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On quaternion valued rectangular S-metric space

S. O. Ayodele [1*], O. K. Adewale [2], B. E. Oyelade [3], O. F. Olayera [4], E. E. Aribike [5]

1,2,4. Tai Solarin University of Education, Ogun State, Nigeria.
3. Bowling Green State University, Ohio, United State.
5. Lagos State University of Science and Technology, Ikorodu, Lagos, Nigeria.
* Corresponding author: andaayo2013@gmail.com*, adewalekayode2@yahoo.com, boyelad@bgsu.edu, olayeraabisola@gmail.com, aribike.ella@yahoo.com

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Abstract

The aim of this paper is made of two part. First, we introduce the concept of quanternion valued rectangular S metric spaces which generalizes both real and complex valued metric spaces. Secondly, we establish and prove some fixed point theorems in the newly introduced spaces. This concept generalizes many known results in literature.

Keywords: Quaternion valued S-metric spaces, S-metric spaces, Usual metric spaces, Fixed point, Contractive maps.
MSC2010: 37C25.

1 Introduction

A metric space can be seen as a distance space having a geometric structure, with only a few axioms. In this paper we introduce the concept of quaternion valued rectangular S metric spaces. The paper treats material concerning quaternion valued rectangular S metric spaces that is important for the study of fixed point theory in Clifford analysis. We introduce the basic ideas of quaternion valued rectangular S metric spaces and Cauchy sequences and discuss the completion of a quaternion valued rectangular S metric space.
In this work, we will work on $\mathbb{H}$, the skew field of quaternions. This means we can write each element $q \in \mathbb{H}$ in the form $q = a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and i, j, and k are the fundamental quaternion units. For these elements we have the multiplication rules $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $kj = -jk = -i$ and $ki = -ik = j$. The conjugate element is given by $\overline{q} = a - bi - cj - dk$. The quaternion modulus has the form of $|q| = \sqrt{a^2 + b^2 + c^2 + d^2}$.
Quaternions can be defined in several different equivalent ways. Quaternion is non commutative in multiplication. There is also more abstract possibilty of treating quaternions as simply quadruples of real numbers $[a, b, c, d]$, with operation of addition and multiplication suitably defined. The components naturally group into the imaginary part (b, c, d), for which we take this part as a vector and the purely real part, a, which called a scalar. Sometimes, we write a quaternion as $[a, V]$ with $V = (b, c, d)$. For more information about metric spaces, its generalization and quaternion analysis,

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see [1–32].
We define a partial order $\preceq$ on $\mathbb{H}$ as follows:
Let $\mathbb{H}$ be the set of quaternions and $q_1, q_2 \in \mathbb{H}$. Define a partial order $\preceq$ on $\mathbb{H}$ as follows:
$q_1 \preceq q_2$ if and only if $Re(q_1) \leq Re(q_2)$ and $Im_s(q_1) \leq Im_s(q_2)$, $q_1, q_2 \in \mathbb{H}$, $s = i, j, k$ where $Im_i = b, Im_j = c, Im_k = d$.
It follows that $q_1 \preceq q_2$, if one of the following conditions is satisfied:

(i) $Re(q_1) = Re(q_2)$ and $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, where $s_1 = j, k$, $Im_i(q_1) < Im_i(q_2)$;

(ii) $Re(q_1) = Re(q_2)$ and $Im_{s_2}(q_1) = Im_{s_2}(q_2)$, where $s_2 = i, k$, $Im_j(q_1) < Im_j(q_2)$;

(iii) $Re(q_1) = Re(q_2)$ and $Im_{s_3}(q_1) = Im_{s_3}(q_2)$, where $s_3 = i, j$, $Im_k(q_1) < Im_k(q_2)$;

(iv) $Re(q_1) = Re(q_2)$ and $Im_{s_1}(q_1) = Im_{s_1}(q_2)$, $Im_i(q_1) = Im_i(q_2)$;

(v) $Re(q_1) = Re(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$ and $Im_j(q_1) = Im_j(q_2)$;

(vi) $Re(q_1) = Re(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$ and $Im_k(q_1) = Im_k(q_2)$;

(vii) $Re(q_1) = Re(q_2)$ and $Im_s(q_1) < Im_s(q_2)$;

(viii) $Re(q_1) < Re(q_2)$ and $Im_s(q_1) = Im_s(q_2)$;

(ix) $Re(q_1) < Re(q_2)$, $Im_{s_1}(q_1) = Im_{s_1}(q_2)$ and $Im_i(q_1) < Im_i(q_2)$;

(x) $Re(q_1) < Re(q_2)$, $Im_{s_2}(q_1) = Im_{s_2}(q_2)$ and $Im_j(q_1) < Im_j(q_2)$;

(xi) $Re(q_1) < Re(q_2)$, $Im_{s_3}(q_1) = Im_{s_3}(q_2)$ and $Im_k(q_1) < Im_k(q_2)$;

(xii) $Re(q_1) < Re(q_2)$, $Im_{s_1}(q_1) < Im_{s_1}(q_2)$ and $Im_i(q_1) = Im_i(q_2)$;

(xiii) $Re(q_1) < Re(q_2)$, $Im_{s_2}(q_1) < Im_{s_2}(q_2)$ and $Im_i(q_1) = Im_i(q_2)$;

(xiv) $Re(q_1) < Re(q_2)$, $Im_{s_3}(q_1) < Im_{s_3}(q_2)$ and $Im_i(q_1) = Im_i(q_2)$;

(xv) $Re(q_1) < Re(q_2)$ and $Im_s(q_1) < Im_s(q_2)$;

(xiv) $Re(q_1) = Re(q_2)$ and $Im_s(q_1) = Im_s(q_2)$.

Conspicuously, we will write $q_1 \precnsim q_2$ if $q_1 \neq q_2$ and one from (i), to (xvi) is satisfied and we will write $q_1 \prec q_2$ if only (xv) is satisfied. It should be noted that

$$q_1 \preceq q_2 \Rightarrow |q_1| \leq |q_2|.$$

2 Main results

We introduce the following:
Definition 2.1 Let X be a non-empty set and $\underline{S} : X^3 \to \mathbb{H}$, a function satisfying the following properties:

(i) $\underline{S}(x, y, z) = 0 \qquad$ if and only if $x = y = z$

(ii) $\underline{S}(x, y, z) \preceq \underline{S}(x, x, a) + \underline{S}(y, y, a) + \underline{S}(z, z, a) \qquad \forall x, y, z \in X$ and all distinct points $a \in X - \{x, y.z\}$..

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Then $(X, \underline{S})$ is called a quanterion valued rectangular S-metric space.
Definition 2.1 extends the work of Adewale and Iluno in [1].

Example 2.2. Let $X = \mathbb{Q}$ and define $\underline{S} : X \times X \times X \to \mathbb{H} \cup \{0\}$ by

$$\underline{S}(a,b,c) = \begin{cases} 0, & a = b = c; \\ 1 + ai + bj + ck, & otherwise \end{cases}$$

Then $(X, \underline{S})$ is a quanterion valued rectangular S-metric space but neither a G-metric space nor rectangular metric space because

$$\underline{S}(a,b,c) \in \mathbb{H}.$$

Example 2.3. Let $X = \mathbb{N} \cup \{0\}$ and define $\underline{S} : X \times X \times X \to \mathbb{H} \cup \{0\}$ by

$$\underline{S}(x,y,z) = \begin{cases} 0, & x = y = z; \\ x + y + z, & otherwise \end{cases}$$

Then $(X, \underline{S})$ is a quanterion valued rectangular S-metric space but neither a G-metric space nor rectangular metric space because

$$G(6,4,2) = G(6,6,2).$$

Example 2.4. Let $X = \mathbb{R}$ and define $\underline{S} : X \times X \times X \to \mathbb{H} \cup \{0\}$ by

$$\underline{S}(x,y,z) = \begin{cases} 0, & x = y = z; \\ \sqrt{x} + \sqrt{y} + \sqrt{z}, & otherwise \end{cases}$$

Then $(X, \underline{S})$ is a quanterion valued rectangular S-metric space but neither a G-metric space nor rectangular metric space because

$$G(x,y,z) \in \mathbb{R}.$$

Definition 2.5. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space. For $y \in X$, $r > 0$, the $\underline{S}$-sphere with centre y and radius r is

$$\underline{S}_S(y,r) = \{z \in X : \underline{S}(y,z,z) < r\}$$

Definition 2.6. Let $(X.\underline{S})$ be a quanterion valued rectangular S-metric space. A sequence $\{x_n\} \subset X$ is $\underline{S}$-convergent to z if it converges to z in the quanterion valued rectangular S-metric topology.

Definition 2.7. Let $(X, \underline{S})$ and $(\overline{X}, \overline{\underline{S}})$ be two quanterion valued rectangular S-metric spaces, a function $T : X \to \overline{X}$ is $\underline{S}$-continuous at a point $x \in X$ if $T^{-1}(\underline{S}_{\overline{S}}(T(x), r)) \in T(X)$, for all $r > 0$. T is $\underline{S}$-continuous if it is $\underline{S}$-continuous at all points of X.

Lemma 2.8. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ converges to x if and only if $\underline{S}(x_n, x, x) \to 0$ as $n \to \infty$.
Proof:
Suppose $\{x_n\}$ converges to x, then given $\epsilon > 0$ there exists α such that $\underline{S}(x_n, x, x) \prec \epsilon$ for all $n \geq \alpha$. So, $\underline{S}(x_n, x, x) \prec \epsilon \implies \underline{S}(x_n, x, x) \to 0$ as $n \to \infty$. It is easy to show that the converse is true.

Lemma 2.9. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be a Cauchy sequence if and only if $\underline{S}(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.
Proof:
Using (ii) of Definition 2.1,

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$\underline{S}(x_n, x_m, x_l) \preceq \underline{S}(x_n, x_n, x) + \underline{S}(x_m, x_m, x) + \underline{S}(x_l, x_l, x)$. The conclusion of the proof is obvious from Lemma 2.8.

Theorem 2.10. Let X be a complete quanterion valued rectangular S-metric space and $T : X \to X$ a map for which there exist the real number, q satisfying $0 \leq q < 0.5$ such that for each pair $x, y, z \in X$.

$$\underline{S}(Tx, Ty, Tz) \preceq q\underline{S}(x, y, z) \tag{2.1}$$

Then T has a unique fixed point.
Proof:
Considering (1),

$$\underline{S}(Tx, Ty, Ty) \preceq q\underline{S}(x, y, y) \tag{2.2}$$

Suppose T satisfies condition (2) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then
$\underline{S}(x_n, x_n, x_{n+1}) = \underline{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) \preceq q\underline{S}(x_{n-1}, x_{n-1}, x_n)$
Setting $H_n = \underline{S}(x_n, x_n, x_{n+1})$ we have

$$H_n \preceq qH_{n-1} \tag{2.3}$$

We deduce that

$$\begin{aligned}
H_n &\preceq qH_{n-1} && (2.4)\\
H_n &\preceq q[QH_{n-2}] && (2.5)\\
H_{n-1} &\preceq q^2 H_{n-2} && (2.6)\\
H_n &\preceq q^3 H_{n-3} && (2.7)\\
H_n &\preceq q^n H_{n-n} && (2.8)\\
H_n &\preceq q^n H_0 \forall n \in \mathbb{N}. && (2.9)
\end{aligned}$$

Suppose there exists $n \in \mathbb{N}$ such that $x_0 = x_n$.

$$\begin{aligned}
\underline{S}(x_0, x_0, Tx_0) &= \underline{S}(x_n, x_n, Tx_n)\\
\underline{S}(x_0, x_0, x_1) &= \underline{S}(x_n, x_n, x_{n+1})\\
H_0 &= H_n\\
H_0 &\preceq q^n H_0.
\end{aligned}$$

Contradiction since $k < 1$. Hence $\forall n \in \mathbb{N}, x_0 \neq x_n$. Repeating this argument, we have that $\forall n, m \in \mathbb{N}$ with $n \neq m, x_n \neq x_m$. Then the terms of a sequence $\{x_n\}$ are distinct.
By repeated use of (ii) in Definition 2.1 and all distinct points $x_{n+1}, x_{n+2}, ..., x_{m-1}$ with $m > n$, we have

$$\begin{aligned}
\underline{S}(x_n, x_m, x_m) &\preceq \underline{S}(x_n, x_n, x_{n+1}) + \underline{S}(x_m, x_m, x_{n+1}) && (2.10)\\
&\quad + \underline{S}(x_m, x_m, x_{n+1}) && (2.11)\\
&= \underline{S}(x_n, x_n, x_{n+1}) + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.12)\\
&= H_n + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.13)\\
&\preceq H_n + 2H_{n+1} + 2^2\underline{S}(x_m, x_m, x_{n+2}) && (2.14)\\
&\preceq H_n + 2H_{n+1} + 2^2 H_{n+2} + 2^3\underline{S}(x_m, x_m, x_{n+3}) && (2.15)\\
&\preceq H_n + 2H_{n+1} + 2^2 H_{n+2} + 2^3 H_{n+3} + ... + 2^{m-1} H_m. && (2.16)\\
&\preceq H_n + 2H_{n+1} + 2^2 H_{n+2} + 2^3 H_{n+3} + ... && (2.17)
\end{aligned}$$

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From (5) and (13), we have

$$\begin{aligned}
\underline{S}(x_n, x_m, x_m) &\preceq q^n H_0 + 2q^{n+1} H_0 + 2^2 q^{n+2} H_0 + ... + 2^{m-1} q^{m-1} H_0 && (2.18)\\
&\preceq [q^n + 2q^{n+1} + 2^2 q^{n+2} + ... + 2^{m-1} q^{m-1}] H_0 && (2.19)\\
&\preceq q^n [1 + 2q + (2q)^2 + ... + (2q)^{m-n-1}] H_0 && (2.20)\\
&\preceq q^n [1 + 2q + (2q)^2 + (2q)^3 + ...] H_0 && (2.21)\\
&\preceq q^n (1 - 2q)^{-1} H_0. && (2.22)
\end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_m)$ as $n, m \to \infty$, we have

$$\lim_{n,m\to\infty} \underline{S}(x_n, x_m, x_m) = \lim_{n,m\to\infty} [q^n (1 - 2q)^{-1}] \underline{S}(x_0, x_0, x_1) = 0 \tag{2.23}$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$\begin{aligned}
\underline{S}(x_n, x_m, x_l) &\preceq \underline{S}(x_n, x_n, x_{n-1}) + \underline{S}(x_m, x_m, x_{n-1}) + && (2.24)\\
&\quad \underline{S}(x_l, x_l, x_{n-1}). && (2.25)
\end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \underline{S}(x_n, x_m, x_l) = 0. \tag{2.26}$$

So, $\{x_n\}$ is a $\underline{S}$-Cauchy Sequence.
By completeness of (X,$\underline{S}$), there exist u $\in X$ such that x_n is $\underline{S}$-convergent to u.
Suppose $Tu \neq u$

$$\underline{S}(x_n, Tu, Tu) \preceq q\underline{S}(x_{n-1}, u, u). \tag{2.27}$$

Taking the limit as n $\to \infty$ and using the fact that function is $\underline{S}$-continuous in its variables, we get

$$\underline{S}(u, Tu, Tu) \preceq q\underline{S}(u, u, u). \tag{2.28}$$

Hence,

$$\underline{S}(u, Tu, Tu) \preceq 0. \tag{2.29}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\underline{S}(Tu, Tv, Tv) \preceq q\underline{S}(u, v, v). \tag{2.30}$$

Since $Tu = u$ and $Tv = v$, we have

$$\underline{S}(u, v, v) \preceq 0. \tag{2.31}$$

which implies that $v = u$.

Remark 2.11. Let $(X, \underline{S})$ be a rectangular S-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = \underline{S}(x, y, y)$, then Theorem 2.10 reduces to Banach contraction principle in rectangular-metric space(an analogue of Banach contraction principle in metric space).

Theorem 2.12. Let X be a complete rectangular S- metric space and $T : X \to X$ a map for which there exist the real number, b satisfying $0 \leq b < 0.2$ such that for each pair $x, y, z \in X$.

$$\underline{S}(Tx, Ty, Tz) \preceq b[\underline{S}(x, Tx, Tx) + \underline{S}(y, Ty, Ty) + \underline{S}(z, Tz, Tz)] \tag{2.32}$$

Then T has a unique fixed point.

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Proof:
Considering (28),

$$\underline{S}(Tx,Ty,Ty) \preceq b[\underline{S}(x,Tx,Tx)+\underline{S}(y,Ty,Ty)+\underline{S}(z,Tz,Tz)]. \tag{2.33}$$

Suppose T satisfies condition (29) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then we have

$$\underline{S}(x_n,x_n,x_{n+1}) \preceq b[\underline{S}(x_{n-1},x_{n-1},x_n)+\underline{S}(x_{n-1},x_{n-1},x_n)+\underline{S}(x_n,x_n,x_{n+1})]. \tag{2.34}$$

We deduce that

$$\underline{S}(x_n,x_n,x_{n+1}) \preceq \frac{2b}{1-b}\underline{S}(x_{n-1},x_{n-1},x_n) \tag{2.35}$$

Let $p = \frac{2b}{1-b} < \frac{1}{2}$

$$\begin{aligned}
\underline{S}(x_n,x_n,x_{n+1}) &\preceq r\underline{S}(x_{n-1},x_{n-1},x_n) &(2.36)\\
&\preceq r^2\underline{S}(x_{n-2},x_{n-2},x_{n-1}) &(2.37)\\
\underline{S}(x_n,x_n,x_{n+1}) &\preceq r^3\underline{S}(x_{n-3},x_{n-3},x_{n-2}) &(2.38)\\
\underline{S}(x_n,x_n,x_{n+1}) &\preceq r^n\underline{S}(x_0,x_0,x_1) &(2.39)\\
H_n &\preceq r^n H_0. &(2.40)
\end{aligned}$$

Suppose there exists $n \in \mathbb{N}$ such that $x_0 = x_n$.

$$\begin{aligned}
\underline{S}(x_0,x_0,Tx_0) &= \underline{S}(x_n,x_n,Tx_n)\\
\underline{S}(x_0,x_0,x_1) &= \underline{S}(x_n,x_n,x_{n+1})\\
H_0 &= H_n\\
H_0 &\preceq r^n H_0.
\end{aligned}$$

Contradiction since $p < \frac{1}{2}$. Hence $\forall n \in \mathbb{N}, x_0 \neq x_n$. Repeating this argument, we have that $\forall n, m \in \mathbb{N}$ with $n \neq m, x_n \neq x_m$. Then the terms of a sequence $\{x_n\}$ are distinct.
By repeated use of (ii) in Definition 2.1 and all distinct points $x_{n+1}, x_{n+2}, ..., x_{m-1}$, we have

$$\begin{aligned}
\underline{S}(x_n,x_m,x_m) &\preceq \underline{S}(x_n,x_n,x_{n+1})+\underline{S}(x_m,x_m,x_{n+1}) &(2.41)\\
&\quad +\underline{S}(x_m,x_m,x_{n+1}) &(2.42)\\
&= \underline{S}(x_n,x_n,x_{n+1})+2\underline{S}(x_m,x_m,x_{n+1}) &(2.43)\\
&= H_n + 2\underline{S}(x_m,x_m,x_{n+1}) &(2.44)\\
&\preceq H_n + 2H_{n+1} + 2^2\underline{S}(x_m,x_m,x_{n+2}) &(2.45)\\
&\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3\underline{S}(x_m,x_m,x_{n+3}) &(2.46)\\
&\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... + 2^{m-1}H_m. &(2.47)\\
&\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... &(2.48)
\end{aligned}$$

From (36) and (44), we have

$$\begin{aligned}
\underline{S}(x_n,x_m,x_m) &\preceq r^nH_0 + 2r^{n+1}H_0 + 2^2r^{n+2}H_0 + ... + 2^{m-1}r^{m-1}H_0 &(2.49)\\
&\preceq [r^n + 2r^{n+1} + 2^2r^{n+2} + ... + 2^{m-1}r^{m-1}]H_0 &(2.50)\\
&\preceq r^n[1 + 2r + (2r)^2 + (2r)^3 + ... + (2r)^{m-n-1}]H_0 &(2.51)\\
&\preceq r^n[1 + 2r + (2r)^2 + (2r)^3 + (2r)^4 + ...]H_0 &(2.52)\\
&\preceq r^n(1-2r)^{-1}H_0. &(2.53)
\end{aligned}$$

Taking the limit of $\underline{S}(x_n,x_m,x_m)$ as $n,m \to \infty$, we have

$$\lim_{n,m\to\infty}\underline{S}(x_n,x_m,x_m) = \lim_{n,m\to\infty}[r^n(1-2r)^{-1}]\underline{S}(x_0,x_0,x_1) = 0. \tag{2.54}$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$\begin{aligned} \underline{S}(x_n, x_m, x_l) &\preceq \underline{S}(x_n, x_n, x_{n-1}) + \underline{S}(x_m, x_m, x_{n-1}) + && (2.55) \\ &\quad \underline{S}(x_l, x_l, x_{n-1}). && (2.56) \end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \underline{S}(x_n, x_m, x_l) = 0. \tag{2.57}$$

So, x_n is a $\underline{S}$-Cauchy Sequence.
By completeness of $(X, \underline{S})$, there exist $u \in X$ such that x_n is $\underline{S}$-convergent to u.
Suppose $Tu \neq u$

$$\begin{aligned} \underline{S}(x_n, Tu, Tu) &\preceq b[\underline{S}(x_{n-1}, x_n, x_n) + \underline{S}(u, Tu, Tu) + \underline{S}(u, Tu, Tu)] && (2.58) \\ &\preceq b[\underline{S}(x_{n-1}, x_n, x_n) + 2\underline{S}(u, Tu, Tu)]. && (2.59) \end{aligned}$$

Taking the limit as n $\to \infty$ and using the fact that function is $\underline{S}$-continuous in its variables, we get

$$\underline{S}(u, Tu, Tu) \preceq 2b\underline{S}(u, Tu, Tu). \tag{2.60}$$

Hence,

$$\underline{S}(u, Tu, Tu) \preceq 0. \tag{2.61}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose v$\neq$ u is such that $Tv = v$, then

$$\underline{S}(Tu, Tv, Tv) \preceq b[\underline{S}(u, Tu, Tu) + \underline{S}(v, Tv, Tv) + \underline{S}(v, Tv, Tv)]. \tag{2.62}$$

Since $Tu = u$ and $Tv = v$, we obtain

$$\underline{S}(u, v, v) \preceq 0 \tag{2.63}$$

and $\underline{S}(u, v, v) \preceq 0$ implies $v = u$.

Remark 2.13. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = \underline{S}(x, y, y)$, then Theorem 2.12 reduces to Kannan's fixed point theorem in rectangular-metric space(an analogue of Kannan's fixed point theorem in metric space).

Theorem 2.14. Let X be a complete quanterion valued rectangular S-metric space and $T : X \to X$ a map for which there exists real numbers a, b, c satisfying $0 \leq a < \frac{1}{2}$, $0 \leq b < \frac{1}{2}$, $0 \leq c < \frac{1}{2}$ with $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ and

$$\phi(t) = \begin{cases} 0, & if \quad t = 0; \\ \frac{t}{3}, & if \quad t \neq 0. \end{cases}$$

such that for each pair $x, y, z \in X$.

$$\underline{S}(Tx, Ty, Tz) \preceq \phi(\delta\underline{S}(x, y, z) + 2\delta\underline{S}(x, x, Tx)) \tag{2.64}$$

Then T has a unique fixed point.

Proof: Considering (60),

$$\underline{S}(Tx, Ty, Ty) \preceq \phi(\delta\underline{S}(x, y, y) + 2\delta\underline{S}(x, x, Tx)) \tag{2.65}$$

Suppose T satisfies condition (61) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} \underline{S}(x_n, x_n, x_{n+1}) &= \underline{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) && (2.66)\\ &\preceq \phi(\delta\underline{S}(x_{n-1}, x_{n-1}, x_n) + 2\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)) && (2.67)\\ &\preceq \phi(3\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)) && (2.68)\\ &\preceq \delta\underline{S}(x_{n-1}, x_{n-1}, x_n) && (2.69) \end{aligned}$$

Setting $H_n = \underline{S}(x_n, x_n, x_{n+1})$, then

$$H_n \preceq \delta H_{n-1}. \qquad (2.70)$$

Deducing

$$\begin{aligned} H_n &\preceq \delta H_{n-1} && (2.71)\\ H_n &\preceq \delta^n H_0 \forall n \in \mathbb{N}. && (2.72) \end{aligned}$$

Suppose there exists $n \in \mathbb{N}$ such that $x_0 = x_n$.

$$\begin{aligned} \underline{S}(x_0, x_0, Tx_0) &= \underline{S}(x_n, x_n, Tx_n)\\ \underline{S}(x_0, x_0, x_1) &= \underline{S}(x_n, x_n, x_{n+1})\\ H_0 &= H_n\\ H_0 &\preceq \delta^n H_0. \end{aligned}$$

Contradiction since $\delta < \frac{1}{2}$. Hence $\forall n \in \mathbb{N} \cup \{0\}, x_0 \neq x_n$. Repeating this argument, $\forall n, m \in \mathbb{N} \cup \{0\}$ with $n \neq m, x_n \neq x_m$. Then the terms of a sequence $\{x_n\}$ are distinct.
By repeated use of (ii) in Definition 2.1 and all distinct points $x_{n+1}, x_{n+2}, ..., x_{m-1}$ with $m > n$,

$$\begin{aligned} \underline{S}(x_n, x_m, x_m) &\preceq \underline{S}(x_n, x_n, x_{n+1}) + \underline{S}(x_m, x_m, x_{n+1}) && (2.73)\\ &\quad + \underline{S}(x_m, x_m, x_{n+1}) && (2.74)\\ &= \underline{S}(x_n, x_n, x_{n+1}) + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.75)\\ &= H_n + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.76)\\ &\preceq H_n + 2H_{n+1} + 2^2\underline{S}(x_m, x_m, x_{n+2}) && (2.77)\\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3\underline{S}(x_m, x_m, x_{n+3}) && (2.78)\\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... + 2^{m-1}H_m. && (2.79)\\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... && (2.80) \end{aligned}$$

From (68) and (76), we have

$$\begin{aligned} \underline{S}(x_n, x_m, x_m) &\preceq \delta^n H_0 + 2\delta^{n+1}H_0 + 2^2\delta^{n+2}H_0 + ... + 2^{m-1}\delta^{m-1}H_0 && (2.81)\\ &\preceq [\delta^n + 2\delta^{n+1} + 2^2\delta^{n+2} + ... + 2^{m-1}\delta^{m-1}]H_0 && (2.82)\\ &\preceq \delta^n[1 + 2\delta + (2\delta)^2 + (2\delta)^3 + ... + (2\delta)^{m-n-1}]H_0 && (2.83)\\ &\preceq \delta^n[1 + 2\delta + (2\delta)^2 + (2\delta)^3 + (2\delta)^4 + ...]H_0 && (2.84)\\ &\preceq \delta^n(1 - 2\delta)^{-1}H_0. && (2.85) \end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_m)$ as $n, m \to \infty$,

$$\lim_{n,m\to\infty} \underline{S}(x_n, x_m, x_m) = \lim_{n,m\to\infty} [\delta^n(1 - 2\delta)^{-1}]\underline{S}(x_0, x_0, x_1) = 0. \qquad (2.86)$$

For $n, m, l \in \mathbb{N} \cup \{0\}$with $n > m > l$,

$$\begin{aligned} \underline{S}(x_n, x_m, x_l) &\preceq \underline{S}(x_n, x_n, x_{n-1}) + \underline{S}(x_m, x_m, x_{n-1}) + && (2.87)\\ &\quad \underline{S}(x_l, x_l, x_{n-1}). && (2.88) \end{aligned}$$

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Taking the limit of $\underline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \underline{S}(x_n, x_m, x_l) = 0. \tag{2.89}$$

So, $\{x_n\}$ is a $\underline{S}$-Cauchy Sequence.
By completeness of (X,$\underline{S}$), there exist u $\in X$ such that x_n is $\underline{S}$-convergent to u.
Suppose $Tu \neq u$

$$\underline{S}(x_n, Tu, Tu) \preceq \phi(\delta\underline{S}(x_{n-1}, u, u) + 2\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)). \tag{2.90}$$

Taking the limit as n $\to \infty$ and using the fact that function is $\underline{S}$-continuous in its variables,

$$\underline{S}(u, Tu, Tu) \preceq \phi(\delta\underline{S}(u, u, u) + 2\delta\underline{S}(u, u, u)). \tag{2.91}$$

Hence,

$$\underline{S}(u, Tu, Tu) \preceq 0. \tag{2.92}$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\underline{S}(Tu, Tv, Tv) \preceq \phi(\delta\underline{S}(u, v, v) + 2\delta\underline{S}(u, u, Tu)). \tag{2.93}$$

Since $Tu = u$ and $Tv = v$, then

$$\underline{S}(u, v, v) \preceq 0. \tag{2.94}$$

which implies that $v = u$

Remark 2.15. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space and $d : X \times X \to [0, \infty)$, a function defined by $d(x, y) = \underline{S}(x, y, y)$ with $\phi(t) = t$, then Theorem 2.14 reduces to Zamfirescu's fixed point theorem in rectangular-metric space(an analogue of Zamfirescu's fixed point theorem in metric space).

Theorem 2.16. Let X be a complete quanterion valued rectangular S- metric space and $T : X \to X$ a map for which there exists real numbers a, b, c satisfying $0 \leq a < \frac{1}{2}$, $0 \leq b < \frac{1}{2}$, $0 \leq c < \frac{1}{2}$ with $\delta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}$ such that for each pair $x, y, z \in X$.

$$\underline{S}(Tx, Ty, Tz) \preceq \phi(\delta\underline{S}(x, y, z)) + \psi(2\delta\underline{S}(x, x, Tx)). \tag{2.95}$$

where $\delta \in [0, 1)$ and functions ϕ, $\psi : \mathbb{H} \to \mathbb{H}$ with $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{4}$ a monotone increasing sequences. Then T has a unique fixed point.

Proof: Considering (91),

$$\underline{S}(Tx, Ty, Ty) \preceq \phi(\delta\underline{S}(x, y, y)) + \psi(2\delta\underline{S}(x, x, Tx)). \tag{2.96}$$

Suppose T satisfies condition (92) and $x_0 \in X$ be an arbitrary point and define a sequence x_n by $x_n = T^n x_0$, then

$$\begin{aligned} \underline{S}(x_n, x_n, x_{n+1}) &= \underline{S}(Tx_{n-1}, Tx_{n-1}, Tx_n) && (2.97)\\ &\preceq \phi(\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)) + \psi(2\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)) && (2.98)\\ &\preceq \delta\underline{S}(x_{n-1}, x_{n-1}, x_n) && (2.99) \end{aligned}$$

Setting $s_n = \underline{S}(x_n, x_n, x_{n+1})$, then

$$H_n \preceq \delta H_{n-1}. \tag{2.100}$$

Deducing

$$\begin{aligned} H_n &\preceq \delta H_{n-1} && (2.101)\\ H_n &\preceq \delta^n H_0 \forall n \in \mathbb{N}. && (2.102) \end{aligned}$$

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Suppose there exists $n \in \mathbb{N}$ such that $x_0 = x_n$.

$$\begin{aligned} \underline{S}(x_0, x_0, Tx_0) &= \underline{S}(x_n, x_n, Tx_n) \\ \underline{S}(x_0, x_0, x_1) &= \underline{S}(x_n, x_n, x_{n+1}) \\ H_0 &= H_n \\ H_0 &\preceq \delta^n H_0. \end{aligned}$$

Contradiction since $\delta < \frac{1}{2}$. Hence $\forall n \in \mathbb{N} \cup \{0\}, x_0 \neq x_n$. Repeating this argument, $\forall n, m \in \mathbb{N} \cup \{0\}$ with $n \neq m, x_n \neq x_m$. Then the terms of a sequence $\{x_n\}$ are distinct.
By repeated use of (ii) in Definition 2.1 and all distinct points $x_{n+1}, x_{n+2}, ..., x_{m-1}$ with $m > n$,

$$\begin{aligned} \underline{S}(x_n, x_m, x_m) &\preceq \underline{S}(x_n, x_n, x_{n+1}) + \underline{S}(x_m, x_m, x_{n+1}) && (2.103) \\ &\quad + \underline{S}(x_m, x_m, x_{n+1}) && (2.104) \\ &= \underline{S}(x_n, x_n, x_{n+1}) + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.105) \\ &= H_n + 2\underline{S}(x_m, x_m, x_{n+1}) && (2.106) \\ &\preceq H_n + 2H_{n+1} + 2^2\underline{S}(x_m, x_m, x_{n+2}) && (2.107) \\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3\underline{S}(x_m, x_m, x_{n+3}) && (2.108) \\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... + 2^{m-1}H_m. && (2.109) \\ &\preceq H_n + 2H_{n+1} + 2^2H_{n+2} + 2^3H_{n+3} + ... && (2.110) \end{aligned}$$

From (98) and (106), we have

$$\begin{aligned} \underline{S}(x_n, x_m, x_m) &\preceq \delta^n H_0 + 2\delta^{n+1}H_0 + 2^2\delta^{n+2}H_0 + ... + 2^{m-1}\delta^{m-1}H_0 && (2.111) \\ &\preceq [\delta^n + 2\delta^{n+1} + 2^2\delta^{n+2} + ... + 2^{m-1}\delta^{m-1}]H_0 && (2.112) \\ &\preceq \delta^n[1 + 2\delta + (2\delta)^2 + (2\delta)^3 + ... + (2\delta)^{m-n-1}]H_0 && (2.113) \\ &\preceq \delta^n[1 + 2\delta + (2\delta)^2 + (2\delta)^3 + (2\delta)^4 + ...]H_0 && (2.114) \\ &\preceq \delta^n(1 - 2\delta)^{-1}H_0. && (2.115) \end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_m)$ as $n, m \to \infty$,

$$\lim_{n,m\to\infty} \underline{S}(x_n, x_m, x_m) = \lim_{n,m\to\infty} [\delta^n(1 - 2\delta)^{-1}]\underline{S}(x_0, x_0, x_1) = 0. \quad (2.116)$$

For $n, m, l \in \mathbb{N} \cup \{0\}$with $n > m > l$,

$$\begin{aligned} \underline{S}(x_n, x_m, x_l) &\preceq \underline{S}(x_n, x_n, x_{n-1}) + \underline{S}(x_m, x_m, x_{n-1}) + && (2.117) \\ &\quad \underline{S}(x_l, x_l, x_{n-1}). && (2.118) \end{aligned}$$

Taking the limit of $\underline{S}(x_n, x_m, x_l)$ as $n, m, l \to \infty$, we have

$$\lim_{n,m,l\to\infty} \underline{S}(x_n, x_m, x_l) = 0. \quad (2.119)$$

So, $\{x_n\}$ is a $\underline{S}$-Cauchy Sequence.
By completeness of $(X, \underline{S})$, there exist u $\in X$ such that x_n is $\underline{S}$-convergent to u.
Suppose $Tu \neq u$

$$\underline{S}(x_n, Tu, Tu) \preceq \phi(\delta\underline{S}(x_{n-1}, u, u)) + \psi(2\delta\underline{S}(x_{n-1}, x_{n-1}, x_n)). \quad (2.120)$$

Taking the limit as n $\to \infty$ and using the fact that function is $\underline{S}$-continuous in its variables,

$$\underline{S}(u, Tu, Tu) \preceq \phi(\delta\underline{S}(u, u, u)) + \psi(2\delta\underline{S}(u, u, u)). \quad (2.121)$$

Hence,

$$\underline{S}(u, Tu, Tu) \preceq 0. \quad (2.122)$$

This is a contradiction. So, $Tu = u$.
To show the uniqueness, suppose v$\neq$ u is such that $Tv = v$, then

$$\underline{S}(Tu, Tv, Tv) \preceq \phi(\delta \underline{S}(u, v, v)) + \psi(2\delta \underline{S}(u, u, Tu)). \tag{2.123}$$

Since $Tu = u$ and $Tv = v$, then

$$\underline{S}(u, v, v) \preceq 0 \tag{2.124}$$

and $\underline{S}(u, v, v) \preceq 0$ implies $v = u$.

Remark 2.17. Let $(X, \underline{S})$ be a quanterion valued rectangular S-metric space and $d : X \times X \to [0, \infty)$ a function defined by $d(x, y) = \underline{S}(x, y, y)$ with $\phi(t) = t$ and $\psi(2\delta \underline{S}(x, Tx, Tx)) = 0$, then Theorem 2.16 reduces to Banach Contraction Principle in rectangular-metric space.

3 Conclusion

In conclusion, a new abstract space is introduced in this research work and some contractive mappings are established and used to prove some fixed point results on the newly introduced space. Examples are given.

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