.\\ &+v\left(486w^2-814w+282\right)+3(94-55w)w-104\big)+3v^2(w(32w-55)+20)+v(3(94-55w)w-104)\\ &+4w(15w-26)+40 \qquad (2.7)\end{aligned}$$

$$\begin{aligned}hy'_{n+1} =\;& y'_n+h^2y''_n+h^3\Big(\tfrac{A_0}{840u^2v^2w^2}f_n+\tfrac{(2v(7w-3)-6w+3)}{840(u-1)^2u^2(u-v)(u-w)}f_{n+u}\\ &+\tfrac{(u(6-14w)+6w-3)}{840(v-1)^2v^2(u-v)(v-w)}f_{n+v}\Big)\\ &+\tfrac{(u(6-14v)+6v-3)}{840(w-1)^2w^2(u-w)(w-v)}f_{n+w}+\tfrac{A_1}{840(u-1)^2(v-1)^2(w-1)^2}f_{n+1}\Big)\\ &+h^4\Big(\tfrac{(2u(7v(3w-1)-7w+3)+v(6-14w)+6w-3)}{840uvw}g_n+\tfrac{(2u(v(7-14w)+7w-4)+2v(7w-4)-8w+5)}{840(u-1)(v-1)(w-1)}g_{n+1}\Big)\end{aligned}$$

$$\begin{aligned}A_0 \;=\;& 2u^2\left(7v^2(w(21w-2)-1)-v\left(14w^2+w-3\right)+(3-7w)w\right)-u(2vw+v+w)(2v(7w-3)-6w+3)\\ &+vw(v(6-14w)+6w-3)\end{aligned}$$

$$\begin{aligned}A_1 \;=\;& 2u^2\left(7v^2(w(9w-16)+6)+v((197-112w)w-75)+42w^2-75w+30\right)+u\left(-2v^2(w(112w-197)+75)\right.\\ &+v\left(394w^2-690w+267\right)-3(w(50w-89)+36)\big)+3\left(v^2(28w^2-50w+20)\right.\\ &+v((89-50w)w-36)+4w(5w-9)+15\big) \qquad (2.8)\end{aligned}$$

$$\begin{aligned}h^2y''_{n+1} =\;& h^2y''_n+h^3\Big(\tfrac{A_0}{420u^2v^2w^2}f0+\tfrac{4-7w+7v(-1+2w)}{420(u-1)^2u^2(u-v)(u-w)}f_{n+u}\\ &-\tfrac{4-7w+7u(-1+2w)}{420(u-v)(v-1)^2v^2(v-w)}f_{n+v}+\tfrac{(-4+u(7-14v)+7v)}{420(u-w)(w-1)^2w^2(w-v)}f_{n+w}\\ &+\tfrac{A_1}{420(u-1)^2(v-1)^2(w-1)^2}f_{n+1}\Big)+h^4\Big(\tfrac{(-4+v(7-14w)+7w+7u(1-2w+v(-2+5w)))}{420uvw}g_n\\ &+\tfrac{(10-14w+7v(-2+3w)-7u(2-3w+v(-3+5w)))}{420(u-1)(v-1)(w-1)}\Big)\end{aligned}$$

$$\begin{aligned}A_0 \;=\;& \left(vw(-4+v(7-14w)+7w)+u\left(v(-4+6w)+t(-4+7w)+v^2\left(7-28w^2\right)\right)\right.\\ &+7u^2\left(v+w-2w^2-4vw^2+v^2\left(-2-4w+30w^2\right)\right)\big)\end{aligned}$$

$$\begin{aligned}A_1 \;=\;& 120+4v(-66+35v)-264w+5(118-63v)vw+7(20+3v(-15+8v))w^2\\ &+7u^2\left(20+2v^2(-2+3w)(-6+5w)\right. +3w(-15+8w)+v(-45+8(13-7w)w))\\ &+u\left(-264+5(118-63w)w-7v^2(45+8w(-13+7w))\right. +2v(295+w(-675+364w))) \qquad (2.9)\end{aligned}$$

formulas in (2)-(2) depends on u, v and w. Their values can be determined by considering the Local Truncation Errors (LTEs) of (2)-(2) respectively, so that one more order for each of the formulas can be gained by setting their principal truncation term to zero. consequently, the values of u, v and w are then optimized. The Obtaining the LTEs for each of u, v and w, we have the following

$$\mathcal{L}(y(t_{n+1}),h)=\tfrac{(2-4t+s(-4+9t)+r(-4+s(9-24t)+9t))y^{(10)}(t_n)h^{10}}{25401600}+O(h)^{11} \qquad (2.10)$$

$$\mathcal{L}(hy'(t_{n+1}),h)=\tfrac{(5-9t+9s(-1+2t)-3r(3-6t+2s(-3+7t)))y^{(10)}(t_n)h^{10}}{12700800}+O(h)^{11} \qquad (2.11)$$

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$$\mathcal{L}(h^2y''(t_{n+1}),h)=\frac{(5-8t+2s(-4+7t)-2r(4-7t+7s(-1+2t)))y^{(10)}(t_n)h^{10}}{4233600}+O(h)^{11} \tag{2.12}$$

Equating the principal terms in (2.10)-(2.12) to zero, then the following system of algebraic equations are obtained

$$\left.\begin{array}{l} 24uvw-9uv-9uw+4u-9vw+4v+4w-2=0\\ 42uvw-18uv-18uw+9u-18vw+9v+9w-5=0\\ 28uvw-14uv-14uw+8u-14vw+8v+8w-5=0 \end{array}\right\} \tag{2.13}$$

which gives the solution where $0<u<v<w<1$ as $u=\frac{1}{6}\left(3-\sqrt{3}\right),\ \ v=\frac{1}{2},\ \ w=\frac{1}{6}\left(3+\sqrt{3}\right)$. Plugging these values into (2)-(2), the main formulae for the approximation of solution of (1.1) are obtained

$$y_{n+1}=y_n+hy_n'+\frac{1}{2}h^2y_n''+h^3\left(\frac{3f_n}{70}+\left(\frac{3\sqrt{3}}{140}+\frac{3}{70}\right)f_{n+u}+\frac{4f_{n+v}}{105}-\left(\frac{3\sqrt{3}}{140}-\frac{3}{70}\right)f_{n+w}\right)+\frac{h^4g_n}{840} \tag{2.14}$$

$$hy_{n+1}'=hy_n'+h^2y_n''+h^3\left(\frac{37f_n}{420}+\left(\frac{3\sqrt{3}}{70}+\frac{9}{70}\right)f_{n+u}+\frac{16f_{n+v}}{105}-\left(\frac{3\sqrt{3}}{70}-\frac{9}{70}\right)f_{n+w}+\frac{f_{n+1}}{420}\right)+\frac{h^4g_n}{420} \tag{2.15}$$

$$h^2y_{n+1}''=h^2y_n''+h^3\left(\frac{19f_n}{210}+\frac{9f_{n+u}}{35}+\frac{32f_{n+v}}{105}+\frac{9f_{n+w}}{35}+\frac{19f_{n+1}}{210}\right)+h^4\left(\frac{g_n}{420}-\frac{g_{n+1}}{420}\right) \tag{2.16}$$

Thus, completing the block scheme, additional formulae are required to approximate the solution of (1.1). These formulae are determined by evaluating the formula in (2.4) and its derivative up to the 2nd at the point $t=t_{n+j}$, $j=u,v,w$, and substituting the optimal values $u=\frac{1}{6}\left(3-\sqrt{3}\right)\simeq 0.211325$, $v=\frac{1}{2}=0.5$, $w=\frac{1}{6}\left(3+\sqrt{3}\right)\simeq 0.788675$ to obtain the following one step formulae

$$\begin{aligned}
y_{n+u} = {} & y_n-\tfrac{1}{6}\left(\sqrt{3}-3\right)hy_n'-\tfrac{1}{12}\left(\sqrt{3}-2\right)h^2y_n''+\tfrac{h^3}{3265920}\left(\left(45036-23564\sqrt{3}\right)f_n\right.\\
& +630\sqrt{3}f_{n+u}+32\left(648-379\sqrt{3}\right)f_{n+v}+18\left(3888-2239\sqrt{3}\right)f_{n+w}\\
& \left.+4\left(81-59\sqrt{3}\right)f_{n+1}+\left(1269-668\sqrt{3}\right)hg_n+\left(20\sqrt{3}-27\right)hg_{n+1}\right)\\
y_{n+v} = {} & y_n+\tfrac{hy_n'}{2}+\tfrac{1}{8}h^2y_n''+\tfrac{h^3}{215040}\left(2208f_n+18\left(64+35\sqrt{3}\right)f_{n+u}\right.\\
& \left.+18\left(64-35\sqrt{3}\right)f_{n+w}-32f_{n+1}+h\left(67g_n+3g_{n+1}\right)\right)\\
y_{n+w} = {} & y_n+\tfrac{1}{6}\left(3+\sqrt{3}\right)hy_n'+\tfrac{1}{12}\left(2+\sqrt{3}\right)h^2y_n''+\tfrac{h^3}{3265920}\left(4\left(11259+5891\sqrt{3}\right)f_n\right.\\
& +18\left(3888+2239\sqrt{3}\right)f_{n+u}+32\left(648+379\sqrt{3}\right)f_{n+v}-630\sqrt{3}f_{n+w}\\
& \left.+4\left(81+59\sqrt{3}\right)f_{n+1}+\left(1269+668\sqrt{3}\right)hg_n-\left(27+20\sqrt{3}\right)hg_{n+1}\right)\\
hy_{n+u}' = {} & hy_n'-\tfrac{1}{6}\left(\sqrt{3}-3\right)h^2y_n''+\tfrac{h^3}{90720}\left(\left(3850-1366\sqrt{3}\right)f_n+630f_{n+u}\right.\\
& +16\left(241-144\sqrt{3}\right)f_{n+v}+18\left(379-216\sqrt{3}\right)f_{n+w}-2\left(19+\sqrt{3}\right)f_{n+1}\\
& \left.+\left(106-35\sqrt{3}\right)hg_n+\left(2+\sqrt{3}\right)hg_{n+1}\right)\\
hy_{n+v}' = {} & hy_n'-\tfrac{1}{2}\,h^2y_n''+\tfrac{h^3}{53760}\left(2342f_n+18\left(105+64\sqrt{3}\right)f_{n+u}+560f_{n+v}\right.\\
& \left.+18\left(105-64\sqrt{3}\right)f_{n+w}+38f_{n+1}+67hg_n-3hg_{n+1}\right)\\
hy_{n+w}' = {} & hy_n'+\tfrac{1}{6}\left(\sqrt{3}+3\right)h^2y_n''+\tfrac{h^3}{90720}\left(2\left(1925+683\sqrt{3}\right)f_n\right.\\
& +18\left(379+216\sqrt{3}\right)f_{n+u}+16\left(241+144\sqrt{3}\right)f_{n+v}+630f_{n+w}\\
& \left.+2\left(\sqrt{3}-19\right)f_{n+1}+\left(106+35\sqrt{3}\right)hg_n-\left(\sqrt{3}-2\right)hg_{n+1}\right)\\
h^2y_{n+u}'' = {} & h^2y_n''+\tfrac{h^3}{15120}\left(2\left(797+44\sqrt{3}\right)f_n+24\left(81-8\sqrt{3}\right)f_{n+u}\right.\\
& +64\left(36-23\sqrt{3}\right)f_{n+v}+24\left(81-43\sqrt{3}\right)f_{n+w}+\left(88\sqrt{3}-226\right)f_{n+1}\\
& \left.+\left(53+6\sqrt{3}\right)hg_n+\left(17-6\sqrt{3}\right)hg_{n+1}\right)\\
h^2y_{n+v}'' = {} & h^2y_n''+\tfrac{h^3}{3360}\left(257f_n+9\left(48+35\sqrt{3}\right)f_{n+u}+512f_{n+v}+9\left(48-35\sqrt{3}\right)f_{n+w}\right.\\
& \left.+47f_{n+1}+4hg_n-4hg_{n+1}\right)\\
h^2y_{n+w}'' = {} & h^2y_n''+\tfrac{h^3}{15120}\left(\left(1594-88\sqrt{3}\right)f_n+24\left(81+43\sqrt{3}\right)f_{n+u}\right.\\
& +64\left(36+23\sqrt{3}\right)f_{n+v}+24\left(81+8\sqrt{3}\right)f_{n+w}-2\left(113+44\sqrt{3}\right)f_{n+1}\\
& \left.+\left(53-6\sqrt{3}\right)hg_n+\left(17+6\sqrt{3}\right)hg_{n+1}\right)
\end{aligned} \tag{2.17}$$

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3 Characteristics of the OSFDBI

In this section the fundamental characteristics of the proposed method is discussed. This is based on the Dalhquist theory on linear-multistep approaches. Mainly discussed are the accuracy, consistency, zero-stability and linear stability analysis of the derived method.

Assume that $z(t)$ is a sufficiently differentiable function, we define the operator

$$\mathcal{L}[z(t);h] = z(t) - \sum_{i=0}^{2} \alpha_i h^i z_n^{(i)}(t) - h^3 \sum_{i=0}^{5} \left(\beta_i z'''(t)_{n+d_i} + h\gamma_0 z^4(t)_n + h\gamma_1 z^4(t)_{n+1}\right) \tag{3.1}$$

where $z^{(i)}(t) = \frac{d^i z}{dt^i}\Big|_{i=0,1,2}$, $d_0 = 0, d_1 = u, d_2 = v, d_3 = w$, $\alpha,\ \beta,\ \gamma$ are real coefficients.

Expanding (3.1) in Taylor series about x_n and collecting all the terms in h, then (3.1) takes the form

$$\mathcal{L}[z(x);h] = R_j h^{q+3} z^{(p+3)}(x) + O(h^{p+4}), \tag{3.2}$$

Here (3.2) is the local truncation error (LTE) with R_j being the principal error constant and q is the order of the corresponding formula (3.1).

In particular, if the formulas in (2.14)-(2.16) are considered, the following is obtained

$$\begin{aligned}
\mathcal{L}[y(x_n);h] = \quad & y(x_n+h) - y(x_n) + hy'(x_n) - \tfrac{1}{2}h^2 y''(x_n) \\
& + h^3\left(\tfrac{3}{70}y'''(x_n) + \left(\tfrac{3\sqrt{3}}{140} + \tfrac{3}{70}\right)y'''(x_n+hu) + \tfrac{4}{150}y'''(x_n+hv)\right. \\
& \left. + \left(\tfrac{3\sqrt{3}}{140} + \tfrac{3}{70}\right)y'''(x_n+hw)\right) - \tfrac{h^4}{840}y^{(iv)}(x_n) \\
\mathcal{L}[y'(x_n);h] = \quad & hy'(x_n+h) - hy'(x_n) - h^2 y''(x_n) \\
& - h^3\left(\tfrac{37}{420}y'''(x_n) + \left(\tfrac{3\sqrt{3}}{70} + \tfrac{9}{70}\right)y'''(x_n+uh) + \tfrac{16}{105}y'''(x_n+vh)\right. \\
& \left. - \left(\tfrac{3\sqrt{3}}{70} - \tfrac{9}{70}\right)y'''(x_n+wh) + \tfrac{y'''(x_n+h)}{420}\right) - \tfrac{h^4}{420}y^{(iv)}(x_n) \\
\mathcal{L}[y''(x_n);h] = \quad & h^2 y''(x_n+h) - h^2 y''(x_n) \\
& + h^3\left(\tfrac{19}{210}y'''(x_n) + \tfrac{9}{35}y'''(x_n+uh) + \tfrac{32}{105}y'''(x_n+vh) + \tfrac{9}{35}y'''(x_n+wh)\right. \\
& \left. + \tfrac{19}{210}y'''(x_n+h)\right) - h^4\left(\tfrac{y^{(iv)}(x_n)}{420} - \tfrac{y^{(iv)}(x_n+h)}{420}\right)
\end{aligned} \tag{3.3}$$

Expanding each operator in (3.3) in Taylor's series about x_n, the following are obtained

$$\begin{aligned}
\mathcal{L}[y(x_n);h] &= \tfrac{h^{11}}{26824089600}y^{(11)}(x_n) + O(h^{12}) \\
\mathcal{L}[y'(x_n);h] &= -\tfrac{h^{11}}{120708403200}y^{(11)}(x_n) + O(h^{12}) \\
\mathcal{L}[y''(x_n);h] &= \tfrac{h^{11}}{1207084032000}y^{(11)}(x_n) + O(h^{12})
\end{aligned} \tag{3.4}$$

For the formulas in (2.17), the LTEs may be obtained in the same way. The order q of the method is uniform, hence the block method has the same order $q = 8$, see [25].

3.1 Zero-stability and Convergence of the OSFDBI

The OSFDBI may be compactly written in the matric form as

$$A_1\bar{Y}_{n+1} = A_0 Y_{n-1} + h^3(B_1\bar{F}_{n+1} + hB_0\bar{G}_n) \tag{3.5}$$

where

$$\bar{Y}_{n+1} = (y_{n+u}, y_{n+v}, y_{n+w}, y_{n+1}, \ldots, y''_{n+1})^T,$$

$$\bar{Y}_{n-1} = (y_n, y_{n-v}, y_{n-w}, y_{n-1}, \ldots, y''_{n-1})^T,$$

$$\bar{F}_{n+1} = (f_n, f_{n+v}, f_{n+w}, f_{n+1}, \ldots, f''_{n-1})^T,$$

$$\bar{G}_n = (G_n, G_{n+v}, G_{n+w}, G_{n+1}, \ldots, G''_{n-1})^T,$$

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and A_0, A_1, B_0, B_1, are matrices of coefficients

Noting that zero-stability of the derived OSFDBI implies the behaviour of (3.1) as $h \to 0$. Thus as $h \to 0$, the scheme in (3.5) becomes

$$A_1 \bar{Y}_{n+1} = A_0 Y_{n-1} \tag{3.6}$$

where

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

, Hence, we compute the characteristic equation given by $|\xi A_1 - A_0| = 0$. The OSFDBI in (3.5) is said to be zero-stable if the roots ξ_j of the characteristic polynomial $\rho(\xi)$ given as $\rho(\xi) = |\xi A_1 - A_0|$ satisfy $|\xi_j| \leq 1$ and for those whose $|\xi_j| = 1$ have multiplicities which are not greater than 3, (see Modebei *et al.* 2019). So that $|\xi A_1 - A_0| = \xi^3(\xi - 1) = 0$. In this case, we have $\xi_j = 1$ for $j = 1$ and $\xi_j = 0$ for $j = 2, 3, 4$. This shows that OSFDBI is zero stable. Condition for convergence for linear multistep method, by Henrici states that, the method must be consistent with order $q > 1$ and at the same time zero stable. From the forgoing analysis, the OSFDBI is of order $q = 8$ and as well, zero stable. Hence the OSFDBI is convergent.

3.2 Computational procedure

The OSFDBI is implemented in block form (3.5) by using MAPLE 18 to find at the beginning of the process the function f and its derivative $f' = g$. this helps not to evaluate the higher derivatives appearing in the method on each step. A simple but efficient algorithm codded in Maple software for the implementation is show below: We begin by noting that the solution of the problem in equation (1.1) is sought in the subintervals $\pi_N = a = x_0 < x_1 < \ldots < x_N = b$, where $h = \frac{b-a}{N}$ is a constant step-size.
Step 1: Use the block method setting $n = 0$, to obtain V_1 in the interval $[y_0, y_1]$, similarly, for, $n = 1$ so that V_2 is obtained in the interval $[y_1, y_2]$, and so on, so that we obtain $V_3, V_4, \ldots, V_N$.
Step 2: Solve the unified block given by the system $V_1 \bigcup V_2 \bigcup, \cdots, V_N$ obtained in step 1. **Step 3**: The solution of equation (1.1) is approximated by the solutions in step 2 as $y_n = [y(x_1); y(x_2) \ldots y(x_n)]^T$ for $n = 1, 2, \ldots N$.

Algorithm 1 Block Algorithm for OSFDBI

1 **begin procedure**ENTER Partitions $(a, b, N, h, \text{variables})$ 2 For $x_n = x_{n-1} + h$, $n = 1, \ldots, N$, $h = \frac{b-a}{N}$ 3 Generate block system 4 Solve [Sysytem, variables] 5 Obtain y_n 6 **end procedure**

4 Numerical Examples

In this part, we implement the OSFDBI using two popular problems in literature so as to substantiate the high level of accuracy of this method. Here, two main numerical examples are presented to show the accuracy of the developed OSFDBI. In the examples considered the absolute errors were obtained as Err $= |y_i - y(x_i)$, where y_i is the approximate solution obtained using OSFDBI or any other numerical approach used for comparisons, and $y(x_i)$ is the exact solution of the problem considered at the grid points x_i $i = 1(1)N$. The approximate solutions with the BHI are compared with the approximate solutions obtained with different methods in the literature. For comparison we have considered the Exponential spline method (ESN) in [2] which is of order 6, the Quintic non-polynomial spline method (QNPSM) in [16] which is of order 4, Exponential quintic spline (EQS) method in [7] with no explicitly stated order, Quartic spline (QS) method in [14] of order 4

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and Quartic B-spline QB-S method in [15] of order 5. We have not found any other higher order methods as derived in this work in the literature for directly solving this singularly perturbed third order BVP. It is also important to note that the comparison has been done based on accuracy and ease of derivation of the method. Therefore, the results in the tables gives an idea of the good performance of the proposed method.

Problem 1. Consider the following singularly perturbed problem

$$\varepsilon y'''(x) + y(x) = f(x); \;\; 0 \leq x \leq 1,$$
$$y(0) = 0, \;\; y(1) = 0, \;\; y'(0) = 9\varepsilon, .$$

where $f(x) = 6\varepsilon x^3(1-x)^5 - 6\varepsilon^2(6(1-x)^5 - 90x(1-x)^4 + 180x^2(1-x)^3 - 60x^3(1-x)^2)$. The exact solution of the problem is $y(x) = 6x^3\varepsilon(1-x)^5$.

Table 1: Comparison of maximum absolute errors obtained for Problem 1.

ε	N=10	N=20	N=40
OSFDBI			
1/16	1.1623×10^{-8}	1.2542×10^{-11}	2.5754×10^{-13}
1/32	1.0232×10^{-9}	2.8357×10^{-12}	1.0777×10^{-13}
1/64	3.0114×10^{-9}	8.4523×10^{-12}	2.4143×10^{-14}
ESM			
1/16	1.0028×10^{-6}	7.7262×10^{-9}	5.9445×10^{-11}
1/32	4.2762×10^{-7}	3.2959×10^{-9}	2.5316×10^{-11}
1/64	1.7929×10^{-7}	1.3211×10^{-9}	1.0236×10^{-11}
EQS			
1/16	4.87×10^{-4}	1.86×10^{-5}	1.95×10^{-5}
1/32	1.95×10^{-4}	8.76×10^{-6}	8.63×10^{-6}
1/64	7.97×10^{-5}	4.00×10^{-6}	3.61×10^{-6}
EQ			
1/16	2.90×10^{-3}	1.20×10^{-4}	6.40×10^{-6}
1/32	9.20×10^{-4}	3.80×10^{-5}	2.10×10^{-6}
1/64	1.40×10^{-4}	6.80×10^{-6}	4.60×10^{-7}
EB-S			
1/16	4.70×10^{-4}	1.10×10^{-4}	2.60×10^{-5}
1/32	1.90×10^{-4}	4.70×10^{-5}	1.20×10^{-5}
1/64	8.00×10^{-5}	1.90×10^{-5}	4.80×10^{-6}

Table 2: Comparison of maximum absolute errors obtained for Problem 1.

ε	N=10	N=20	N=40	N=80	N=160
OSFDBI					
10^{-1}	2.54×10^{-10}	5.48×10^{-13}	6.65×10^{-14}	2.18×10^{-13}	2.71×10^{-14}
10^{-2}	1.24×10^{-10}	2.14×10^{-13}	7.12×10^{-15}	1.47×10^{-14}	2.58×10^{-15}
10^{-3}	5.51×10^{-11}	1.21×10^{-13}	1.54×10^{-16}	1.26×10^{-16}	4.16×10^{-16}
10^{-4}	4.18×10^{-12}	1.31×10^{-14}	1.12×10^{-16}	5.14×10^{-16}	2.32×10^{-18}
10^{-5}	1.11×10^{-14}	1.78×10^{-15}	3.21×10^{-17}	3.71×10^{-18}	3.41×10^{-20}
10^{-6}	1.29×10^{-15}	6.25×10^{-16}	4.44×10^{-19}	9.12×10^{-20}	5.50×10^{-22}
10^{-7}	1.47×10^{-16}	4.25×10^{-17}	2.15×10^{-19}	1.24×10^{-21}	2.01×10^{-22}
QNPSM					
10^{-1}	1.20×10^{-5}	2.02×10^{-7}	3.20×10^{-9}	4.76×10^{-11}	5.56×10^{-13}
10^{-2}	6.68×10^{-7}	1.13×10^{-8}	1.78×10^{-10}	2.71×10^{-12}	3.66×10^{-14}
10^{-3}	3.30×10^{-8}	5.24×10^{-10}	8.55×10^{-12}	1.32×10^{-13}	1.96×10^{-15}
10^{-4}	1.20×10^{-9}	2.70×10^{-11}	3.89×10^{-13}	6.31×10^{-15}	9.68×10^{-17}
10^{-5}	1.69×10^{-11}	8.96×10^{-13}	2.10×10^{-14}	2.85×10^{-16}	4.48×10^{-18}
10^{-6}	1.76×10^{-13}	1.19×10^{-14}	6.26×10^{-16}	1.56×10^{-17}	2.20×10^{-19}
10^{-7}	1.76×10^{-15}	1.23×10^{-16}	7.93×10^{-18}	4.21×10^{-19}	1.34×10^{-20}

The Table 1-2 shows the results obtained using OSFDBI and compared to the methods in [2] and [16] respectively. The OSFDBI compared favourably with both methods.

Problem 2. Consider the following singularly perturbed problem

$$-\varepsilon y'''(x) + y(x) = f(x); \quad 0 \leq x \leq 1,$$
$$y(0) = 0, \quad y(1) = 3\varepsilon \sin(3), \quad y'(0) = 9\varepsilon, .$$

where $f(x) = 81\varepsilon^2 \cos(3x) + 3\varepsilon \sin(3x)$. The exact solution of the problem is $y(x) = 3\varepsilon \sin(3x)$.

Table 3: Comparison of maximum absolute errors obtained for Problem 2.

ε	N=10	N=20	N=40
OSFDBI			
$1/16$	3.2543×10^{-10}	4.4587×10^{-13}	1.4781×10^{-14}
$1/32$	2.1245×10^{-10}	2.1451×10^{-13}	2.1457×10^{-15}
$1/64$	1.2450×10^{-10}	2.1124×10^{-13}	1.2145×10^{-15}
ESM			
$1/16$	4.4336×10^{-8}	2.0866×10^{-10}	1.0750×10^{-12}
$1/32$	1.8916×10^{-8}	8.8814×10^{-11}	4.5211×10^{-13}
$1/64$	7.9396×10^{-9}	3.5668×10^{-11}	1.8448×10^{-13}
EQS			
$1/16$	2.32×10^{-4}	6.12×10^{-5}	1.52×10^{-5}
$1/32$	9.77×10^{-5}	2.59×10^{-5}	6.45×10^{-6}
$1/36$	3.78×10^{-5}	1.00×10^{-6}	2.50×10^{-6}
EB-S			
$1/16$	2.40×10^{-4}	6.10×10^{-5}	1.50×10^{-5}
$1/32$	1.00×10^{-4}	2.60×10^{-5}	6.40×10^{-6}
$1/36$	4.00×10^{-5}	1.00×10^{-6}	2.50×10^{-6}

Table 4: Comparison of maximum absolute errors obtained for Problem 2.

ε	N=10	N=20	N=40	N=80	N=160
OSFDBI					
10^{-1}	1.78×10^{-9}	3.25×10^{-10}	2.48×10^{-15}	2.01×10^{-14}	1.98×10^{-15}
10^{-2}	3.25×10^{-9}	2.22×10^{-12}	1.11×10^{-15}	1.28×10^{-14}	3.61×10^{-15}
10^{-3}	2.77×10^{-10}	1.79×10^{-12}	2.57×10^{-15}	9.11×10^{-16}	5.17×10^{-17}
10^{-4}	8.14×10^{-12}	9.48×10^{-14}	3.45×10^{-15}	2.74×10^{-16}	1.23×10^{-19}
10^{-5}	2.54×10^{-13}	9.12×10^{-15}	4.12×10^{-17}	2.27×10^{-19}	3.24×10^{-20}
10^{-6}	2.84×10^{-15}	8.27×10^{-18}	5.55×10^{-19}	6.67×10^{-20}	3.57×10^{-22}
10^{-7}	1.10×10^{-16}	1.23×10^{-19}	2.79×10^{-20}	1.22×10^{-21}	2.54×10^{-23}
Method in [16]					
10^{-1}	5.45×10^{-7}	8.51×10^{-9}	1.27×10^{-10}	3.83×10^{-12}	2.56×10^{-12}
10^{-2}	3.12×10^{-5}	4.82×10^{-10}	7.18×10^{-12}	1.00×10^{-13}	4.76×10^{-14}
10^{-3}	1.66×10^{-9}	2.32×10^{-11}	3.53×10^{-13}	5.21×10^{-15}	4.02×10^{-16}
10^{-4}	6.05×10^{-11}	1.30×10^{-12}	1.67×10^{-14}	2.56×10^{-16}	7.28×10^{-18}
10^{-5}	9.57×10^{-13}	4.24×10^{-14}	9.58×10^{-16}	1.19×10^{-17}	1.34×10^{-19}
10^{-6}	1.28×10^{-14}	6.89×10^{-16}	2.84×10^{-17}	6.93×10^{-19}	8.60×10^{-21}
10^{-7}	1.15×10^{-16}	7.78×10^{-18}	4.59×10^{-19}	1.87×10^{-20}	4.99×10^{-22}

As observed in the exponential spline method (ESM) in [2] and the Quintic non-polynomial spline method (QNPSM) in [16], the absolute errors shows a decrease as the perturbation parameter ε decreases and as well the values of N increases. The results obtained in Tables 1-4 by the proposed method have been compared with the numerical results obtained in [2] and [16]. It can be seen that the proposed method performed better and shows more accuracy. .

5 Conclusion

In this paper, the One-step fourth derivative block integrator (OSFDBI) have been derived and implemented on two most common problems in literature to obtain the numerical solution of singularly perturbed third-order boundary value problems. The one-step idea in this paper was to economize the number of function evaluation so as to reduce the cost of implementation in terms of CPU time. In the course of implementation, the perturbation parameter as well as the number of subinterval have been varied in other to know the extents with which the proposed method can be more accurate with the compared methods. The tables of absolute errors shows clearly established the fact that the proposed method out performed the other two methods compared in literature.

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