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Nonassociative Moufang Loops of Odd Order $p_1p_2^2p^4$ Properties and Characterizations

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Abstract

We investigate nonassociative Moufang loops of odd order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes satisfying $p_1 < p_2 < q$. We show that there exists a normal Hall subloop of order q^4 in L, and further prove that subject to some extra conditions the Moufang loop of this order is associative. Notably, the Loops Package gap4r5 does not contain Moufang loops of order $p_1p_2^2q^4$, indicating a gap in the current knowledge of higher-order Moufang loops. Our results provide new insights into the structure and properties of nonassociative Moufang loops, building upon existing research.

Keywords: Loop, Maximal Subloop, Order, Nonassociative.
MSC2010: 20N05.

1 Introduction

A loop $\langle L, *\rangle$ is said to be a Moufang loop if for any $u, v, w \in L$ the identity $(u * v) * (w * u) = (u*(v * w))*u$ is satisfied. Moufang loops are nonassociative algebraic structures with unique properties, making them essential in various mathematical contexts. The works of [1] and [2] respectively proved that all Moufang loops of order 3^3 and $p^4, p \geq 3$ (where p is a prime) are associative. But then, nonassociative Moufang loops of order 3^4 [1] and p^5, $p \geq 3$ [3] are known to exist. Thus, the highest possible order based on these results would be all Moufang loops of order $3^3p_1^4p_2^4 \cdots p_n^4$ and is likely to be associative if $p_1, p_2, \cdots p_n, p_i > 3, i \in \{1, 2, \cdots n\}$ are distinct odd primes. This appeared more and more likely due to progressives works as seen in [5–14].
Notably, constructing an example of nonassociative Moufang loops of odd order $p_1p_2^2q^4$ remains an elusive goal, justifying the existence of Moufang loops of that order is still to be achieved.
Recent studies have expanded the understanding of Moufang loops. Building on Bruck's foundational text [1], researchers have identified additional characterizations. Specifically, [15, 16] have contributed significantly to the field. Furthermore, [17] have discovered four new loop identities that individually characterize Moufang loops, bringing the total to eight.
However, the study of higher-order loops remains an open area of research. In the recent work of [14], some characterization of Moufang loops of odd order $p_2^2q^4$ were explored. This work extends

the exploration by investigating the possible classification and determination of Moufang loops of odd order $p_1p_2^2q^4$. This clearly is an open area of research as the exact number of Moufang loops of certain orders, for instance for orders less than or equal to 64, 81, and 243 are known and documented in the Loops Package [18] but higher orders are yet to be documented. This work focuses on nonassociative Moufang loops of odd order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes satisfying $p_1 < p_2 < q$, by exploring the properties of these loops, including the order of the associator subloop, the existence of normal Hall subloops and the possibility of it being associative.

2 Definitions and Notations

Some very standard definition for easy understanding of this work are as follows:

1. A loop $\langle L, *\rangle$, is a binary system, satisfying the following two conditions: (i) specification of any two of the elements u, v, w in the equation $u * v = w$ uniquely determines the third element, (ii) the binary system contains an identity element.

2. A Moufang loop is a loop $\langle L, *\rangle$ such that $(u * v)(w * u) = (u*(v * w))*u$ for any $u, v, w \in L$.

3. The associator subloop of L is denoted as $L_a = (L, L, L) = \langle (l_1, l_2, l_3)|l_i \in L\rangle$. In a Moufang Loop, L_a is the subloop generated by all the associators $(u, v, w) \in L$ such that $(u, v, w) = (u * vw)^{-1}(uv * w)$. L is associative if and only if $L_a = \{1\}$.

4. If M is a normal subloop of L, then

(a) M is a proper normal subloop of L, if $M \neq L$.

(b) L/M is a proper quotient loop of L, if $M \neq \{1\}$.

3. If M is a normal subloop of L, then

(a) M is a minimal normal subloop of L, if M is non-trivial and contains no proper nontrivial subloop which is normal in L. In other words, if there exists $H \lhd L$ with $\{1\} < H < M$, then $H = \{1\}$ or M.

(b) M is a maximal normal subloop of L, if M is not a proper subloop of every other proper normal subloop of L. In other words, if there exists $H \lhd L$ such that $M < H$, then $M = H$ or $H = L$.

Note that one can refer to [1,4], for further details.

3 Basic properties and known results

Let L be a Moufang loop.

Lemma 3.1. $N = N(L)$ is a normal subloop of L. Clearly N, is a group, by the definition of N. [1]

The following properties hold only for finite Moufang loops L.

Lemma 3.2. Suppose K is a subloop of L. Then $|K|$ divides $|L|$. [1]

Lemma 3.3. Suppose K is a subloop of L, and π is a set of primes. Then

a. L is solvable.

b. If K is a minimal normal subloop of L, then K is an elementary abelian group and $(K, K, L) = \langle (k_1, k_2, l)|k_i \in K, l \in L\rangle = \{1\}$.

c. K is a normal subloop of $L, (K, K, L) = \{1\}$ and $(|K|, |L/K|) = 1$ implies $K \subset N$.

d. L contains a Hall π -subloop.

[13]

Lemma 3.4. Suppose $|L|$ is odd and every proper subloop of L is associative. If there exists a normal Sylow subloop, minimal in L, then L is associative. [1]

Lemma 3.5. If there exist H, K in L such that the two conditions; (i) $H \lhd K \lhd L$ and (ii) $(|H|, |K/H|) = 1$ hold, then $H \lhd L$. [13]

Lemma 3.6. Let L be of odd order, K, a normal subloop, minimal in L such that (i) $K \subset L$, and (ii) Q, a Hall subloop of L. If, (i) every proper subloops and proper quotient loops of L are associative, (ii) $(|K|, |Q|) = 1$ and (iii) $Q \lhd KQ$, then L is associative. [1]

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Lemma 3.7. Let L be of odd order and K a normal Hall subloop of L. If, (i) $K = \langle x\rangle L_a$ for some $x \in K\backslash L_a$ and (ii) $L_a \subset N$, then $K \subset N$. [8]

Lemma 3.8. If $|L|$ is odd and every proper subloop of L is associative. Also, if in addition, N contains a Hall subloop of L, then L is associative. [12]

Lemma 3.9. If $|L| = p^{\alpha}m$ where p is the least prime dividing $|L|$ such that $(p, m) = 1$ and $\alpha \in \{1, 2\}$. Then there exists a subloop M of order m that is normal in L. [7]

Lemma 3.10. If L is Moufang loop of finite order such that its order is coprime to six. Then any normal subloop of L, that is also minimal in L will be a subset of the nucleus of L. [6]

Lemma 3.11. If L is nonassociative and of odd order, and M a maximal normal subloop of L. Also, if all proper subloops and proper quotient loops of L are associative and, L_a is a Sylow subloop of N, then $L_a = N$. [8]

Lemma 3.12 Let L be a Moufang loop of odd order and R be a normal Hall subloop of L. Then there does not exist any element $x \in R/N$ such that $R \subset \langle x\rangle N$. [9]

Lemma 3.13 Suppose $K \lhd L$. Then L/K is a group implies $L_a \subset K$. [12]

Lemma 3.14 is diassociative, that is, $\langle u, v\rangle$ a group for any $u, v \in L$. Moreover, if $(u, v, w) = 1$ for some $u, v, w \in L$, then $\langle u, v, w\rangle$ is a group. [1]

Lemma 3.15 Let G be a group of order pq such that $p \neq q$. Then there exists normal subgroups of order p and q in G. [8]

Lemma 3.16 Assume $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is a product of distinct odd primes, then any group of order m is abelian if and only if the following two conditions are satisfied (i) for all $i \in \{1, \cdots k\}$ $\alpha_i \leq 2$ and (ii) for any i and j, $p_i \not| (p_j^{\alpha_j} - 1)$. [5]

4 Main result

The following result proves the existence of a normal Hall subloop of order q^4 in L.

Lemma 4.1 *Let L be a nonassociative Moufang loop of odd order $p_1p_2^2q^4$ where p_1, p_2, q are distinct odd primes satisfying $p_1 < p_2 < q$ with $(p_1p_2^2, q) = 1$ and $q \not\equiv 1 (mod\ p_i)$ for every prime divisor p_i of $p_1p_2^2$. Then there exists a normal Hall subloop of order q^4 in L.*

Proof:

By Lemma(3.9), there exists a normal subloop M of order $p_2^2q^4$ in L. Since $|L/M| = p_1$, it follows that M is a maximal normal subloop of L. Using the same Lemma (3.9), there exists a normal subloop Q of order q^4 in M. Also, since $|Q| = q^4$, it follows that Q is a Hall subloop normal in M. It is therefore clear that $Q \lhd M \lhd L$ and that $(|Q|, |M/Q|) = 1$. So by Lemma (3.5), it follows that Q is a normal Hall subloop of L.

The next result investigates the possible order for the associator subloop of the Moufang loop under consideration.

Theorem 4.2 *Let L be a nonassociative Moufang loop of order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes satisfying $p_1 < p_2 < q$, $q \not\equiv 1 (mod\ p_i)$ and $p_2 \not\equiv 1 (mod\ p_1)$. Assume all proper subloops and proper quotient loops of L are groups and $(|L|, 6) = 1$, then $|L_a| = q^2$.*

Proof:

Assuming that every proper subloop and proper quotient loops of L are groups.

Now L_a, by Lemma (3.3)(b), is a minimal normal subloop of L and by the same Lemma (3.3)(b) is also an elementary abelian group.

So, $|L_a| = p_i^{\alpha_i}, q, q^2, q^3$ or q^4, for some $1 \leq \alpha_i \leq 2$, also by Lemma (3. 4), L_a is not a Sylow subloop of L. So $\alpha_i \neq 2$ and $|L_a| \neq q^4$. Hence, $|L_a| = p_i^{\alpha_i}, q, q^2, q^3$ with $\alpha_i = 1$.

4.0.1 Case 1: Suppose $|L_a| = p_i^{\alpha_i}$ with $\alpha_i = 1$.

There exists P_2 a Sylow p_2-subloop of order p_2^2 in L by Lemma (3.3)(d).

Now $L_a \lhd L$, so $L_aP_2 < L$ and$|L_aP_2| = \frac{|L_a||P_2|}{|L_a \cap P_2|} = p_ip_2^2$. Furthermore, since P_2 is a Sylow p_2-subloop of L_aP_2, $P_2 \lhd L_aP_2$ by Lemma (3.9). Also, since $(|P_2|, |L_a|) = (p_2^2, p_i^{\alpha_i}) = 1$, L is associative by Lemma (3.6). This negates the assumption of the Theorem.

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Therefore $|L_a| \neq p_i^{\alpha_i}$ with $\alpha_i = 1$.

4.0.2 Case 2: Suppose $|L_a| = q, q^2$ or q^3.

Case 2.1: Suppose $|L_a| = q$ Then there exists P_1 a Hall subloop of order p_1 in L by Lemma (3.3)(d). Now $L_a \lhd L$ so $L_aP_1 < L$ and $|L_aP_1| = \frac{|L_a||P_1|}{|L_a \cap P_1|} = p_1 q$. Furthermore, since P_1 is a Sylow p_1-subloop of L_aP_1, $P_1 \lhd L_aP_1$ by Lemma (3.15). Also, since $(|P_1|, |L_a|) = (p_1, q) = 1$, L is associative by Lemma (3.6). This also negates the assumption of the Theorem.

Therefore $|L_a| \neq q$.

Case 2.2: Suppose $|L_a| = q^3$ Now, since the order of L is coprime to 6, it follows from Lemma (3.10), that L_a is contained in the nucleus of L. Using Lemma (3.2), it follows that $|L_a|$ divides $|N|$, meaning that $q^3||N|$.

Clearly, by Lemma(3.8), N cannot contain a Hall subloop, so $q^4 \nmid |N|$. Thus since $|L_a| = q^3$ and $L_a \subset N$, it follows that L_a is a Sylow subloop of N. Now by Lemma (3.11), it follows that $L_a = N$, meaning $|L_a| = q^3 = |N|$.

Now by Lemma (4.1), there exists a normal hall subloop in Q of order q^4 in L. Suppose that NQ. Then NQ would be a subloop having order greater than q^4, which clearly contradicts Lemma (3.2). Hence, $N \subset Q$.

So, $|Q| / \mid N| = q$ meaning $\mid$ N$| < |Q|$. So there exists $x \in Q/N$ such that $\langle x \rangle N < L$ since $N \lhd L$ by Lemma (3.1). So $|N| < | \langle x \rangle N|$. Clearly, since $|N| < | \langle x \rangle N|$ and $\mid$ N$| < |Q|$, it follows that $|\langle x \rangle N| = |Q|$, meaning $Q = \langle x \rangle N$ for some $x \in Q/N$. This is a contradiction of Lemma (3.12).

Therefore $|L_a| \neq q^3$.

From all the cases and subcases proven above, it follows that $|L_a| = q^2$.

Finally the result below proves that subject to some extra conditions the Moufang loop of odd order $p_1p_2^2q^4$ is associative.

Theorem 4.3 *Let L be a Moufang loop of odd order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes satisfying $p_1 < p_2 < q$, $q \not\equiv \pm 1 (mod\ p_i)$ and $p_2 \not\equiv 1 (mod\ p_1)$. Assume that all proper subloops and proper quotient loops of L are groups and $(|L|, 6) = 1$. Then L is associative.*

Proof:

Assume L is not associative. Then there exist a nonassociative Moufang loop of odd order $p_1p_2^2q^4$, where p_1, p_2, q are distinct odd primes with $p_i < q$, $q \not\equiv \pm 1 (mod\ p_i)$ and $p_2 \not\equiv 1 (mod\ p_1)$. Assume also, that every proper subloop of L are associative. Then by Theorem (4.2), $|L_a| = q^2$. Now L_a, by Lemma (3.3)(b), is a minimal normal subloop of L and by the same Lemma (3.3)(b) is also an elementary abelian group. So L/L_a is a group and by Lemma (3.3)(d) there exists a subloop P/L_a of order $p_1p_2^2$ in L/L_a. In fact by Lemma (3.14), since $q \not\equiv \pm 1 (mod\ p_i)$ and $p_2 \not\equiv 1 (mod\ p_1)$ and the maximum power of each of the prime factors is 2, it follows that P/L_a is normal in L/L_a. Thus P is a normal Hall subloop of L of order $p_1p_2^2q^2$. Furthermore, by Lemma (3.14), there exists a subloop P^* of order $p_1p_2^2$ normal in P. Hence, by Lemma (3.5), since $P^* \lhd P \lhd L$ and $(|P^*|, |P/P^*|) = (p_1p_2^2, q^2) = 1$, it follows that P^* is a normal subloop of L. Now, since L/P^* is a quotient subloop(subgroup), it follows that $L_a \subset P^*$. So by Lemma (3.2), $|L_a|$ divides $|P^*|$. This is a contradiction as $|P^*| = p_1p_2^2$ and $|L_a| = q^2$.

It follows that L is associative.

Conclusion

In this work, we have investigated nonassociative moufang loops of odd order $p_1p_2^2q^4$, we saw that they exhibit unique properties and characteristics that distinguish them from other algebraic structures subject to some conditions on the order of the loop. Further research directions may include investigating further if the Moufang loop of this particular order without the extra conditions, would be associative. Also, the possible exploration of connections between Moufang loops and other algebraic structures can be investigated.

References

[1] Bruck, R. H. A, (1976). Survey of Binary Systems, Springer–Verlag, New York, 1971.

[2] F. Leong, The devil and angel of loops, Proc. Amer. Math. Soc., 54, 32–34.

[3] Wright, C. et al. (1965). Nilpotency conditions for finite loops, Illinois J. Math 9: 399– 409.

[4] Bol, G. (1937). Gewebe und gruppen, Math. Ann., 114, 414–431.

[5] Chein, O., and Rajah, (2000). Possible orders of nonassociative Moufang loops. *Commentationes Mathematicae Universitatis Carolinae*, 41(2):237–244.

[6] Gagola, S. M. III. (2014) On the nuclei of moufang loops with orders coprime to six. Journal of Algebra, 402:280-293.

[7] Ademola, L. A. and Rajah, A (2016). Moufang loops of odd order $p_1^3 p_2^3 \cdots p_n^3$. *Journal of Progressive Research in Mathematics.* 9(3) 1395-1405.

[8] Rajah, A and Ademola, L. A. (2017). Moufang loops of odd order $p^4 q^3$. *Bulletin of the Malaysian Mathematical Sciences Society.* DOI:10.1007/s40840-017-0471-2.

[9] Ademola, L. A. (2017). Ph. D. Thesis, Associativity of Moufang Loop of Odd Order, Universiti Sains Malaysia.

[10] Rajah, A and Ademola, L. A. (2018). Moufang loops of odd order $p^4 q_1^3 \cdots q_n^3$. *Bulletin of the Malaysian Mathematical Sciences Society.* DOI:10.1007/s40840-017-0471-2.

[11] Ademola, L. A., Bimwarit R, and Hosea G.G (2019). Determination of Normal Subgroups of Groups of Prime Orders $p^\alpha q^\beta$, $\alpha, \beta \in \{1, 2\}$. *Journal of the Nigerian Association of Mathematical Physics.* 9(March & May) 1-4.

[12] Ademola L. A. and Zaku G. G. (2021). Investigation On Lesser Prime Order Subloops Of Odd Order Moufang Loops. Asian journal of Pure and applied mathematics. 3(2)30-33

[13] Ademola L. A. and Jelten J. N. (2022). Maximal normal subloops of Moufang loop of odd order $p^\alpha q^4$ Asian journal of Pure and applied mathematics. 4(3)42-47.

[14] Zaku G. G. and Ademola L. A. (2023). Existence of An Associator Subloop For Moufang Loops Of Odd Order $p^2 q^4$ With $N \neq \emptyset$. *International Journal of Innovative Mathematics, Statistics & Energy Policies.* 11(1), 32-35.

[15] Jaiyeola, T. G. (2009). A study of new concepts in Smarandache quasigroups and loops. *ProQuest Information and Learning (ILQ)*, Ann Arbor, USA, 127pp.

[16] Solarin, A. R. T., Adeniran, J. O., Jaiyeola, T. G., Isere, A. O., and Oyebo, Y. T. (2023). Some Varieties of Loops (Bol-Moufang and Non-Bol-Moufang Types). *In M.N. Hounkonnou et al. (Eds.), Algebra without Borders: Classical and Constructive Nonassociative Algebraic Structures (STEAM-H)*. Springer, Cham.

[17] George, O.O., and Jaiyeola, T. G. (2022). Nuclear identification of some new loop identities of length five. *Buletinul Academiei de Ştiinţe a Republicii Moldova*. Matematic 99(2), 39–58.

[18] Nagy, G. P and Vojtechovsky, P., LOOPS: Computing with quasigroups and loops in GAP, version 1.0.0 computational package for GAP. Loops Package. *https://www.math.du.edu/loop*.