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Inertial Iterative Algorithm of Generalized f - Projection Method for Fixed point Problem, Maximal Monotone Operators and Generalized Mixed Equilibrium Problems

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Abstract

The aim of this article is to investigate fixed point problem, maximal monotone operators and generalized mixed equilibrium problems by considering the generalized $f-$ projection technique. We propose a modified inertial based algorithm for finding a common solution in respect of this problem. Also, we prove a strong convergence of the sequence generated by the proposed modified inertial iterative algorithm in uniformly smooth and uniformly convex Banach spaces. Finally, we give some applications of our theorem.

Keywords: Inertial, Generalized f - Projection, Fixed Point Problem, Maximal Monotone Operators, Generalized Mixed Equilibrium Problems, Strong Convergence.
MSC2010: 47H09, 47J25.

1 introduction

The fixed point theory has played an important role in the field of Mathematics, especially in the area of nonlinear analysis and other related areas in pure and applied mathematics. Due to its importance, many researchers have considered it as one of the most interesting area in mathematics. Many authors have developed several iterative processes for approximating fixed points of nonexpansive mappings including their generalizations: for more detail see [1–13] and the reference therein.

Let E a real Banach space with its dual as E^*, $\mathbb{R}$ denote the set of real numbers and Q be a nonempty closed convex subset of E. We consider $GMEP$ [1] as the generalized mixed equilibrium problem: find $\omega \in Q$ such that

$$B(\omega,\vartheta) + \langle A\omega, \vartheta - \omega\rangle + b(\omega,\vartheta) - b(\omega,\omega) \geq 0, \forall \vartheta \in Q,$$

where $B, b : Q \times Q \longrightarrow \mathbb{R}$ are bifunctions and $A : Q \longrightarrow E^*$ is a nonlinear mapping. The set of solutions of generalized mixed equilibrium problem is denoted by

$$GMEP(B, A, b) = \{\omega \in Q : B(\omega, \vartheta) + \langle A\omega, \vartheta - \omega\rangle + b(\omega, \vartheta) - b(\omega, \omega) \geq 0, \forall \vartheta \in Q\}.$$

Furthermore, if $A \equiv 0$ and $b \equiv 0$, then the generalized mixed equilibrium problem $(GMEP)$ reduces to equilibrium problem, denoted by EP [14] which is defined as to find an element $\omega \in Q$ such that

$$B(\omega, \vartheta) \geq 0, \forall \vartheta \in Q.$$

The solutions set of equilibrium problems is given by

$$EP(B) = \{\omega \in Q : B(\omega, \vartheta) \geq 0, \forall \vartheta \in Q\}.$$

Generalized mixed equilibrium problems have been considered as a cornerstone for research in the field of science and engineering. Also, it is used in structural analysis, physics, economics and other science and social sciences. More so, it is found in optimization problems, Nash equilibrium problem in non cooperative game, variational inequality problem, variational inclusion problem, fixed point problem etc (for details see [1,3,8]).
For the purpose of fast convergence of the iterative algorithm, an inertial- type extrapolation technique was first introduced by Polyak [15] as a process of accelerating the rate of convergence of the sequence. Due to the importance of this technique along this direction, many authors have been studied this techniques extensively (for details see [4,5,16,17]).
Consider E as a Banach space and S as a maximal monotone operator then the problem for solving a zero point of a maximal monotone operator: $u^* \in E$ such that

$$0 \in S(u^*).$$

$S^{-1}0$ denotes the set of all point $u^* \in E$ such that $0 \in S(u^*)$. This considered as efficient tool for solving problems arising in optimization, analysis and other related field of research.
By considering $\omega \in E$ and $\omega^* \in E^*$, then $\langle \omega, \omega^*\rangle$ is the set valued of ω^* at ω. Therefore $J : E \longrightarrow 2^{E^*}$ denoted as the normalized duality mapping and defined by

$$J(\omega) = \big\{\omega^* \in E^* : \langle \omega, \omega^*\rangle = \|\omega\|^2, \|\omega^*\| = \|\omega\|\big\}, \forall \omega \in E.$$

For E as a Hilbert space, we observe that $J = I$, where I denote the identity map. The Lyapunov functional $\phi : E \times E \longrightarrow \mathbb{R}$ defined by

$$\phi(\vartheta, \omega) = \|\vartheta\|^2 - 2\langle \vartheta, J\omega\rangle + \|\omega\|^2, \forall \omega, \vartheta \in E. \tag{1.1}$$

An operator $S \subset E \times E^*$ is called monotone if $\langle \omega - \vartheta, \omega^* - \vartheta^*\rangle \geq 0$, whenever $(\omega, \omega^*), (\vartheta, \vartheta^*) \in S$. A monotone S is called maximal if its graph $G(S)$ is not properly contained in the graph of any other monotone operator. A mapping $T : Q \longrightarrow Q$ is said to be nonexpansive [2, 18] if $\| T\omega - T\vartheta \| \leq \| \omega - \vartheta \|, \forall \omega, \vartheta \in Q$, we denote $F(T) = \{\omega \in Q : T\omega = \omega\}$ as the set of fixed point of T. A point $p \in Q$ is said to be an asymptotic fixed point of T, if Q contains a sequence $\{\omega_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \| \omega_n - T\omega_n \| = 0$. $\hat{F}(T)$ denote the set of asymptotic fixed point of T, and a mapping T is said to be $L-Lipschitz$ *continuous* if there exists a constant $L > 0$ such that $\| T\omega - T\vartheta \| \leq L \| \omega - \vartheta \|, \forall \omega, \vartheta \in Q$. S is called closed if for any sequence $\{\omega_n\} \subset Q$ with $\omega_n \longrightarrow \omega$ and $S\omega_n \longrightarrow \vartheta$ then $\vartheta = S\omega$.

Definition 1.1. *Let $\{T_i\}_{i=1}^{\infty} : Q \longrightarrow Q$ be a sequence of mapping. Then $\{T_i\}_{i=1}^{\infty}$ is said to be:*

(1) A family of uniformly quasi-ϕ-asymptotically nonexpansive [2, 18], if $\Gamma := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ such that for each $i \geq 1$

$$\phi(p, T_i^n\omega) \leq k_n\phi(p, \omega), \quad \forall \omega \in Q, p \in \Gamma, n \geq 1;$$

(2) A family of uniformly total quasi-ϕ-asymptotically nonexpansive [2, 18], if $\Gamma := \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists nonnegative real sequences ζ_n, μ_n with $\zeta_n \longrightarrow 0$, $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and strictly increasing continuous function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\psi(0) = 0$ such that for each $i \geq 1$

$$\phi(p, T_i^n \omega) \leq \phi(p, \omega) + \zeta_n \psi\big(\phi(p, \omega)\big) + \mu_n, \forall \omega \in Q, p \in \Gamma, n \geq 1.$$

(3) A mappings $T : Q \longrightarrow Q$ is said to be uniformly L- Lipschits continuous, if there exists a constant $L > 0$ such that

$$\| T^n \omega - T^n \vartheta \| \leq L \| \omega - \vartheta \|, \forall \omega, \vartheta \in Q, \forall n \geq 1.$$

Alber [19] introduced and studied that the generalised projection $\Pi_Q : E \longrightarrow Q$ is a map assigns to an arbitrary point $\omega \in E$ the minimum point of the functional $\phi(\vartheta, \omega)$; that is, $\Pi_Q(\omega) = \omega^*$, where ω^* is the solution to the minimization problem

$$\phi(\omega^*, \omega) = \min_{\omega_0 \in Q} \phi(\vartheta, \omega).$$

Existence and the uniqueness of the operator Π_Q follows from the strict monotonicity of the mapping J and properties of the functional $\phi(\vartheta, \omega)$. If E is a real Hilbert space H, then $\phi(\vartheta, \omega) = \|\vartheta - \omega\|^2$ and Π_Q become the metric projection of E onto Q (for details see [5, 20, 21]).

In 2006, using the technique of generalized $f-$ projection in Banach space, Wu and Huang [22] established properties of the generalized $f-$ projection operator as well as extended the definition of generalized projection operator which was proposed and studied by Alber [19]
Consider the functional $G : Q \times E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ defined by:

$$G(\vartheta, q) = \| \vartheta \|^2 - 2\langle \vartheta, q \rangle + \| q \|^2 + 2\varpi f(\vartheta), \tag{1.2}$$

where $\vartheta \in Q, q \in E^*, \varpi$ is positive number and $f : Q \longrightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi continuous. By considering the definitions G and f, the following properties studied by Wu and Huang [22] hold:
i) $G(\vartheta, q)$ is convex and continuous with respect to q when ϑ is fixed;
ii) $G(\vartheta, q)$ is convex and lower semicontinuous with respect to ϑ when q is fixed.

Definition 1.2. *Let Q be a nonempty closed convex subset of a real Banach space E with E^* as its dual. Then an operator $\Pi_Q^f : E^* \longrightarrow 2^Q$ is called generalized f- projection if*

$$\Pi_Q^f q = \{v \in Q : G(v, q) = \inf_{y \in Q} G(\vartheta, q), \forall q \in E^*\}.$$

In 2010, Li et al [23] proposed generalized $f-$ projection operator and proved the strong convergence theorem for relatively nonexpansive mapping. Later Siwaporn and Kumam [24] introduced hybrid algorithm of generalized $f-$ projection operator for finding the solution of generalized Kly Fan inequalities and fixed point problem in Banach space.
In 2013, Siwaporm et al [18] consider the following Mann type iterative algorithm for approximating the totally quasi -ϕ- asymptotically nonexpansive maps by the method of hybrid generalized $f-$ projection.

$$\begin{cases} Q_{1,j} = Q, \ \forall j \geq 1; \\ y_{n,j} = J^{-1}(\beta_n J\omega_n + (1 - \beta_n) J T_j^n \omega_n); \\ Q_{n+1,j} = \{u \in Q_n : G(u, J y_{n,j}) \leq G(u, J\omega_n) + \delta_n\}; \\ Q_{n+1} = \cap_{j=1}^{\infty} Q_{n+1,j}; \\ \omega_{n+1} = \Pi_{Q_{n+1}}^f \omega_1, \forall n \geq 1. \end{cases}$$

They proved that $\{\omega_n\}$ converges strongly to $\Pi_\Gamma^f \omega_1$.
In 2014, Jingling et al [11] considered the following algorithm for approximating the common element

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of generalized mixed equilibrium problem, maximal monotone operator and relatively nonexpansive map in Banach space.

$$\begin{cases} x \in C_0, \ arbitrarily; \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JS_n x_n); \\ z_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)JJ_{r_n} y_n); \\ u_n \in C \ such \ that f(u_n, y) + \psi(y) - \psi(u_n) + \langle Au_n, y - u_n \rangle \\ +\frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n \rangle \geq 0, \forall y \in C; \\ C_{n+1} = \{u \in C_n : G(u, Ju_n) \leq \gamma_n G(u, Jx_n) + (1-\gamma_n)G(u, Jy_n) \leq G(u, Jx_n)\}; \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \geq 0. \end{cases}$$

It has been proved that $\{x_n\}$ generated by the scheme above converges strongly to $\Pi^f_\Omega x_0$.
In 2021, Hammad et al [6] constructed a hybrid iterative algorithm for solving maximal monotone operators and fixed point problem in Banach space. From the notion of generalized $f-$ projection, Siwaporn Soewan [25] proposed and studied hybrid algorithm for finding a maximal monotone operator in Banach space, using the following iterative algorithm:

$$\begin{cases} x_1 \in C, \ C_1 = C, \\ z_n = J^{-1}(\gamma_n Jx_n + (1-\gamma_n)JJ_{r_n} x_n), \\ C_{n+1} = \{z \in C : G(z, Jz_n) \leq G(z, Jx_n), \\ x_{n+1} = \Pi^f_{C_{n+1}} x_1, \forall n \geq 1. \end{cases}$$

The authors proved that $\{x_n\}$ converges strongly to $\Pi^f_\Omega x_1$.

Motivated by the results Siwaporm et al [18], Jingling et al [11], In this article, we propose and study a modified inertial iterative algorithm for approximating a common fixed point of total quasi -ϕ- asymptotically nonexpansive mappings, maximal monotone operators and a system of generalized mixed equilibrium problems. We prove a strong convergence theorem of the proposed modified inertial iterative algorithm in Banach spaces. The results presented in this work, extend and improve the results of Siwaporm et al [18], Jingling et al [11] and many other results in the literature.

2 Preliminaries

In this section, we consider some preliminary definitions and Lemmas that led to the proving of our main result.

Let E be a real Banach space with $\| \ . \ \|$ and E^* as the norm and dual space of E respectively, $K := \{\omega \in E : \|\omega\| = 1\}$ be the unit sphere of E. E is said to be smooth if the $\lim_{t \to 0} \frac{\| \omega + t\vartheta \| - \| \omega \|}{t}$ exists for all $\omega, \vartheta \in K$, it is also said to be uniformly smooth if the limit exists uniformly in $\omega, \vartheta \in K$. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ defined by

$$\rho_E(t) = sup\{\frac{\| \omega + \vartheta \| + \| \omega - \vartheta \|}{2} - 1; \|\omega\| = 1, \|\vartheta\| \leq t\}.$$

A Banach space E said to be strictly convex if $\frac{\| \omega + \vartheta \|}{2} < 1$ for all $\omega, \vartheta \in K$ with $\|\omega\| = \|\vartheta\| = 1$ and $\omega \neq \vartheta$ and E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\| \omega + \vartheta \|}{2} \leq 1 - \delta$ for all $\omega, \vartheta \in K$ with $\|\omega\| = \|\vartheta\| = 1$ and $\|\omega - \vartheta\| \geq \varepsilon$. The modulus of convexity of E is the function $\delta : [0, 2] \longrightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = inf\{1 - \|\frac{\omega + \vartheta}{2}\| : \omega, \vartheta \in K, \|\omega\| = \|\vartheta\| = 1, \|\omega - \vartheta\| \geq \varepsilon\}.$$

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It follows from (1.1) that

$$(\|\vartheta\| - \|\omega\|)^2 \le \phi(\omega, \vartheta) \le (\|\vartheta\| + \|\omega\|)^2, \quad \forall \omega, \vartheta \in E; \tag{2.1}$$

$$\phi(\omega, \vartheta) = \phi(\omega, z) + \phi(z, \vartheta) + 2\langle \omega - z, Jz - J\vartheta\rangle, \quad \forall \omega, \vartheta, z \in E; \tag{2.2}$$

and

$$\phi(\omega, \vartheta) \le \|\omega\|\|J\omega - J\vartheta\| + \|\vartheta\|\|\omega - \vartheta\|, \quad \forall \omega, \vartheta, \in E. \tag{2.3}$$

Remark 2.1. *We observe from the basic properties of E, E^* and J that the following holds [25]:*

i) If E is a smooth, then J is single valued and semi continuous;
ii) If E is uniformly smooth, then E is smooth and reflexive;
iii) If E is an arbitrary Banach space, then J is monotone and bounded;
iv) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subset of E.
v) E is uniformly smooth if and only if E^* is uniformly convex;
vi) If E is a strictly convex, then J is strictly monotone;
vii) If E is reflexive, smooth and strictly convex, then the normalized duality mapping J is single valued, one-to-one and onto.

Remark 2.2. *If E is a reflexive, strictly convex and smooth Banach space, for each $\omega, \vartheta \in E$, $\phi(\omega, \vartheta) = 0$ if and only if $\omega = \vartheta$. It is enough to conclude that for $\phi(\omega, \vartheta) = 0$, then we have $\omega = \vartheta$. By (i), we notice that $\| \omega \|^2 = \| \vartheta \|^2$. This gives $\langle \omega, J\vartheta\rangle = \|\omega\|^2 = \|J\vartheta\|^2$. Observe by definition of J that $J\omega = J\vartheta$. Hence, this lead to $\omega = \vartheta$ (see for example [25, 31] and therein)*

Lemma 2.3. *(see [26]) Let E be a smooth and uniformly convex Banach space and let $\{\omega_n\}$ and $\{\vartheta_n\}$ be sequences in E such that either $\{\omega_n\}$ or $\{\vartheta_n\}$ is bounded. If $\lim_{n\to\infty} \phi(\omega_n, \vartheta_n) = 0$, then $\lim_{n\to\infty} \| \omega_n - \vartheta_n \| = 0$.*

Remark 2.4. *If $\{\omega_n\}$ and $\{\vartheta_n\}$ are bounded, from (2.3) it is obvious that the converse of Lemma 2.3 is also true.*

Lemma 2.5. *(see [2]) Let Q be a nonempty closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $T : Q \longrightarrow Q$ be a closed and total quasi-ϕ-asymptotically nonexpansive mapping with sequences $\{\zeta_n\}$, $\{\mu_n\}$ of nonnegative real numbers with $\zeta_n \longrightarrow 0, \mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\psi(0) = 0$. If $\mu_1 = 0$, then the fixed point set $F(T)$ is a closed convex subset of Q.*

Lemma 2.6. *(see [22]) Let E be a reflexive Banach space with its dual E^* and Q be a nonempty closed convex subset of E. The following statements hold:*
i) $\Pi_Q^f q$ is nonempty closed convex subset of Q for all $q \in E^*$;
ii) If E is smooth, then for all $q \in E^*, \omega \in \Pi_Q^f q$ if and only if

$$\langle \omega - \vartheta, q - J\vartheta\rangle + \varpi f(\vartheta) - \varpi f(\omega) \ge 0, \forall \vartheta \in Q;$$

iii) If E is strictly convex and $f : Q \longrightarrow \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(\xi\omega) = \xi f(\omega)$ for all $\xi > 0$ such that $\xi\omega \in Q$ where $\omega \in Q$), then Π_Q^f is single valued mapping.

Lemma 2.7. *(see [27]) Let Q be nonempty closed convex subset of a reflexive Banach space E and E^* be the dual space of E. If E is strictly convex, then $\Pi_Q^f q$ is single valued.*

Recall that if E is a smooth Banach space, then J is single valued mapping. Therefore, there exists a unique element $q \in E^*$ such that $q = J\omega$ for $\omega \in E$. Now, by substituting $q = J\omega$ in (1.2), we obtain

$$G(\vartheta, J\omega) = \| \vartheta \|^2 - 2\langle \vartheta, J\omega\rangle + \|J\omega\|^2 + 2\varpi f(\vartheta). \tag{2.4}$$

It follows from the definition of G that

$$G(\vartheta, J\omega) = G(\vartheta, Jz) + \phi(z, \omega) + 2\langle \vartheta - z, Jz - J\omega \rangle, \forall \omega, \vartheta, z \in E. \tag{2.5}$$

Also, by the notion of the second generalized f- projection in Banach spaces,

Definition 2.8. *(see [23]) Let Q be a nonempty closed convex subset of a real smooth Banach space E. An operator $\Pi_Q^f : E \longrightarrow 2^Q$ is said to be generalized f- projection if*

$$\Pi_Q^f \omega = \{v \in Q : G(v, J\omega) = \inf_{\vartheta \in Q} G(\vartheta, J\omega), \forall \omega \in E\}.$$

Lemma 2.9. *(see [28]) Let E be a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicountinuous convex functional. There exists $z^* \in E^*$, $\eta \in \mathbb{R}$ such*

$$f(\omega) \geq \langle \omega, z^* \rangle + \eta, \forall \omega \in E.$$

Lemma 2.10. *(see [23]) Let Q be a nonempty closed convex subset of a reflexive smooth Banach space E. Then, the following statements hold:*
i) $\Pi_Q^f \omega$ is nonempty closed convex subset of Q for all $\omega \in E$;
ii) for all $\omega \in E$, $\hat{\omega} \in \Pi_Q^f$ if and only if

$$\langle \hat{\omega} - \vartheta, J\omega - J\hat{\omega} \rangle + \varpi f(\vartheta) - \varpi f(\hat{\omega}) \geq 0, \forall \vartheta \in Q;$$

iii) If E is strictly convex, then Π_Q^f is single valued mapping.

Lemma 2.11. *(see [23]) Let Q be a nonempty closed convex subset of a reflexive smooth Banach space E. and $\hat{\omega} \in \Pi_Q^f$ for all $\omega \in E$. Then*

$$\phi(\vartheta, \hat{\omega}) + G(\hat{\omega}, J\omega) \leq G(\vartheta, J\omega), \forall \vartheta \in Q.$$

Remark 2.12. *Let E be a uniformly smooth and uniformly convex Banach space, $f(\omega) = 0, \forall \omega \in E$. It follows from Alber [19] that Lemma 2.11 reduces to the property of the generalized projection operator.*

If $f(\vartheta) \geq 0, \forall \vartheta \in C$ and $f(0) = 0$, then it follows from the definition of totally quasi-ϕ-asymptotically nonexpansive mapping T that T is equivalent to the following:
If $F(T) \neq \emptyset$ and there exists nonnegative real sequences $\{\zeta_n\}$, $\{\mu_n\}$ with $\zeta_n \longrightarrow 0$, $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and a strictly increasing continuous function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$G(p, JT^n\omega) \leq G(p, J\omega) + \zeta_n \psi(G(p, J\omega)) + \mu_n, \quad \forall \omega \in Q, p \in F(T), n \geq 1.$$

Lemma 2.13. *(see [23]) Let E be a Banach space and $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicountinuous mapping with domain $D(f)$. If $\{\omega_n\} \subset D(f)$ such that $\omega_n \rightharpoonup \hat{\omega} \in D(f)$ and $G(\omega_n, J\vartheta) \longrightarrow G(\hat{\omega}, J\vartheta)$ (as $n \to \infty$), then $\| \omega_n \| \longrightarrow \| \hat{\omega} \|$ (as $n \to \infty$).*

Lemma 2.14. *(see [29]) Let Q be a nonempty closed convex subset of strictly convex, smooth and reflexive Banach space E, let $S \subset E \times E^*$ be a monotone operator satisfying $D(S) \subset Q \subset J^{-1}(\cap_{r>0} R(J + rS))$. Let J_r and S_r, for all $r > 0$ be the resolvent and the Yosida approximation of S, respectively. The following statements hold:*
i) $\phi(v, J_r\omega) + \phi(J_r\omega, \omega) \leq \phi(v, \omega), \forall \omega \in Q, v \in S^{-1}0$;
ii) $(J_r\omega, S_r\omega) \in S, \forall \omega \in Q$, where $(\omega, \omega^*) \in S$ denotes the value of ω^* at $\omega(\omega^* \in S\omega)$ iii) $F(J_r) = S^{-1}0$.

Lemma 2.15. *(see [25]) Let E be a strictly convex, smooth and reflexive Banach space, $S \subset E \times E^*$ be a monotone operator with $S^{-1}0 \neq \emptyset$, and for each $r > 0$, $J_r = (J + rS)^{-1}J$. Then*

$$G(q, JJ_r\omega) + \phi(J_r\omega, \omega) \leq G(q, J\omega), \forall \omega \in E, q \in S^{-1}0.$$

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Lemma 2.16. *(see [14, 21]) Let E be a smooth, strictly convex and reflexive Banach space, and Q be a nonempty closed convex subset of E. Let $B : Q \times Q \longrightarrow \mathbb{R}$ be a bifunction satisfying the conditions $(B_1) - (B_4)$. Let $r > 0$ be any given number and $\omega \in E$ be any given point. Then, there exists $z \in Q$ such that*

$$B(z, \vartheta) + \frac{1}{r}\langle \vartheta - z, Jz - J\omega\rangle \geq 0, \forall \vartheta \in Q.$$

By replacing ω with $J^{-1}(J\omega - rA\omega)$, where A is a monotone mapping from Q into E^*, then there exists $z \in Q$ such that

$$B(z, \vartheta) + \langle \vartheta - z, Az\rangle + \frac{1}{r}\langle \vartheta - z, Jz - J\omega\rangle \geq 0.\forall \vartheta \in Q.$$

***Assumption B**: Consider the bifunction $B : Q \times Q \longrightarrow \mathbb{R}$ satisfies the following assumptions:*
(B_1) $B(\omega, \omega) = 0, \forall \omega \in Q$;
(B_2) B is monotone, 1.e, $B(\omega, \vartheta) + B(\vartheta, \omega) \leq 0,\ \forall \omega, \vartheta \in Q$;
(B_3) for each $\omega, \vartheta, z \in Q, \limsup_{\pi \to 0} B(\pi z + (1 - \pi)\omega, \vartheta) \leq B(\omega, \vartheta)$;
(B_4) for each $\omega \in Q, \vartheta \mapsto B(\omega, \vartheta)$ is convex and lower semicontinuous.

***Assumption b**: Also consider $b : Q\times Q \longrightarrow \mathbb{R}$ as a bifunction satisfying the following assumptions:*
(b_1) b is skew-symmetric, i.e., $b(\omega, \omega) - b(\omega, \vartheta) - b(\vartheta, \omega) + b(\vartheta, \vartheta) \geq 0, \forall \omega, \vartheta \in Q$;
(b_2) b is convex in the second argument;
(b_3) b is continuous.

Lemma 2.17. *(see [10, 30]) Let Q be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E . Let $A : Q \longrightarrow E^*$ be a continuous and monotone mapping, $B : Q \times Q \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumptions $(B_1) - (B_4)$ and $b : Q \times Q \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumptions $(b_1) - (b_3)$. For any given number $r > 0$ and $\omega \in E$, define a mapping $T_r : E \longrightarrow Q$ by*

$$T_r(\omega) = \{z \in Q : B(z, \vartheta) + \langle \vartheta - z, Az\rangle + \frac{1}{r}\langle \vartheta - z, Jz - J\omega\rangle + b(z, \vartheta) - b(z, z) \geq 0, \forall \vartheta \in Q\},$$

$\forall \omega \in E$.
The mapping T_r has the following properties:
(p_1) T_r is single-valued;
(p_2) T_r is a firmly nonexpansive - type mapping, for all $\omega, \vartheta \in E$

$$\langle T_r\omega - T_r\vartheta, JT_r\omega - JT_r\vartheta\rangle \leq \langle T_r\omega - T_r\vartheta, J\omega - J\vartheta\rangle$$

(p_3) $F(T_r) = GMEP(B, A, b)$;
(p_4) $GMEP(B, A, b)$ is a closed convex set of Q.
(p_5) $\phi(p, T_r\omega) + \phi(T_r\omega, \omega) \leq \phi(p, \omega),\quad \forall p \in F(T_r),\quad \omega \in E.$

3 Main result

Theorem 3.1. *Let Q be a nonempty closed and convex subset of a uniformly smooth and uniformly convex real Banach space E. Let $f : E \longrightarrow \mathbb{R}$ be a convex and lower semicontinuous function with $Q \subset int(D(f))$, where $D(f)$ is the domain of f and $S_i \subset E \times E^*, i = 1, 2, 3, ...$ be a sequence of maximal monotone operators satisfying $D(S_i) \subset Q$ and $J_{r_n} = (J + r_nS_i)^{-1}J$, for all $r_n > 0$ and $i = 1, 2, 3, ...$ Let $B_i : Q \times Q \longrightarrow \mathbb{R}, i = 1, 2, 3, ...$ be a sequence of bifunctions satisfying assumptions $(B_1) - (B_4)$, $b_i : Q \times Q \longrightarrow \mathbb{R}, i = 1, 2, 3, ...$ be a sequence of bifunctions satisfying assumptions $(b_1) - (b_3)$ and $A_i : Q \longrightarrow E^*, i = 1, 2, 3, ...$ be a sequence of continuous monotone maps. Let $\{T_i\}_{i=1}^{\infty} : Q \longrightarrow Q$ be an infinite family of closed uniformly L- Lipschitz continuous and uniformly total quasi -ϕ-asymptotically nonexpansive mappings with nonnegative real numbers*

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sequences ζ_n, μ_n such that $\zeta_n \longrightarrow 0$, $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and stricly increasing continuous function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\psi(0) = 0$. Assume that $\Gamma := \big(\cap_{i=1}^{\infty} F(T_i)\big) \cap \big(\cap_{i=1}^{\infty} S_i^{-1}0\big) \cap \big(\cap_{i=1}^{\infty} GMEP(B_i, A_i, b_i)\big) \neq \emptyset$ Let $\{x_n\}$ be a sequence defined as follows:

$$\begin{cases} \omega_1 \in Q_1 = E; \\ w_n = \omega_n + \alpha_n(\omega_n - \omega_{n-1}); \\ \vartheta_n = J^{-1}(\rho_{0,n} J w_n + \displaystyle\sum_{i=1}^{\infty} \rho_{i,n} J T_i^n w_n); \\ z_n = J^{-1}(\gamma_n J w_n + (1-\gamma_n) J J_{r_n} \vartheta_n); \\ u_n \in C \text{ such that } B_i(u_n, \vartheta) + \langle A_i u_n, \vartheta - u_n\rangle \\ + \dfrac{1}{r_{i,n}} \langle \vartheta - u_n, J u_n - J z_n\rangle + b_i(u_n, \vartheta) - b_i(u_n, u_n) \geq 0, \forall \vartheta \in Q; \\ Q_{n+1} = \{u \in Q_n : G(u, J u_n) \leq G(u, J w_n) + \delta_n\}; \\ \omega_{n+1} = \Pi^f_{Q_{n+1}} \omega_1, \ \forall n \geq 1, \end{cases} \tag{3.1}$$

where $\alpha_n \subset (0,1)$, $\{\gamma_n\}$ and $\{\rho_{i,n}\} \subset [0,1]$ such that $\sum_{i=0}^{\infty} \rho_{i,n} = 1$, $\{r_n\}$ is a sequence in $(0,\infty)$ with $\{r_{i,n}\} \subset [a, \infty)$ for some $a > 0$, $\forall i = 1,2,3,....$ and $\delta_n = \zeta_n \psi\big(G(\hat{p}, J w_n)\big) + \mu_n$, $\hat{p} \in \Gamma$. Assume that $\liminf_{n\to\infty} \rho_{0,n}\rho_{i,n} > 0, \forall i \geq 1$, $\liminf_{n\to\infty}(1-\gamma_n) > 0$ and $\lim_{n\to\infty} r_n = \infty$. Then $\{\omega_n\}$ converges strongly to $\Pi^f_\Gamma \omega_1$, where Π^f_Γ is the generalized f- projection of E onto Γ.

Proof. Consider $\Omega_i : Q \times Q \longrightarrow \mathbb{R}$ and $T_{i,r} : E \longrightarrow Q$ as functions defined by

$$\Omega_i(z, \vartheta) = B_i(z, \vartheta) + \langle A_i z, \vartheta - z\rangle, \ \forall z, \vartheta \in Q$$

and

$$T_{i,r}(\omega) = \{z \in Q : \Omega_i(z, \vartheta) + \frac{1}{r_{i,n}} \langle \vartheta - z, Jz - J\omega\rangle + b_i(z, \vartheta) - b_i(z,z) \geq 0, \quad \forall \vartheta \in Q\}, \qquad \forall i \geq 1, \ \omega \in E,$$

respectively. We present the functions Ω_i which satisfies Assumptions $(B1)-(B4)$ and the functions $T_{i,r}$ which satisfies properties $(p_1) - (p_5)$ of Lemma 2.17 (see [10, 21] for more details).

We present the proof in the following steps:

Step 1 : we show that for all $n \geq 1$, Q_{n+1} is closed and convex. Clearly $Q_1 = Q$ is closed and convex. Supposed that Q_n is closed and convex for all $n \in \mathbb{N}$. For any $u \in Q_n$, the inequality below is from definition of Q_{n+1} :

$$G(u, J u_n) - G(u, J w_n) \leq \delta_n,$$

which implies that

$$\| u \|^2 - 2\langle u, J u_n\rangle + \|u_n\|^2 + 2\varpi f(u) - \| u \|^2 + 2\langle u, J w_n\rangle - \|w_n\|^2 - 2\varpi f(u) \leq \delta_n.$$

which gives

$$2\langle u, J w_n\rangle - 2\langle u, J u_n\rangle + \|u_n\|^2 - \|w_n\|^2 \leq \delta_n,$$

hence, we have

$$2\langle u, J w_n - J u_n\rangle \leq \| w_n \|^2 - \| u_n \|^2 + \delta_n.$$

Therefore, Q_{n+1} is closed and convex, $\forall n \geq 1$. Which lead to $\Pi^f_{C_{n+1}} \omega_1$ is well defined.

Step 2 : We show by induction that $\Gamma \subset Q_n, \forall n \geq 1$. It obvious that $\Gamma \subset Q_1 = Q$. Suppose that $\Gamma \subset Q_n$ for some $n \geq 1$. Assume that $u_n = T_{i,r_n} z_n$ for all $i \geq 1$ and $v_n = J_{r_n} \vartheta_n$ for all $n \geq 1$. Let $\hat{p} \subset \Gamma$ and by Lemma 2.11, we get the following estimate:

$$\begin{aligned}
G(\hat{p}, Ju_n) &= G(\hat{p}, JT_{i,r_n} z_n) \\
&\leq G(\hat{p}, Jz_n) \\
&= G(\hat{p}, \gamma_n Jw_n + (1-\gamma_n) Jv_n) \\
&= \| \hat{p} \|^2 - 2\langle \hat{p}, \gamma_n Jw_n + (1-\gamma_n) Jv_n \rangle \\
&+ \|\gamma_n Jw_n + (1-\gamma_n) Jv_n \|^2 + 2\varpi f(\hat{p}) \\
&\leq \| \hat{p} \|^2 - 2\gamma_n \langle \hat{p}, Jw_n \rangle - 2(1-\gamma_n)\langle \hat{p}, Jv_n \rangle + \gamma_n \| Jw_n \|^2 \\
&+ (1-\gamma_n) \| Jv_n \|^2 + 2\varpi f(\hat{p}) \\
&= \gamma_n G(\hat{p}, Jw_n) + (1-\gamma_n) G(\hat{p}, Jv_n) \\
&= \gamma_n G(\hat{p}, Jw_n) + (1-\gamma_n) G(\hat{p}, JJ_{r_n} \vartheta_n) \\
&\leq \gamma_n G(\hat{p}, Jw_n) + (1-\gamma_n) G(\hat{p}, J\vartheta_n) \qquad (3.2)
\end{aligned}$$

Now, from the fact that $\{T_i\}, \forall i \geq 1$ is total quasi-ϕ-asymptotically nonexpansive maps, then we obtain

$$\begin{aligned}
G(\hat{p}, J\vartheta_n) &= G(\hat{p},\ \rho_{0,n} Jw_n + \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n) \\
&= \| \hat{p} \|^2 - 2\langle \hat{p},\ \rho_{0,n} Jw_n + \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n \rangle + \| \rho_{0,n} Jw_n + \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n \|^2 \\
&+ 2\varpi f(\hat{p}) \\
&\leq \| \hat{p} \|^2 - 2\rho_{0,n} \langle \hat{p},\ Jw_n \rangle - 2\sum_{i=1}^{\infty} \rho_{i,n} \langle \hat{p}, JT_i^n w_n \rangle + \rho_{i,n} \|Jw_n\|^2 \\
&+ \sum_{i=1}^{\infty} \rho_{i,n} \|JT_i^n w_n\|^2 + 2\varpi f(\hat{p}) \\
&= \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} G(\hat{p}, JT_i^n w_n) \\
&\leq \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} \big[G(\hat{p}, Jw_n) + \zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n \big] \\
&\leq \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} \big[\zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n \big] \\
&= \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} G(\hat{p}, Jw_n) + (1-\rho_{0,n}) \big[\zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n \big] \\
&\leq \rho_{i,n} G(\hat{p}, Jw_n) + \sum_{i=1}^{\infty} \rho_{i,n} G(\hat{p}, Jw_n) + \zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n \\
&= G(\hat{p}, Jw_n) + \zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n. \qquad (3.3)
\end{aligned}$$

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Therefore, putting (3.3) in (3.2), we get

$$\begin{aligned} G(\hat{p}, Ju_n) &\leq \gamma_n G(\hat{p}, Jw_n) + (1-\gamma_n)\big[G(\hat{p}, Jw_n) + \zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n\big] \\ &= \gamma_n G(\hat{p}, Jw_n) + (1-\gamma_n) G(\hat{p}, Jw_n) + (1-\gamma_n)\big[\zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n\big] \\ &= G(\hat{p}, Jw_n) + (1-\gamma_n)\big[\zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n\big] \\ &\leq G(\hat{p}, Jw_n) + \zeta_n \psi\big(G(\hat{p}, Jw_n)\big) + \mu_n \\ &= G(\hat{p}, Jw_n) + \delta_n. \end{aligned} \tag{3.4}$$

Which lead to $\hat{p} \in Q_{n+1}$, gives that $\Gamma \subset Q_{n+1}$, therefore $\Gamma \subset Q_n, \forall n \geq 1$.

Step 3 : we show that the sequence $\{\omega_n\}$ is cauchy and $\omega_n \longrightarrow \hat{\omega}$ as $(n \longrightarrow \infty)$.
Now, since $f : E \longrightarrow \mathbb{R}$ is convex and lower semi continuous mapping, then, by Lemma 2.9 there exists $z^* \in E^*$ and $\eta \in \mathbb{R}$ such that

$$f(\omega) \geq \langle \omega, z^* \rangle + \eta, \forall \omega \in E.$$

Therefore, for $\omega_n \in E$, we have

$$\begin{aligned} G(\omega_n, J\omega_1) &= \| \omega_n \|^2 - 2\langle \omega_n, J\omega_1 \rangle + \|\omega_1\|^2 + 2\varpi f(\omega_n) \\ &\geq \| \omega_n \|^2 - 2\langle \omega_n, J\omega_1 \rangle + \|\omega_1\|^2 + 2\varpi \langle \omega_n, z^* \rangle + 2\varpi\eta \\ &= \| \omega_n \|^2 - 2\langle \omega_n, J\omega_1 - \varpi z^* \rangle + \|\omega_1\|^2 + 2\varpi\eta \\ &\geq \| \omega_n \|^2 - 2 \| \omega_n \| \| J\omega_1 - \varpi z^* \| + \| \omega_1 \|^2 + 2\varpi\eta \\ &= (\| \omega_n \| - \| J\omega_1 - \varpi z^* \|)^2 + \| \omega_1 \|^2 - \| J\omega_1 - \varpi z^* \|^2 + 2\varpi\eta. \end{aligned} \tag{3.5}$$

Hence, from the definition of Q_n and (3.5), since $\hat{p} \in \Gamma$ and $\omega_n = \Pi^f_{Q_n}\omega_1$, then we obtain

$$\begin{aligned} G(\hat{p}, J\omega_1) &\geq G(\omega_n, J\omega_1) \\ &\geq (\| \omega_n \| - \| J\omega_1 - \varpi z^* \|)^2 + \| \omega_1 \|^2 - \| J\omega_1 - \varpi z^* \|^2 + 2\varpi\eta. \end{aligned}$$

Implies $\{\omega_n\}$ is bounded and so are $\{u_n\}$, $\{z_n\}$, $\{\vartheta_n\}$, $\{w_n\}$, and $\{G(\omega_n, J\omega_1)\}$.
Therefore, since $\omega_{n+1} = \Pi^f_{Q_{n+1}}\omega_1 \in Q_{n+1} \subset Q_n$, $\omega_n = \Pi^f_{Q_n}\omega_1$, then by Lemma 2.11, we obtain

$$\begin{aligned} 0 &\leq (\| \omega_{n+1} - \omega_n \|)^2 \\ &\leq \phi(\omega_{n+1}, \omega_n) \\ &\leq G(\omega_{n+1}, J\omega_1) - G(\omega_n, J\omega_1). \end{aligned} \tag{3.6}$$

Which implies that $\{G(\omega_n, J\omega_1)\}$ is non decreasing. Therefore $\lim_{n\to\infty} G(\omega_n, J\omega_1)$ exists. Now, since $\omega_n = \Pi^f_{Q_n}\omega_1, \quad \omega_m = \Pi^f_{Q_m}\omega_1 \in Q_m \subset Q_n$, for any $m > n$, then from (3.6) we get

$$\phi(\omega_m, \omega_n) \leq G(\omega_m, J\omega_1) - G(\omega_n, J\omega_1).$$

By taking $m, n \longrightarrow \infty$, we conclude that

$$\lim_{n\to\infty} \phi(\omega_m, \omega_n) = 0.$$

Then, by Lemma 2.3, we have

$$\lim_{n\to\infty} \| \omega_m - \omega_n \| = 0.$$

This shows $\{\omega_n\}$ is cauchy. Therefore using the fact Q is closed subset of Banach space E and Q_n is closed and convex, we can assume that there exists an element $\hat{\omega} \in Q$ such that

$$\lim_{n\to\infty} \omega_n = \hat{\omega}. \tag{3.7}$$

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Step 4 : we show that $\hat{\omega} \in \Gamma$. Now, since $\lim_{n\to\infty} G(\omega_n, J\omega_1)$ exists from *Step* 3 then, it follows from (3.6) that

$$\lim_{n\to\infty} \phi(\omega_{n+1}, \omega_n) = 0. \tag{3.8}$$

Using Lemma 2.3, we get

$$\lim_{n\to\infty} \| \omega_{n+1} - \omega_n \| = 0. \tag{3.9}$$

Taking advantage of J as uniformly norm-to-norm continuous on bounded subsets of E, we conclude that

$$\lim_{n\to\infty} \| J\omega_{n+1} - J\omega_n \| = 0. \tag{3.10}$$

Observe that, by the definition of w_n from (3.1), we get

$$\| w_n - \omega_n \| = \| \alpha_n(\omega_n - \omega_{n-1}) \| \leq \| \omega_n - \omega_{n-1} \| .$$

Gives

$$\lim_{n\to\infty} \| w_n - \omega_n \| = 0. \tag{3.11}$$

Notice that (3.7) and (3.11), we get

$$\lim_{n\to\infty} w_n = \hat{\omega}. \tag{3.12}$$

Since $\{\omega_n\}$ is bounded, by considering Remark 2.4 and (3.11), we obtain

$$\lim_{n\to\infty} \phi(w_n, \omega_n) = 0. \tag{3.13}$$

From (3.9) and (3.11), we have

$$\lim_{n\to\infty} \| \omega_{n+1} - w_n \| = 0. \tag{3.14}$$

Also, since J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \| J\omega_{n+1} - Jw_n \| = 0. \tag{3.15}$$

By Remark 2.4 and (3.14), we obtain

$$\lim_{n\to\infty} \phi(\omega_{n+1}, w_n) = 0. \tag{3.16}$$

Also, by (3.12) and (3.14), we conclude that

$$\lim_{n\to\infty} \omega_{n+1} = \hat{\omega}. \tag{3.17}$$

Observe that from the definition of Q_{n+1} in (3.1) and $\omega_{n+1} = \Pi^f_{Q_{n+1}} \omega_1$, we get

$$G(\omega_{n+1}, Ju_n) \leq G(\omega_{n+1}, Jw_n).$$

Which lead to

$$\begin{aligned} \| \omega_{n+1} \|^2 \quad - & \quad 2\langle \omega_{n+1}, Ju_n \rangle + \|u_n\|^2 + 2\varpi f(\omega_{n+1}) \\ \leq & \quad \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n \rangle + \|w_n\|^2 + 2\varpi f(\omega_{n+1}). \end{aligned}$$

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This gives

$$\begin{aligned} \| \omega_{n+1} \|^2 &- 2\langle \omega_{n+1}, Ju_n\rangle + \|u_n\|^2 \\ &\leq \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n\rangle + \|w_n\|^2, \end{aligned}$$

which implies that

$$\phi(\omega_{n+1}, u_n) \leq \phi(\omega_{n+1}, w_n).$$

By considering (3.16), we observe that

$$\lim_{n\to\infty} \phi(\omega_{n+1}, u_n) = 0.$$

Now using Lemma 2.3, we obtain

$$\lim_{n\to\infty} \| \omega_{n+1} - u_n \| = 0. \tag{3.18}$$

Following from the fact that J is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\lim_{n\to\infty} \| J\omega_{n+1} - Ju_n \| = 0. \tag{3.19}$$

By triangular inequality, we have

$$\| \omega_n - u_n \| \leq \| \omega_n - \omega_{n+1} \| + \| \omega_{n+1} - u_n \| . \tag{3.20}$$

Now. putting (3.9) and (3.18) in (3.20), we get

$$\lim_{n\to\infty} \| \omega_n - u_n \| = 0. \tag{3.21}$$

Using (3.7) and (3.21), we obtain

$$\lim_{n\to\infty} u_n = \hat{\omega}. \tag{3.22}$$

We also observe that from (3.11) and (3.21), we get

$$\lim_{n\to\infty} \| w_n - u_n \| = 0. \tag{3.23}$$

From J is uniformly continuous on bounded subset of E, we conclude that

$$\lim_{n\to\infty} \| Jw_n - Ju_n \| = 0. \tag{3.24}$$

Noticing that by definition of Q_{n+1} and $\omega_{n+1} = \Pi^f_{Q_{n+1}}\omega_1$, we have

$$G(\omega_{n+1}, Jz_n) \leq G(\omega_{n+1}, Jw_n).$$

Equivalent to

$$\begin{aligned} \| \omega_{n+1} \|^2 &- 2\langle \omega_{n+1}, Jz_n\rangle + \|z_n\|^2 + 2\varpi f(\omega_{n+1}) \\ &\leq \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n\rangle + \|w_n\|^2 + 2\varpi f(\omega_{n+1}), \end{aligned}$$

gives

$$\begin{aligned} \| \omega_{n+1} \|^2 &- 2\langle \omega_{n+1}, Jz_n\rangle + \|z_n\|^2 \\ &\leq \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n\rangle + \|w_n\|^2, \end{aligned}$$

implies that

$$\phi(\omega_{n+1}, z_n) \leq \phi(\omega_{n+1}, w_n).$$

By (3.16), we get

$$\lim_{n\to\infty} \phi(\omega_{n+1}, z_n) = 0.$$

Using Lemma 2.3, we obtain

$$\lim_{n\to\infty} \| \omega_{n+1} - z_n \| = 0. \tag{3.25}$$

By considering J as uniformly norm-to-norm continuous on bounded subsets of E, we conclude that

$$\lim_{n\to\infty} \| J\omega_{n+1} - Jz_n \| = 0. \tag{3.26}$$

Taking into account that

$$\| \omega_n - z_n \| \leq \| \omega_n - \omega_{n+1} \| + \| \omega_{n+1} - z_n \| . \tag{3.27}$$

Using (3.9) and (3.25) in (3.27), we have

$$\lim_{n\to\infty} \| \omega_n - z_n \| = 0. \tag{3.28}$$

From (3.7) and (3.28), we conclude that

$$\lim_{n\to\infty} z_n = \hat{\omega}. \tag{3.29}$$

Also, since J is uniformly norm-to- norm continuous on bounded subsets of E and by (3.28), we have

$$\lim_{n\to\infty} \| J\omega_n - Jz_n \| = 0. \tag{3.30}$$

We also observe that by (3.19) and (3.26), we get

$$\lim_{n\to\infty} \| Ju_n - Jz_n \| = 0. \tag{3.31}$$

Also, from $\omega_{n+1} = \Pi^f_{Q_{n+1}}\omega_1$ and by definition of Q_{n+1}, we have

$$G(\omega_{n+1}, J\vartheta_n) \leq G(\omega_{n+1}, Jw_n).$$

Then, we get that

$$\begin{aligned} \| \omega_{n+1} \|^2 &- 2\langle \omega_{n+1}, J\vartheta_n\rangle + \|\vartheta_n\|^2 + 2\varpi f(\omega_{n+1}) \\ &\leq \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n\rangle + \|w_n\|^2 + 2\varpi f(\omega_{n+1}), \end{aligned}$$

therefore, we have

$$\begin{aligned} \| \omega_{n+1} \|^2 &- 2\langle \omega_{n+1}, J\vartheta_n\rangle + \|\vartheta_n\|^2 \\ &\leq \| \omega_{n+1} \|^2 - 2\langle \omega_{n+1}, Jw_n\rangle + \|w_n\|^2, \end{aligned}$$

hence

$$\phi(\omega_{n+1}, \vartheta_n) \leq \phi(\omega_{n+1}, w_n).$$

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By (3.16), we have

$$\lim_{n\to\infty} \phi(\omega_{n+1}, \vartheta_n) = 0.$$

Also by Lemma 2.3, we conclude that

$$\lim_{n\to\infty} \| \omega_{n+1} - \vartheta_n \| = 0. \tag{3.32}$$

From the fact that J is uniformly norm-to-norm continuous on bounded subsets of E, we get that

$$\lim_{n\to\infty} \| J\omega_{n+1} - J\vartheta_n \| = 0. \tag{3.33}$$

We also notice that

$$\| \omega_n - \vartheta_n \| \leq \| \omega_n - \omega_{n+1} \| + \| \omega_{n+1} - \vartheta_n \| . \tag{3.34}$$

Now, using (3.9) and (3.32) in (3.34), we get

$$\lim_{n\to\infty} \| \omega_n - \vartheta_n \| = 0. \tag{3.35}$$

We observe that by (3.7) and (3.35), we conclude that

$$\lim_{n\to\infty} \vartheta_n = \hat{\omega}. \tag{3.36}$$

From (3.1), we have the following estimate:

$$\begin{aligned}
\| J\omega_{n+1} - J\vartheta_n \| &= \| J\omega_{n+1} - \big(\rho_{0,n} Jw_n + \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n\big) \| \\
&= \| J\omega_{n+1} - \rho_{0,n} Jw_n - \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n \| \\
&= \| J\omega_{n+1} + \sum_{i=1}^{\infty} \rho_{i,n} J\omega_{n+1} - \sum_{i=1}^{\infty} \rho_{i,n} J\omega_{n+1} + \rho_{0,n} J\omega_{n+1} \\
&- \rho_{0,n} J\omega_{n+1} - \rho_{0,n} Jw_n - \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n \| \\
&= \| \sum_{i=1}^{\infty} \rho_{i,n} J\omega_{n+1} - \sum_{i=1}^{\infty} \rho_{i,n} JT_i^n w_n + \rho_{0,n} J\omega_{n+1} - \rho_{0,n} Jw_n \| \\
&= \| \sum_{i=1}^{\infty} \rho_{i,n} \big(J\omega_{n+1} - JT_i^n w_n\big) + \rho_{0,n} \big(J\omega_{n+1} - Jw_n\big) \| \\
&= \| \sum_{i=1}^{\infty} \rho_{i,n} \big(J\omega_{n+1} - JT_i^n w_n\big) - \rho_{0,n} \big(Jw_n - J\omega_{n+1}\big) \| \\
&\geq \sum_{i=1}^{\infty} \rho_{i,n} \| J\omega_{n+1} - JT_i^n w_n \| - \rho_{0,n} \| Jw_n - J\omega_{n+1} \|,
\end{aligned}$$

this implies that

$$\begin{aligned}
\| J\omega_{n+1} - JT_i^n w_n \| \leq \frac{1}{\sum_{i=1}^{\infty} \rho_{i,n}} \big[& \| J\omega_{n+1} - J\vartheta_n \| \\
& + \rho_{0,n} \| Jw_n - J\omega_{n+1} \| \big].
\end{aligned} \tag{3.37}$$

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Using (3.15), (3.33) and $\liminf\limits_{n\to\infty}\sum\limits_{i=1}^{\infty}\rho_{i,n}>0$ in (3.37), we obtain

$$\lim_{n\to\infty}\| J\omega_{n+1}-JT_i^n w_n \|=0,\quad \forall i\geq 1. \tag{3.38}$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded subsets of E, we get

$$\lim_{n\to\infty}\| \omega_{n+1}-T_i^n w_n \|=0,\quad \forall i\geq 1. \tag{3.39}$$

From the triangular inequality that for each $i\geq 1$, we have

$$\| w_n-T_i^n w_n \|\leq\| w_n-\omega_{n+1} \| + \| \omega_{n+1}-T_i^n w_n \| . \tag{3.40}$$

Putting (3.14) and (3.39) in (3.40), we obtain

$$\lim_{n\to\infty}\| w_n-T_i^n w_n \|=0,\quad \forall i\geq 1. \tag{3.41}$$

Taking into account that for each $i\geq 1$, we have

$$\| T_i^n w_n-\hat{\omega} \|\leq\| T_i^n w_n-w_n \| + \| w_n-\hat{\omega} \| . \tag{3.42}$$

Using (3.12) and (3.41) in (3.42), we conclude that

$$\lim_{n\to\infty}\| T_i^n w_n-\hat{\omega} \|=0,\quad \forall i\geq 1. \tag{3.43}$$

Furthermore, using assumption that T_i is uniformly L_i- Lipschitz continuous for each $i\geq 1$, lead to

$$\begin{aligned}\| T_i^{n+1}w_n-T_i^n w_n \| &\leq \| T_i^{n+1}w_n-T_i^{n+1}w_{n+1} \| + \| T_i^{n+1}w_{n+1}-w_{n+1} \|\\ &+ \| w_{n+1}-w_n \| + \| w_n-T_i^n w_n \|\\ &\leq \big(L_i+1\big)\| w_{n+1}-w_n \| + \| T_i^{n+1}w_{n+1}-w_{n+1} \| + \| w_n-T_i^n w_n \|\end{aligned} \tag{3.44}$$

Therefore, by using (3.9) and (3.41) in (3.44), we obtain

$$\lim_{n\to\infty}\| T_i^{n+1}w_n-T_i^n w_n \|=0,\quad \forall i\geq 1.$$

Hence, from (3.43), it yield that

$$\lim_{n\to\infty}\| T_i^{n+1}w_n-\hat{\omega} \|=0,\quad \forall i\geq 1.$$

Implies that $T_iT_i^n w_n\longrightarrow\hat{\omega}$ as $n\longrightarrow\infty$. Therefore, in view of the Closedness of T_i, we conclude that $T_i\hat{\omega}=\hat{\omega},\quad \forall i\geq 1$. Hence,

$$\hat{\omega}\in\cap_{i=1}^{\infty}F(T_i).$$

Taking the advantage of (3.2) that

$$\begin{aligned}G(\hat{p},Ju_n) &= G(\hat{p},JT_{i,r_n}z_n),\quad \forall i\geq 1.\\ &\leq G(\hat{p},Jz_n)\\ &\leq \gamma_n G(\hat{p},Jw_n)+(1-\gamma_n)G(\hat{p},Jv_n),\end{aligned}$$

gives

$$G(\hat{p},Jv_n)\geq\frac{1}{1-\gamma_n}\big(G(\hat{p},Ju_n)-\gamma_n G(\hat{p},Jw_n)\big)$$

Moreover, by Lemma 2.15, we observe that

$$\begin{aligned}
\phi(v_n, w_n) &= \phi(J_{r_n} w_n, w_n) \\
&\leq G(\hat{p}, Jw_n) - G(\hat{p}, JJ_{r_n} w_n) \\
&= G(\hat{p}, Jw_n) - G(\hat{p}, Jv_n) \\
&\leq G(\hat{p}, Jw_n) - \frac{1}{1-\gamma_n}\big(G(\hat{p}, Ju_n) - \gamma_n G(\hat{p}, Jw_n)\big) \\
&= \frac{1}{1-\gamma_n}\big(G(\hat{p}, Jw_n) - G(\hat{p}, Ju_n)\big) \\
&= \frac{1}{1-\gamma_n}\big(\| w_n \|^2 - \| u_n \|^2 - 2\langle \hat{p}, Jw_n - Ju_n\rangle\big) \\
&\leq \frac{1}{1-\gamma_n}\big(\| w_n \|^2 - \| u_n \|^2 + 2|\langle \hat{p}, Jw_n - Ju_n\rangle|\big) \\
&\leq \frac{1}{1-\gamma_n}\big((\| w_n - u_n \|)(\| w_n \| + \| u_n \|) + 2 \| \hat{p} \| \| Jw_n - Ju_n \| \big)
\end{aligned} \tag{3.45}$$

Since $\liminf_{n\to\infty}(1-\gamma_n) > 0$, now by using (3.23) and (3.24) in (3.45), we obtain

$$\lim_{n\to\infty} \phi(v_n, w_n) = 0. \tag{3.46}$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \| w_n - v_n \| = 0. \tag{3.47}$$

Since J is uniformly norm-to- norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} \| Jw_n - Jv_n \| = 0. \tag{3.48}$$

It also follows from (3.12) and (3.47) that

$$\lim_{n\to\infty} v_n = \hat{\omega}.$$

Now, since $r_n \geq a, \;\; v_n = J_{r_n} w_n$ and by (3.47), we get

$$\lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - Jv_n \| = 0. \tag{3.49}$$

Then

$$\begin{aligned}
\lim_{n\to\infty} \| S_{i,r_n} w_n \| &= \lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - JJ_{r_n} w_n \| \\
&= \lim_{n\to\infty} \frac{1}{r_n} \| Jw_n - Jv_n \| \\
&= 0, \quad \forall i \geq 1.
\end{aligned}$$

Consider $(\sigma, \sigma^*) \in S_i$, monotonicity of S_i and by Lemma 2.14, we get

$$\langle \sigma - v_n, \; \sigma^* - S_{i,r_n} w_n \rangle \geq 0, \quad \forall n \geq 0, \quad i \geq 1.$$

Now, taking the limit as $n \longrightarrow \infty$, we have $\langle \sigma - \hat{\omega}, \sigma^* \rangle \geq 0$. It follows from the maximality of S_i that $\hat{\omega} \in S_i^{-1}0, \;\; \forall i \geq 1$. Hence

$$\hat{\omega} \in \cap_{i=1}^{\infty} S_i^{-1}0.$$

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Next, we show that $\hat{\omega} \in \big(\cap_{i=1}^{\infty} GMEP(B_i, A_i, b_i)\big)$. From the equation $u_n = T_{i,r_n} z_n$, (3.31) and $\{r_{in}\} \subset [a, \infty)$ for some $a > 0$, we observe that

$$\lim_{n\to\infty} \parallel \frac{Ju_n - Jz_n}{r_{in}} \parallel = 0, \quad \forall i \geq 1. \tag{3.50}$$

By $u_n = T_{i,r_n} z_n$, we obtain

$$\Omega_i(u_n, \vartheta) + \frac{1}{r_{in}} \langle \vartheta - u_n, Ju_n - Jz_n\rangle + b_i(\vartheta, u_n) - b_i(u_n, u_n) \geq 0, \ \forall \vartheta \in Q, \ i \geq 1.$$

Where

$$\Omega_i(u_n, \vartheta) = B_i(u_n, \vartheta) + \langle A_i u_n, \vartheta - u_n\rangle,$$

We also observe that by the assumption B_2, we get

$$\begin{aligned} \frac{1}{r_{in}} \langle \vartheta - u_n, Ju_n - Jz_n\rangle &\geq -\Omega_i(u_n, \vartheta) - b_i(\vartheta, u_n) + b_i(u_n, u_n) \\ &\geq \Omega_i(\vartheta, u_n) - b_i(\vartheta, u_n) + b_i(u_n, u_n) \end{aligned}$$

By taking $n \longrightarrow \infty$, (3.50) and the lower semicontinuity of $\vartheta \longrightarrow f(\vartheta, \ .)$, we conclude that

$$\Omega_i(\vartheta, \hat{\omega}) - b_i(\vartheta, \hat{\omega}) + b_i(\hat{\omega}, \hat{\omega}), \ \forall y \in Q, \ i \geq 1.$$

Consider $\vartheta_\pi := \pi\vartheta + (1-\pi)\hat{\omega}, \ \forall \pi \in (0, 1]$, then $\vartheta_\pi \in Q$, hence

$$\Omega_i(\vartheta_\pi, \hat{\omega}) - b_i(\vartheta_\pi, \hat{\omega}) + b_i(\hat{\omega}, \hat{\omega}) \leq 0, \ i \geq 1.$$

Also, by the assumptions $(B_1) \ - \ (B_4)$ for all $i \geq 1$, we obtain

$$\begin{aligned} 0 &= \Omega_i(\vartheta_\pi, \vartheta_\pi) \\ &\leq \pi\Omega_i(\vartheta_\pi, \vartheta) + (1-\pi)\Omega_i(\vartheta_\pi, \hat{\omega}) \\ &\leq \pi\Omega_i(\vartheta_\pi, \vartheta) + (1-\pi)\big[b_i(\vartheta_\pi, \hat{\omega}) - b_i(\hat{\omega}, \hat{\omega})\big] \\ &\leq \pi\Omega_i(\vartheta_\pi, \vartheta) + (1-\pi)\big[b_i(\vartheta, \hat{\omega}) - b_i(\hat{\omega}, \hat{\omega})\big]. \end{aligned}$$

Letting $\pi > 0$, it follows from the assumption (B_3) that

$$\Omega_i(\hat{\omega}, \vartheta) + b_i(\vartheta, \hat{\omega}) - b_i(\hat{\omega}, \hat{\omega}) \geq 0, \ \forall \vartheta \in Q, \ i \geq 1.$$

This implies that

$$\hat{\omega} \in \big(GMEP(B_i, A_i, b_i)\big), \quad i \geq 1.$$

Hence

$$\hat{\omega} \in \big(\cap_{i=1}^{\infty} GMEP(B_i, A_i, b_i)\big).$$

Therefore

$$\hat{\omega} \in \Gamma$$

Step 5 : we show that $\hat{\omega} = \Pi_\Gamma^f \omega_1$. Since Γ is closed and convex set, by Lemma 2.10, we have that $\Pi_\Gamma^f \omega_1$ is single-valued denoted by x^*. Also from the definition of $\omega_n = \Pi_{Q_n}^f \omega_1$ and $x^* \in \Gamma \subset Q_n$, we get that

$$G(\omega_n, J\omega_1) \leq G(x^*, J\omega_1), \quad \forall n \geq 1.$$

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From the definition of G and f, we note that for any given $\omega \in E$, $G(\vartheta, J\omega)$ is convex and lower semi continuous with respect to ϑ. Then

$$\begin{aligned} G(\hat{\omega}, J\omega_1) &\leq \liminf_{n\to\infty} G(\omega_n, J\omega_1) \\ &\leq \limsup_{n\to\infty} G(\omega_n, J\omega_1) \\ &\leq G(x^*, J\omega_1). \end{aligned}$$

By definition of $\Pi_\Gamma^f \omega_1$ and $\hat{\omega} \in \Gamma$, we conclude that $x^* = \hat{\omega} = \Pi_\Gamma^f \omega_1$ and $\omega_n \longrightarrow \hat{\omega}$ as $n \to \infty$. This completes the proof. □

4 APPLICATION

Some applications of theorem 3.1 are to be present in this section as follows:

4.1 Countable family of total quasi-phi-asymptotically nonexpansive maps, maximal monotone operator and system of generalized equilibrium problems.

We observe that $\{\omega_n\}$ defined in theorem 3.1 converges strongly to $\Pi_\Gamma^f \omega_1$ by setting $A \equiv 0$ in theorem 3.1, where $\Gamma := \big(\cap_{i=1}^\infty F(T_i)\big) \cap \big(\cap_{i=1}^\infty S_i^{-1}0\big) \cap \big(\cap_{i=1}^\infty GEP(B_i, b_i)\big)$ and $GEP(B, b)$ is the set of solutions of the generalized equilibrium problem for B and b.

4.2 Countable family of total quasi-phi-asymptotically nonexpansive maps, maximal monotone operators and system of variational inequalities problems.

We observe that $\{\omega_n\}$ defined in theorem 3.1 converges strongly to $\Pi_\Gamma^f \omega_1$ by setting $B \equiv 0, b \equiv 0$ in theorem 3.1, where $\Gamma := \big(\cap_{i=1}^\infty F(T_i)\big) \cap \big(\cap_{i=1}^\infty S_i^{-1}0\big) \cap \big(\cap_{i=1}^\infty VIP(A_i)\big)$ and $VIP(Q, A)$ is the set of solutions of variational inequality problem for A over Q.

Application in Hilbert space
We also present the application of theorem 3.1 in Hilbert space as follows:

Theorem 4.1. *Let Q be a nonempty closed and convex subset of a Hilbert space H. Let $S_i \subset E \times E^*, i = 1, 2, 3, \ldots$ be a sequence of maximal monotone operators satisfying $D(S_i) \subset Q$ and $J_{r_n} = (J + r_n S_i)^{-1} J$, for all $r_n > 0$, $i = 1, 2, 3, \ldots$ Let $B_i : Q \times Q \longrightarrow \mathbb{R}, i = 1, 2, 3, \ldots$ be a sequence of bifunctions satisfying assumptions $(B_1) - (B_4)$, $b_i : Q \times Q \longrightarrow \mathbb{R}, i = 1, 2, 3, \ldots$ be a sequence of bifunctions satisfying assumptions $(b_1) - (b_3)$ and $A_i : Q \longrightarrow E^*, i = 1, 2, 3, \ldots$ be a sequence of continuous monotone maps. Let $\{T_i\}_{i=1}^\infty : Q \longrightarrow Q$ be an infinite family of closed uniformly L- Lipschitz continuous and uniformly total quasi-asymptotically nonexpansive mappings with the sequences ζ_n, μ_n of nonnegative real numbers with $\zeta_n \longrightarrow 0$, $\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$ and stricly increasing continuous function $\psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\psi(0) = 0$. Assume that $\Gamma := \big(\cap_{i=1}^\infty F(T_i)\big) \cap \big(\cap_{i=1}^\infty S_i^{-1}0\big) \cap \big(\cap_{i=1}^\infty GMEP(B_i, A_i, b_i)\big) \neq \emptyset$. Let $\{\omega_n\}$ be a sequence defined*

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as follows:

$$\begin{cases} \omega_1 \in Q_1 = E; \\ w_n = \omega_n + \alpha_n(\omega_n - \omega_{n-1}); \\ \vartheta_n = J^{-1}(\rho_{0,n} J w_n + \sum_{i=1}^{\infty} \rho_{i,n} J T_i^n w_n); \\ z_n = J^{-1}(\gamma_n J w_n + (1-\gamma_n) J J_{r_n} \vartheta_n); \\ u_n \in Q \text{ such that } B_i(u_n, \vartheta) + \langle A_i u_n, \vartheta - u_n \rangle \\ + \frac{1}{r_{i,n}} \langle \vartheta - u_n, J u_n - J z_n \rangle + b_i(u_n, \vartheta) - b_i(u_n, u_n) \geq 0, \forall \vartheta \in Q; \\ Q_{n+1} = \{u \in Q_n : \| u - u_n \|^2 \leq \| u - w_n \|^2 + \delta_n\}; \\ \omega_{n+1} = P_{Q_{n+1}} \omega_1, \ \forall n \geq 1, \end{cases}$$

where $\alpha_n \subset (0,1)$, $\{\gamma_n\}$ *and* $\{\rho_{i,n}\} \subset [0,1]$ *such that* $\sum_{i=0}^{\infty} \rho_{i,n} = 1$, $\{r_n\}$ *is a sequence in* $(0,\infty)$ *with* $\{r_{i,n}\} \subset [a,\infty)$ *for some* $a > 0$, $\forall i = 1,2,3,....$ *and* $\delta_n = \zeta_n \psi(\| w_n - \hat{p} \|^2) + \mu_n$, $\hat{p} \in \Gamma$. *Assume that* $\liminf_{n\to\infty} \rho_{0,n}\rho_{i,n} > 0, \forall i \geq 1$, $\liminf_{n\to\infty}(1-\gamma_n) > 0$ *and* $\lim_{n\to\infty} r_n = \infty$. *Then,* $\{\omega_n\}$ *converges strongly to* $P_\Gamma \omega_1$, *where* P_C *is the metric projection of* H *onto* C.

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