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Approximating Fixed Point of Generalized C-class Contractivity Conditions

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Abstract

In this study, we introduced novel class of contractivity conditions called C-class Akram contraction and C-class generalized M_J-Contraction and established the convergence of Picard and Jungck iterations to the unique fixed point and unique common fixed point respectively. Our results generalizes and extends some existing related results in literature.

Keywords: Fixed Point, $C-$Class Function, C-Class Akram Contraction, C-Class Generalized M_J-Contraction.
MSC2010: 47H10, 54H25.

1 Introduction and Preliminaries

Fixed point theory is a very useful tool in the fields of Mathematics, Engineering and several others. This is becuase, diverse real life problems in these fields may be formulated as systems of non-linear mathematical equations, such as optimization problem, differential equation, integro-diffrential equation, amongst others. These problems may be solved using fixed point approach. What then is fixed point and its approach?

Let X be a nonempty set and J a self-map on X. Any element $x \in X$ is a fixed point of J, if $J(x) = x$ and we denote the set of all fixed points of J by $F_J = \{x \in X : J(x) = x\}$.

Given a complete metric space M and a continuous self-map J on M such that

$$d(Jx, Jy) \leq ad(x, y), \quad \forall x, y \in M; \quad \text{with} \quad a \in [0, 1) \quad \text{fixed}, \tag{1.1}$$

Banach [1] established that J has a unique fixed point in M, and that Picard iteration $x_{n+1} = Jx_n$ converges to this unique fixed point.

In generalizing Banach's contractivity condition and some of its earlier generalizations, the following definitions was introduced by Akram et al.

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Definition 1.1. *[2]: An operator $J : X \to X$ of a metric space (X, d) is said to be A-contraction if:*

$$d(Jx, Jy) \leq \phi\Big(d(x, y), d(x, Jx), d(y, Jy)\Big) \tag{1.2}$$

for all $x, y \in X$ and some $\phi \in (A)$, where (A) is the set of all functions $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$ satisfying:
i) ϕ is continuous on the set $\mathbb{R}^3_+$ (with respect to Euclidean metric on $\mathbb{R}^3$);
ii) if any of the conditions $a \leq \phi(a, b, b)$, or $a \leq \phi(b, b, a)$, or $a \leq \phi(b, a, b)$ holds for some $a, b \in \mathbb{R}_+$, then there exist $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Literature abounds with several generalizations and extensions of classical Banach's fixed point theorem, interested reader can see [3–5] and the references there in.

More over, in 1976, Gerald Jungck [6] established the notion of common fixed point of mappings. Given a pair of self-mapping (J, P) on a complete metric space (X, d), with $J(X) \subset P(X)$ where P is continuous. Then, J *and* P have a unique common fixed point, if there exists $\aleph \in (0, 1)$ such that,

$$d(Jx, Jy) \leq \aleph d(Px, Py) \qquad \forall \qquad x, y \in X, \tag{1.3}$$

Definition 1.2. *[6] Let M be a complete metric space, and suppose $J, P : M \to M$. For $x_0 \in M$, sequence $\{Px_n\}_{n=0}^{\infty} \subset M$ defined by*

$$Px_{n+1} = Jx_n, \;\; n \geq 0, \tag{1.4}$$

is called Jungck iterative process.

Using idea of Jungck, many authors have improved on the existing iterative techniques.

Definition 1.3. *[7] Given a Banach space M, and the pair of operator $J, P : M \to M$. For any generic $x_0 \in M$, the sequence $\{Px_n\}_{n=0}^{\infty}$, defined by*

$$Px_{n+1} = (1 - \alpha)Px_n + \alpha Jx_n, \quad n \geq 0, \quad \alpha \in (0, 1). \tag{1.5}$$

is called Jungck-Schaefer iteration.

For more on Jungck-type iterative algorithms, interested reader can see [8–13] and references therein.

Recently, Olatinwo and Omidire [9] established unique common fixed point of generalized M_J contraction.

Definition 1.4. *[9]: Let (X, d) be a metric space and $J, P : X \to X$ such that*

$$\begin{aligned} d(Jx, Jy) \quad &\leq \quad \phi\Big(d(Px, Py), d(Px, Jx), d(Py, Jy), \\ &\quad\quad [d(Px, Jx)]^r [d(Py, Jx)]^p d(Px, Jy), d(Py, Jx)[d(Px, Jx)]^m\Big) \\ &\quad\quad \forall \; x, \;\; y \in X; r, p, m \in \mathbb{R}_+ \end{aligned}$$

for some $\phi \in \Phi$ where Φ is the set of all functions satisfying $\phi : \mathbb{R}^5_+ \to \mathbb{R}_+$ such that:
(i) ϕ is continuous on the set $\mathbb{R}^5_+$ (with respect to Euclidean metric on $\mathbb{R}^5$);
(ii) if any of the conditions $a \leq \phi(a, b, b, b, b)$, or $a \leq \phi(b, b, a, b, b)$, or $a \leq \phi(b, b, a, c, c)$ holds for some $a, b, c \in \mathbb{R}_+$, then there exists a constant $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Meanwhile, in 2014, Ansari introduced the concept of $C-$class functions as defined below:

Definition 1.5. *[14] A mapping $G : [0, \infty)^2 \to \mathbb{R}$ is called $C-$class function if it is continuous and satisfies the following axioms:*
(1) $G(\mu, \aleph) \leq \mu$;
(2) $G(\mu, \aleph) = \mu$ implies that either $\mu = 0$ or $\aleph = 0$; for all $\mu, \aleph \in [0, \infty)$.

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Note for some G we have that $G(0,0)=0$.
We denote the set of $C-$class functions by $\mathcal{C}$.
See [14] for example of C-class functions.

Remark 1.6. *Motivated by Definition 1.5 many researchers have improved on existing contractive definitions in other spaces and classes, like cone C-class, inverse C-class, multiplicative C-class function etc.*

Definition 1.7. *[14, 15] A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:*
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t)=0$ if and only if $t=0$.

We denote the set of altering distance functions by Ψ.

Definition 1.8. *[14] Let Φ denote the class of functions $\phi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:*
(i) ϕ is continuous;
(ii) $\phi(t)>0, \quad t>0$ and $\phi(0)\geq 0$.

Definition 1.9. *[14]A tripled (ψ,ϕ,G) where $\psi\in\Psi$, $\phi\in\Phi$ and $G\in\mathcal{C}$ is said to be monotone if for any $x,y\in[0,\infty)$*

$$x\leqslant y\Longrightarrow G(\psi(x),\phi(x))\leqslant G(\psi(y),\phi(y)).$$

Lemma 1.10. *[16]Suppose (X,d) is a metric space. Let $\{x_n\}$ be a sequence in Xsuch that $d(x_n,x_{n+1})\to 0$ as $n\to\infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon>0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with*

$m(k)>n(k)>k$ such that $d(x_{m(k)},x_{n(k)})\geq\varepsilon$, $d(x_{m(k)-1},x_{n(k)})<\varepsilon$ and
(i) $\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)+1})=\varepsilon$;
(ii) $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)})=\varepsilon$;
(iii).$\lim_{k\to\infty}d(x_{m(k)-1},x_{n(k)})=\varepsilon$

We note also that, $\lim_{k\to\infty}d(x_{m(k)+1},x_{n(k)+1})=\varepsilon$ and $\lim_{k\to\infty}d(x_{m(k)},x_{n(k)-1})=\varepsilon$

In this paper, it is our intention to give a more robust and novel contractivity definitions using $C-$ class function in generalizing A-contraction and M_J contraction which are generalizations and extensions of Banach, Jungck, Kannan, Akram et al., Olatinwo et al. contractivity definitions and many more in literature. Also, the convergence of Picard and Jungck iterative schemes to the unique fixed point and unique common fixed point of these novel contractivity definitions shall be established.

2 Preliminary

We introduce the following definitions which are generalization of Definitions 1.1 and 1.4 as well as some others in literature.

Definition 2.1. *Let M be a metric space. A mapping $J:M\to M$ will be called a* ***C-Class Akram Contraction*** *if $\forall\ x,y\in M$,*

$$d(Jx,Jy)\leq A(d(x,y),d(x,Jx),d(y,Jy)) \tag{2.1}$$

and some $(\psi,\phi,G)\in\Psi\times\Phi\times\mathcal{C}$ where $A:\mathbb{R}^3_+\to\mathbb{R}^2_+$ is a set of functions satisfying:

1. *A is continuous on the set $\mathbb{R}^3_+$ (with respect to the Euclidean metric on $\mathbb{R}^3$); and*

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2. *If any of the conditions $a \leq A(a,b,b)$ or $a \leq A(b,b,a)$ or $a \leq A(b,a,b)$ holds for some $a,b \in \mathbb{R}_+$, then $\exists\ G \in \mathcal{C}$ such that $\psi(a) \leq G(\psi(b),\phi(b))$.*

Definition 2.2. *Let M be a metric space. A function $J,P: M \to M$ will be called a **C-Class Generalized M_J Contraction** if*

$$\begin{aligned} d(Jx,Jy) \quad \leq \quad & A\Big(d(Px,Py), d(Px,Jx), d(Py,Jy), \\ & [d(Px,Jx)]^r[d(Py,Jx)]^p d(Px,Jy), d(Py,Jx)[d(Px,Jx)]^m\Big) \\ & \forall\ x,\ y \in X; r,p,m \in \mathbb{R}_+ \end{aligned} \tag{2.2}$$

and that some $(\psi,\phi,G) \in \Psi \times \Phi \times \mathcal{C}$ and where A is the set of all function satisfying $A: \mathbb{R}^5_+ \to \mathbb{R}^2_+$ such that:
(i) A is continuous on the set $\mathbb{R}^5_+$ (with respect to Euclidean metric on $\mathbb{R}^5$);
(ii) if any of the conditions $a \leq A(a,b,b,b,b)$, or $a \leq A(b,b,a,b,b)$, or $a \leq A(b,b,a,c,c)$ holds for some $a,b,c \in \mathbb{R}_+$, then there exists a function $G \in \mathcal{C}$ such that $\psi(a) \leq G(\psi(b),\phi(b))$.

Remark 2.3. *Definition 2.1 is a generalization of the Definition 1.1 (see [2]) which is a generalization of Banach, Kannan, and many more in literature.*

Remark 2.4. *Definition 2.2 generalizes Definition 1.4 see [9] which is a generalization of Jungck and many others in literature.*

The following shall be required in the sequel:

Definition 2.5. *[17] Two self-mappings J and P on a metric space X are weakly commuting if*

$$||JPu - PJu|| \leq ||Ju - Pu||,\ \forall\ u \in X.$$

Definition 2.6. *[18] Self mappings J and P on a metric space X are compatible if and only if*

$$\lim_{n\to\infty} ||JPu_n - PJu_n|| = 0$$

whenever $\{u_n\}$ is a sequence in X, such that

$$\lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} P(u_n) = w$$

for some $w \in X$.

Remark 2.7. *Commuting mappings are weakly commuting and the reverse is not true, see [17] for example. Weakly commuting mappings are compatible, but compatible mappings may not be weakly commuting see [18] for illustration.*

Lemma 2.8. *[17, 18]: Let J and P be two compatible self-mappings on a complete metric space M. If $Jx^* = Px^*$, then $JPx^* = PJx^*$.*
Assume that $\lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Px_n = w$ for some $w \in M$.
(a) If J is continuous at w, then

$$\lim_{n\to\infty} PJx_n = Jw.$$

If P is continuous at w, then

$$\lim_{n\to\infty} JPx_n = Pw.$$

(b) If J and P are continuous at w, then

$$Jw = Pw \quad \text{and} \quad PJw = JPw.$$

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3 Results

Theorem 3.1. *Let M be a complete metric space and let $J : M \to M$ be a mapping satisfying Definition 2.1, Then:*
(i) J has a unique fixed point $x^* \in M$; and
(ii) For any initial guess $x_0 \in M$, the Picard iteration converges to the unique fixed point of $J \in M$.

Proof. Using Picard iteration, $x_1 = Jx_0$ and Definition 2.1, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Jx_{n-1}, Jx_n) \\
&\leq A\big(d(x_{n-1}, x_n), d(x_{n-1}, Jx_{n-1}), d(x_n, Jx_n)\big) \\
&= A\big(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\big) \\
& \quad \text{then, there exists a function } G \in \mathcal{C} \text{ such that} \\
\psi(d(x_n, x_{n+1})) &\leq G(\psi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n))) \qquad (3.1) \\
& \quad \text{using Definition 1.5, we arrived at} \\
d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_n).
\end{aligned}$$

We deduce that $\{d(x_n, x_{n+1})\}$ is a monotonic decreasing sequence of positive numbers.
And since every monotone decreasing sequence in a metric space is bounded by zero, so there exists $l \geq 0$ such that

$$d(x_n, x_{n+1}) \leq l$$

Letting $n \to \infty$ in (3.1) and using continuity of functions G, ψ and ϕ, we obtain

$$\psi(l) \leq G(\psi(l), \phi(l)),$$

and by Definition 1.5, we arrived at $\psi(l) = 0$ or $\phi(l) = 0$.
Thus, we have that $l = 0$ and, then

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \qquad (3.2)$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence, If $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.10 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ε as $k \to \infty$:

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon \ , \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \qquad (3.3)$$

Using (3.1) together with (2.1), we have

$$d(x_{m(k)+1}, x_{n(k)+1})) \leq G(\psi(p(x_{m(k)}, x_{n(k)})), \phi(p(x_{m(k)}, x_{n(k)}))), \qquad (3.4)$$

and

$$\begin{aligned}
d(x_{m(k)+1}, x_{n(k)+1}) &= d(Jx_{m(k)}, Jx_{n(k)}) \\
&\leq A\big(d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, Jx_{m(k)}), d(x_{n(k)}, Jx_{n(k)})\big) \\
&= A\big(d(x_{m(k)}, x_{n(k)}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{n(k)+1})\big) \\
& \quad as\ k \to \infty, \text{ we have} \\
\varepsilon &\leq A\big(\varepsilon, 0, 0\big),
\end{aligned}$$

and using Definitions 1.5, 1.7 and 2.1, we arrived at the following

$$\psi(\varepsilon) \leq G(\psi(0), \phi(0)) \leq \psi(0) = 0$$

which implies that $\psi(\varepsilon) = 0$, thus $\varepsilon = 0$. This shows that the sequence $\{x_n\}$ is Cauchy. Then, there exists $x^* \in M$ such that $x_n \to x^*$ as $n \to \infty$.
Now, let $x = x^*$ and $y = x_n$ in Definition 2.1, we have:

$$\begin{aligned}
d(Jx^*, Jx_n) &\leq A\Big(d(x^*, x_n), d(x^*, Jx^*), d(x_n, Jx_n)\Big) \\
d(Jx^*, x_{n+1}) &\leq A\Big(d(x^*, x_n), d(x^*, Jx^*), d(x_n, x_{n+1})\Big)
\end{aligned}$$

taking limits as $n \to \infty$ together with Definition 2.1 we have

$$\begin{aligned}
d(Jx^*, x^*) &\leq A\Big(d(x^*, x^*), d(x^*, Jx^*), d(x^*, x^*)\Big) \\
&= A\Big(0, d(x^*, Jx^*), 0\Big) \\
&\quad \text{then, there exists a } G \in \mathcal{C} \text{ such that} \\
\psi(d(Jx^*, x^*)) &\leq G(\psi(0), \phi(0)) \leq \psi(0) = 0 \\
&\quad \text{that is} \\
\psi(d(Jx^*, x^*)) &= 0 \\
&\quad \text{Using Definition 1.7, we arrived at the following} \\
d(Jx^*, x^*) &= 0.
\end{aligned}$$

Therefore, $Jx^* = x^*$.

For uniquness of the fixed point. If $p \in M$ exists satisfies $Jp = p, \quad p \neq x^*$ then taking $x = p$ and $y = x^*$ in Definition 2.1, we get

$$\begin{aligned}
d(p, x^*) &= d(Jp, Jx^*) \\
&\leq A\big(d(p, x^*), d(p, Jp), d(x^*, Jx^*)\big) \\
&= A\big(d(p, x^*), d(p, p), d(x^*, x^*)\big) \\
&\quad A\big(d(p, x^*), 0, 0\big) \\
&\quad \text{then,} \\
\psi(d(p, x^*)) &\leq G(\psi(0), \phi(0)) \leq \psi(0) = 0 \\
&\quad \text{and, we arrived at} \\
d(p, x^*) &= 0.
\end{aligned}$$

Hence, we have that $\quad p = x^*$. □

Theorem 3.2. *Let M be a complete metric space and let $J, P : M \to M$ be compatible mappings satisfying Definition 2.2. If P is continuous and $J(M) \subset P(M)$, then:*
(i) J and P have a unique common fixed point $x^* \in M$; and
(ii) For any generic point $x_0 \in M$, the Jungck iteration (1.4) converges to the unique common fixed point x^* (say) of J and P.

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Proof. By Jungck iteration (1.4) and Definition (2.2), we have

$$\begin{aligned}
d(Px_n, Px_{n+1}) &= d(Jx_{n-1}, Jx_n) \\
&\leq A\Big(d(Px_{n-1}, Px_n), d(Px_{n-1}, Jx_{n-1}), d(Px_n, Jx_n), \\
&\quad [d(Px_{n-1}, Jx_{n-1})]^r [d(Px_n, Jx_{n-1})]^p d(Px_{n-1}, Jx_n), \\
&\quad d(Px_n, Jx_{n-1})[d(Px_{n-1}, Jx_{n-1})]^m\Big) \\
&= A\Big(d(Px_{n-1}, Px_n), d(Px_{n-1}, Px_n), d(Px_n, Px_{n+1}), \\
&\quad [d(Px_{n-1}, Px_n)]^r [d(Px_n, Px_n)]^p d(Px_{n-1}, Jx_n), \\
&\quad d(Px_n, Px_n)[d(Px_{n-1}, Px_n)]^m\Big) \\
&= A\Big(d(Px_{n-1}, Px_n), d(Px_{n-1}, Px_n), d(Px_n, Px_{n+1}), 0, 0\Big) \\
&\quad \text{then, there exists a } G \in \mathcal{C} \text{ such that} \\
\psi\Big(d(Px_n, Px_{n+1})\Big) &\leq G\Big(\psi(d(Px_{n-1}, Px_n)), \phi(d(Px_{n-1}, Px_n))\Big) \qquad (3.5) \\
&\quad \text{using Definition 1.5, we arrived at the following} \\
d(Px_n, Px_{n+1}) &\leq d(Px_{n-1}, Px_n).
\end{aligned}$$

And we deduce that $\{d(Px_n, Px_{n+1})\}$ is a monotonic decreasing sequence of positive numbers. And since every monotononic decreasing sequence in a metric space is bounded by zero, so there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(Px_n, Px_{n+1}) = l$$

Letting $n \to \infty$ in (3.5) and using continuity of functions G, ψ and ϕ, we obtain

$$\psi(l) \leq G(\psi(l), \phi(l)),$$

which implies that $\psi(l) = 0$ or $\phi(l) = 0$. Thus $l = 0$ and

$$\lim_{n\to\infty} d(Px_n, Px_{n+1}) = 0 \qquad (3.6)$$

Now, we shall prove that $\{Px_n\}$ is a Cauchy sequence. Suppose not, then by Lemma 1.10 there exist $\varepsilon > 0$ and two sequences $\{Pm_k\}$ and $\{Pn_k\}$ of positive integers such that the following sequences tend to ε as $k \to \infty$:

$$\lim_{k\to\infty} d(Px_{m(k)+1}, Px_{n(k)+1}) = \varepsilon \ , \lim_{k\to\infty} d(Px_{m(k)}, Px_{n(k)}) = \varepsilon \qquad (3.7)$$

From (3.5) together with (2.2), we have

$$d(Px_{m(k)+1}, Px_{n(k)+1})) \leq G(\varphi(d(Px_{m(k)}, Px_{n(k)})), \psi(d(Px_{m(k)}, Px_{n(k)}))), \qquad (3.8)$$

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and

$$\begin{aligned}
d(Px_{m(k)+1}, Px_{n(k)+1}) &= d(Jx_{m(k)}, Jx_{n(k)}) \\
&\leq A\Big(d(Px_{m(k)}, Px_{n(k)}), d(Px_{m(k)}, Jx_{m(k)}), d(Px_{n(k)}, Jx_{n(k)}), \\
&\quad [d(Px_{m(k)}, Jx_{m(k)})]^r [d(Px_{n(k)}, Jx_{m(k)})]^p d(Px_{m(k)}, Jx_{n(k)}), \\
&\quad d(Px_{n(k)}, Jx_{m(k)})[d(Px_{m(k)}, Jx_{m(k)})]^m\Big) \\
&= A\Big(d(Px_{m(k)}, Px_{n(k)}), d(Px_{m(k)}, Px_{m(k)+1}), d(Px_{n(k)}, Px_{n(k)+1}), \\
&\quad [d(Px_{m(k)}, Px_{m(k)+1})]^r [d(Px_{n(k)}, Px_{m(k)+1})]^p d(Px_{m(k)}, Px_{n(k)+1}), \\
&\quad d(Px_{n(k)}, Px_{m(k)+1})[d(Px_{m(k)}, Px_{m(k)+1})]^m\Big) \\
&\quad \text{as } k \to \infty, \text{we have} \\
\varepsilon &\leq A\big(\varepsilon, 0, 0, 0, 0\big) \\
&\quad \text{then, there exists a } G \in \mathcal{C} \text{ such that} \\
\psi(\varepsilon) &\leq G(\psi(0), \phi(0)). \\
&\quad \text{And using Definitions 1.5 and 1.7, we arrived at the following} \\
\psi(\varepsilon) &\leq G(\psi(0), \phi(0)) \leq \psi(0) = 0,
\end{aligned}$$

which implies that $\psi(\varepsilon) = 0$, thus $\varepsilon = 0$. This shows that the sequence $\{Px_n\}$ is Cauchy. Therefore, there exists $x^* \in M$ such that $Px_n = Jx_{n-1} \to x^*$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Jx_{n-1} = x^* \tag{3.9}$$

By Compatibility of J and P, continuity of P (continuity of P implies continuity of J as well, since $J(M) \subset P(M)$) and Lemma 2.8, we have:

$$\lim_{n\to\infty} P(Px_{n-1}) = Px^*, \ \lim_{n\to\infty} J(Px_{n-1}) = Px^* \tag{3.10}$$

and

$$\lim_{n\to\infty} P(Jx_n) = \lim_{n\to\infty} J(Px_{n-1}) = Px^*. \tag{3.11}$$

From Definition (2.2), with $x = Px_n$ and $y = x_n$, we have

$$\begin{aligned}
d(J(Px_n), Jx_n) &\leq A\Big(d(P(Px_n), Px_n), d(P(Px_n), J(Px_n)), d(Px_n, Jx_n), \\
&\quad [d(P(Px_n), J(Px_n))]^r [d(Px_n, J(Px_n))]^p d(P(Px_n), Jx_n), \\
&\quad d(Px_n, J(Px_n))[d(P(Px_n), J(Px_n))]^m\Big),
\end{aligned}$$

as $n \to \infty$, using equations (3.9), (3.10), (3.11), we have

$$\begin{aligned}
d(Px^*, x^*) &\leq A\Big(d(Px^*, x^*), d(Px^*, Px^*), d(x^*, x^*), \\
&\quad [d(Px^*, Px^*)]^r [d(x^*, Px^*)]^p d(Px^*, x^*), \\
&\quad d(x^*, Px^*)[d(Px^*, Px^*)]^m\Big) \\
&= A\Big(d(Px^*, x^*), 0, 0, 0, 0\Big) \\
&\quad \text{then, there exists a } G \in \mathcal{C} \text{ such that} \\
\psi(d(Px^*, x^*)) &\leq G(\psi(0), \phi(0)) \leq \psi(0) = 0. \\
&\quad \text{And using Definition 1.7, we arrived at the following} \\
(d(Px^*, x^*) &= 0.
\end{aligned}$$

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Therefore, $Px^* = x^*$.

Again by Lemma 2.8 (b), together with the fact that P is continuous and $J(M) \subseteq P(M)$, implies J is also continuous. Then, we have

$$Jx^* = Px^* = x^*, \quad \text{and} \quad PJx^* = JPx^*$$

For the uniqueness of this common fixed point. Suppose not, then there exists $x^* \neq y^*,\ x^*, y^* \in \{F_{P\cap J}\}$(common fixed point set of operators J and P) such that
$Px^* = Jx^* = x^*$, $Py^* = Jy^* = y^*$, and we have

$$\begin{aligned}
d(x^*, y^*) &= d(Jx^*, Jy^*) \\
&\leq A\Big(d(Px^*, Py^*), d(Px^*, Jx^*), d(Py^*, Jy^*), \\
&\quad [d(Px^*, Jx^*)]^r[d(Py^*, Jx^*)]^p d(Px^*, Jy^*), d(Py^*, Jx^*)[d(Px^*, Jx^*)]^m\Big), \\
&= A\Big(d(x^*, y^*), d(x^*, x^*), d(y^*, y^*), \\
&\quad [d(x^*, x^*)]^r[d(y^*, x^*)]^p d(x^*, y^*), d(y^*, x^*)[d(x^*, x^*)]^m\Big) \\
&= A\Big(d(x^*, y^*), 0, 0, 0, 0\Big) \\
&\quad \text{then, there exists a } G \in \mathcal{C} \text{ such that} \\
\psi(d(x^*, y^*)) &\leq G(\psi(0), \phi(0)) \leq \psi(0) = 0 \\
&\quad \text{and, we arrived at} \\
d(x^*, y^*) &= 0.
\end{aligned}$$

Hence, $x^* = y^*$ □

Below is a result on commutativity of mappings which is a stronger condition than compatibility (see Remark 2.7) and we omit the prove as it is similar to what is obtained in the prove of Theorem 3.2.

Corollary 3.3. *Let M be a complete metric space and let $J, P : M \to M$ be commuting (or weakly commuting) mappings satisfying Definition 2.2. If P is continuous and $J(M) \subset P(M)$, then:*
(i) J and P have a unique common fixed point $x^* \in M$;
(ii) The unique common fixed point x^* (say) can be approximated by Jungck iteration for any generic point $x_0 \in M$.

Remark 3.4. *Theorem 3.1 is a generalization of the results in Akram et al. (see [2]) which is a generalization of Banach, Kannan, and many more in literature.*

Remark 3.5. *Theorem 3.2 generalizes the work of Olatinwo et al. [9] which is a generalization of Jungck and many others in literature.*

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