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Improved Finite Difference Methods for Solving Second-Order Boundary Value Problems of Ordinary Differential Equations Using Chebyshev Polynomials

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Abstract

In this paper, two advance numerical techniques for solving second-order boundary value problems in ordinary differential equations (ODEs) are presented. The first method, the Chebyshev Finite Difference Method (CFDM), which enhances the traditional Finite Difference Method by utilizing Chebyshev Polynomials as basis functions, resulting in improved computational performance. The second method developed is the Perturbed Chebyshev Finite Difference Method (Perturbed CFDM), which incorporates perturbation techniques to further enhance the accuracy and efficiency of the method. Both methods were applied to homogeneous and non-homogeneous linear boundary value problems, with numerical results demonstrating that the Perturbed CFDM significantly outperforms both standard CFDM and the traditional finite difference method in terms of accuracy and computational efficiency. These findings establish the Perturbed CFDM as a powerful and reliable tool for solving boundary value problems. All computations were carried out using MATLAB, ensuring accurate approximation and numerical solutions of the tested problems.

Keywords: Boundary Value Problems, Chebyshev Polynomials, Finite Difference Method, Ordinary Differential Equations, Perturbation.
MSC2010: 34B05, 33C45, 15A06, 65D25, 65L80.

1 Introduction

Ordinary Differential Equations (ODEs) are essential tools for modeling diverse phenomena across fields such as social sciences, natural sciences, and engineering [1]. They describe systems like beam depletion, disease epidemiology, and population dynamics, with solutions often dependent on specific initial and boundary conditions [2]. Initial Value Problems (IVPs) define conditions at a single point [3], while Boundary Value Problems (BVPs) involve conditions, at more than one points within the domain of problem solution [4]. Analytical solutions to some BVPs are often challenging due to their complexity, making numerical approximations a practical alternative [5].

Over the years, numerous numerical methods including the Collocation Method, Finite Difference Method (FDM), Galerkin Method, Shooting Method, Tau Method, Finite Element Method (FEM), Spectral Method and Least Squares Method have been developed to approximate solutions to differential equations, particularly for problems lacking analytical solutions. These methods have demonstrated increasing accuracy and efficiency. Among them, the FDM, particularly the central forward difference, has been widely employed for solving both ODEs and Partial Differential Equations (PDEs). However, despite its accuracy, the FDM has limitations, such as its inability to provide solutions between grid points and the high computational cost required for greater precision [6].

Recent advancements in numerical methods have sought to improve accuracy and computational efficiency through the use of polynomial functions. Orthogonal polynomials, including Chebyshev, Legendre, and Laguerre polynomials, have gained significant attention due to their ability to enhance approximation accuracy and convergence rates. Researchers have applied these polynomials in combination with numerical techniques to achieve substantial improvements. For example, the author(s) in [7] utilized Bernstein polynomials with least squares approximation to solve Volterra-Fredholm integro-differential equations, achieving high precision. Abdurkadir [8] employed a generalized shifted Legendre polynomial approximation to solve second-order nonlinear BVPs, demonstrating efficiency through MATLAB simulations. Similarly, Negarchi and Nouri [9] applied Hermite polynomials with a modified variational iteration method to tackle tenth-order BVPs, reducing computational time while improving accuracy. Rubayyi [10] combined the spectral collocation method with Legendre polynomials to solve nonlinear fractional Klein-Gordon equations. In [11], the author(s) integrated the Galerkin method with Chebyshev polynomials to solve some boundary value problems in of ordinary differential equations. Author(s) in [12] demonstrated the efficacy of Chebyshev polynomials in a modified variational iteration method for solving 12th-order BVPs. Hence, these studies underscore the effectiveness of combining numerical methods with orthogonal polynomials to address the solution of complex differential equations.

In this study, an improved finite difference method incorporating Chebyshev polynomials and perturbation is developed for solving second-order boundary value problems of ordinary differential equations. The paper is organized as follows: Section 1 provides an introduction to the study, section 2 presents preliminary definitions and discussion on ODEs, finite difference methods, and Chebyshev polynomials. In Section 3, the formulation of the improved finite difference methods; the Chebyshev Finite Difference Method (CFDM) and the Perturbed Chebyshev Finite Difference Method (PCFDM) are presented. Numerical examples, results, discussions, and conclusions are provided in Section 4.

2 Preliminary

In this section, the fundamental definitions and concepts essential for the application of Chebyshev finite difference approximation methods to boundary value problems of ordinary differential equations are presented. [Differential Equation] A differential equation is an equation that relates the derivative of a dependent function say y to an independent function say x or t. The derivative can either be a total derivative: $\frac{dy}{dx}$ or a partial derivative: $\frac{\partial y}{\partial x}$ [13]. [Second Order Boundary Value Problem] [14] described a general form of second order boundary value problem of ODEs as follows:

$$\frac{d^2y}{dx} = f(x, y, dy/dx), \quad a \leq x \leq b, \tag{2.1}$$

where the boundary conditions on the interval $[a, b]$ is written as $y(a) = \alpha$ and $y(b) = \beta$. For $\alpha,\ \beta$ are some constant values. [Finite Difference Approximation Method] The finite difference approximation methods (backward, central and forward), are numerical schemes for solving differential equations by replacing each of the derivatives in the differential equation with an appropriate difference-quotient approximation. The difference approximation methods are thus known as Finite or Discretization Methods [15]. [Polynomial Function] An nth-degree polynomial of a variable x

is a linear combination of constant value function c's Funaro_Pol_approx_1994, written as

$$y(x) = c_n x^n + c_{(n-1)} x^{(n-1)} + \cdot + c_1 x + c_0 \tag{2.2}$$

[Orthogonal Polynomials] Orthogonal polynomials are set of polynomials say $\{P_n(x)\}$ that are defined over a range of interval $[a, b]$ and obey orthogonality property [16]

$$\int_a^b W(x) P_n(x) P_m(x) dx = \delta_{m,n} \lambda_n, \tag{2.3}$$

where $W(x)$ is a weighting function and $\delta_{m,n}$ is Kronecker delta. Then if $\lambda_n = 1$, the polynomials are not only orthogonal but also known to be orthonormal.

2.1 Central Finite Difference over Backward and Forward Finite Difference

Finite difference methods (backward, forward, and central) are widely used in science and engineering to numerically approximate several type of differential equations with or without analytical solution. Among these, the central finite difference method stands out due to its distinct advantages. Firstly, its symmetric nature ensures higher accuracy compared to forward and backward difference methods [15]. Secondly, it offers greater stability and reduces numerical dispersion, making it more robust in simulations. Thirdly, the central method exhibits faster convergence rates for iterative solutions, which enhances computational efficiency in large-scale problems. Lastly, it is flexible enough to achieve higher-order accuracy by incorporating additional node points, making it ideal for precision-demanding applications [17]. These properties make the central finite difference method a reliable and widely preferred tool for solving ordinary and partial differential equations.

2.2 Second Order Ordinary Differential Equations

An ordinary differential equation with derivative of the unknown function y in order two, say $\frac{d^2(y)}{dx^2}$ is called second order ordinary differential equation. Where x is an independent variable. The second order ODE can either be equipped with initial value conditions known as initial value problem (IVP) or boundary conditions known as boundary value problems (BVPs). A general form of second order ODE is of the form [14].

$$P(x)\frac{d^2(y)}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y(x) = f(x), a < x < b, \tag{2.4}$$

where $P(x)$, $Q(x)$, $R(x)$ are constant or variable coefficient of the ODE and if $f(x) = 0$, the equation (2.4) is homogeneous or else non homogeneous. Equation (2.4) is called a second order boundary value problem if the unknown function y is equipped with the interval $[a, b] \in x$ as

$$y(a) = y_0,\ y(b) = y_c \tag{2.5}$$

2.3 Chebyshev Equation and Polynomials

Chebyshev differential equation is one of the special case of Sturm Liouvile boundary value problem. The orthogonality property, generating function and Parseval's identity are some important properties of the polynomials. Compare to other interpolation functions, Chebyshev polynomial have been extensively used and is said to be more accurate in approximating polynomial functions [18]. The Sturm Liouvile ODE and its boundary conditions is defined on an interval $[a, b] \in x$ as

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + [q(x) + \lambda r(x)]y = 0, \tag{2.6}$$

$$c_1 y(a) + c_2 y'(b) = 0,$$
$$c_3 y(b) + c_4 y'(b) = 0,$$

where c_1, c_2, c_3 and c_4 are constants and λ is a parameter to be determine. Suppose the interval $a = 1$ and $b = -1$, set $p(x) = \sqrt{1-x^2}$, $q(x) = 0$, $r(x) = \frac{1}{\sqrt{1-x^2}}$ and $\lambda = n^2$, then equation (2.6) can be written as

$$\frac{d}{dx}\left((1-x^2)\frac{d^2y}{dx}\right) + n^2\frac{1}{\sqrt{1-x^2}}y = 0 \tag{2.7}$$

2.3.1 Chebyshev differential equation [19]

The ordinary differential equation (2.6) written in the form (2.7) is called Chebyshev differential equation with $|x| < 1$ and $n \in N_0$.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y(x) = 0 \tag{2.8}$$

The solution of the ordinary differential equation (2.8), which exhibits singularities at $x = [1, -1]$, would be represented as the Chebyshev polynomials.

2.3.2 Orthogonality property of Chebyshev polynomials

The polynomials $T_n(x)$ form complete orthogonal set on the interval $[1, -1] \in x$ with the weighted function $w(x) = \frac{1}{\sqrt{1-x^2}}$ [19], such that

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{matrix} 0, \ m \neq n \\ \pi, \ m = n \\ \frac{\pi}{2}, \ m = n = 1, 2\cdots \end{matrix} \tag{2.9}$$

2.3.3 Parseval's identity property

The parseval's identity that relates the series $y(x) = \sum_{n=0}^{infty} a_n T_n(x)$ with Chebyshev polynomials $T_n(x)$ is defined as

$$\int_{-\infty}^{\infty} w(x)[y(x)]^2\, dx = \pi a_o^2 + \frac{\pi}{2}\sum_{n=1}^{\infty} a_n^2 \tag{2.10}$$

2.3.4 Oscillation property

Among the orthogonal polynomials that are bounded by the interval $[-1, 1]$, the Chebyshev polynomial $T(x)$ oscillates with smallest maximum deviation in the interval $[-1, 1]$.

2.3.5 Chebyshev polynomial of first kind

The Chebyshev polynomials $T_n(x)$ can be obtained by means of Rodrigue's formula

$$T_n(x) = \frac{(-2)^n n!}{(2n)!}\sqrt{1-x^2}\frac{d^n}{dx^n}(1-x^2)^{n-1/2}, \quad n = 0, 1, 2, 3, ...$$

The Chebyshev first kind generating function is given by

$$\frac{1-zx}{1-2zx+z^2} = \sum_{n=0}^{\infty} T_n(x)z^n \tag{2.11}$$

When the first two Chebyshev polynomials $T_0(x)$ and $T_1(x)$ are known, all other polynomials $T_n(x)$, $n \geq 2$ can be obtained by means of recurrence formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{2.12}$$

2.3.6 Chebyshev polynomial of second kind

The generating function of Chebyshev polynomials of second kind for $x \in [-1,1]$ and $n = 1,2,3,\cdots$ is given by

$$\frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^2, \quad \forall\, |x| < 1, \quad |t| < 1 \tag{2.13}$$

And satisfies the recurrence relation given by

$$\begin{aligned} U_0(x) &= 1 \qquad\qquad U_1(x) = 2x \\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x), \ \forall \geq 1 \end{aligned} \tag{2.14}$$

2.3.7 Shifted Chebyshev Polynomials

For analytical and numerical studies, it is often convenient to shift the interval of Chebyshev polynomials from the full range $[-1,1]$ to the half interval $0 \leq x \leq 1$. This adjustment is particularly useful for boundary value problems that are defined solely on the half interval [20]. Thus, the shifted Chebyshev polynomials are defined by

$$T_n^*(x) = (2x-1)T_n(x).$$

The first five Shifted Chebyshev polynomials are given as

$$\begin{gathered} T0 = 1; \\ T1 = 2x - 1; \\ T2 = 8x^2 - 8x + 1; \\ T3 = 32x^3 - 48x^2 + 18x - 1; \\ T4 = 128x^4 - 256x^3 + 160x^2 - 32x + 1; \\ T5 = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1; \end{gathered}$$

The recurrence formula of shifted Chebyshev polynomials is defined as

$$T_{n+1}^*(x) = (4x-2)T_n^*(x) - T_{n-1}^*(x), \qquad T_0^* = 1 \tag{2.15}$$

2.4 Central Finite Difference Approximation Method for Second Order ODEs

The central difference approximations of first and second order ordinary differential equations are obtained as follows [15]: The central difference of first derivative is defined by

$$\frac{dy}{dx} = \frac{y(x+h) - y(x-h)}{2h} + O(h^2) \tag{2.16}$$

The central difference of second derivative is given by

$$\frac{d^2y}{dx^2} = \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + O(h^2) \tag{2.17}$$

where $0(h^2)$ is the truncation error. The $y_1, y_2, ..., y_{n-1}$ can be calculated as; At the mesh point $x = x_i$.

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A general second-order ordinary differential equation is given by:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y(x) = f(x), a \leq x \geq b \tag{2.18}$$

The central forward difference representation of equation (2.18) is given by

$$P(x_i)\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + Q(x_i)\frac{y_{i+1} - y_{i-1}}{2h} + R(x_i)y_i = f(x_i), (i = 1, 2, \cdots, n+1) \tag{2.19}$$

$$2P(x_i)(y_{i+1} - 2y_i + y_{i-1}) + hQ(x_i)(y_{i+1} - y_{i-1}) + 2h^2R(x_i)y_i = 2h^2f(x_i), (i = 1, 2, ..., n+1) \tag{2.20}$$

Equation (2.20) is the central forward approximation of the second order ODE (2.18) at the node points of x_i, for $i = 1, 2, \cdots n$.

3 Improved Finite Difference Methods for Boundary Value Problems

3.1 Chebyshev-Finite Difference Method (CFDM)

The approach in this section is designed to harness the properties of Chebyshev Polynomial in the application of finite difference method. The usual central finite differences are replaced by Chebyshev basis functions $T_r(x)$ of degree n, that is

$$y'(x) = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$y''(x) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

The above components of the finite differences are replaced with the following:

$$y_i = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_i$$

$$y_{i+1} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1}$$

$$y_{i-1} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}$$

$$y_{i-2} = \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-2}$$

These yield the following:

$$y(x) = \sum_{r=0}^{n} a_r T_r(x) \tag{3.1}$$

$$y'(x) = \frac{1}{2h}\left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1} - \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right) \tag{3.2}$$

$$y''(x) = \frac{1}{h^2}\left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1} - 2\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i} + \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right) \tag{3.3}$$

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3.1.1 CFDM for Second Order Ordinary Differential Equations

Consider the general form of the second order ordinary differential equation defined by equation (2.18) as:

$$P(x)y''(x) + Q(x)y'(x) + R(x)y(x) = f(x), a \leq x \geq b \tag{3.4}$$

Subject to the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta \tag{3.5}$$

The approximate solution is designed to takes the form:

$$y(x) = \sum_{r=0}^{n} a_r T_r(x) \tag{3.6}$$

The derivatives in equation (3.4) are replaced by Chebyshev approximate derivatives in equation (3.2) and (3.3) and yields:

$$\begin{array}{c} \frac{P(x)}{h^2}\left(\left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1}\right) - 2\left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i}\right) + \left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right)\right) + \\ \frac{Q(x)}{2h}\left(\left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i+1}\right) - \left(\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right)\right) + R(x)\left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i} = f(x) \end{array} \tag{3.7}$$

For $n = \frac{b-a}{h}$. Where a and b are the boundary points, and h is the step size. $T_r(x)$ is the Chebyshev polynomials Shifted into the interval $[a, b]$ of the given problem, That is, $a \leq x \leq b$. The a_r is called the adjustable parameters. Suppose $n = 4$, equation (3.7) is expanded as follows:

$$\begin{array}{c} \frac{P(x)}{h^2}\left(\left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i+1} - 2\left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i} + \left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i-1}\right) + \\ \frac{Q(x)}{2h}\left(\left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i+1} - \left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i-1}\right) + R(x)\left[\sum_{r=0}^{4} a_r T_r(x)\right]_{i} = f(x) \end{array} \tag{3.8}$$

$\Rightarrow$

$$\begin{array}{c} \frac{1}{h^2}P(x)\left([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{i+1}\right) - 2\frac{1}{h^2}P(x)\left([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{i}\right) + \\ \frac{1}{h^2}P(x)\left([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{i-1}\right) + \\ \frac{1}{2h}Q(x)\left([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{i+1} - [a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{i-1}\right) + \\ R(x)\left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{i} = f(x) \end{array} \tag{3.9}$$

Multiply through the terms by h^2, yields

$$\begin{array}{c} P(x_i)([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_{i+1}} - 2\left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{x_i} + \\ \left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{x_{i-1}}) + \\ \frac{h}{2}Q(x_i)\left(\left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{x_{i+1}} - \left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{x_{i-1}}\right) + \\ h^2R(x_i)\left[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4\right]_{x_i} = h^2f(x_i) \end{array} \tag{3.10}$$

To derive the matrix form of equation (3.10) for any nth degree of Chebyshev polynomials, the following working steps are applied:

Let denote the nth vector form of the adjustable parameters by:

$$\mathbf{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{bmatrix}$$

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$\mathbf{T}_i = \begin{bmatrix} T_0(x_i) & T_1(x_i) & T_2(x_i) & T_3(x_i) & T_4(x_i) & \cdots & T_n(x_i) \end{bmatrix}$ denote the nth row vector of Chebyshev polynomials evaluated at x_i.

$\mathbf{T} = \begin{bmatrix} \mathbf{T}_0 \\ \mathbf{T}_1 \\ \vdots \\ \mathbf{T}_n \end{bmatrix}$ is the matrix of Chebyshev polynomials evaluated at all x_i

$\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ are diagonal matrices with elements $P(x_i)$, $Q(x_i)$, and $R(x_i)$, respectively.

$\mathbf{f}$ represents the vector of function values $f(x_i)$.

$\mathbf{D}_1$ denote the first difference matrix:

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix} \tag{3.11}$$

$\mathbf{D}_2$ denote the second difference matrix:

$$\mathbf{D}_2 = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{bmatrix} \tag{3.12}$$

Therefore, the nth matrix form of equation (3.10) is defined as by:

$$\left(\mathbf{PD}_2 + \frac{h}{2}\mathbf{QD}_1 + h^2\mathbf{R}\right)\mathbf{Ta} = h^2\mathbf{f} \tag{3.13}$$

The derived matrix equation (3.13) is the Chebyshev-Finite Difference Method (CFDM) of approximation for solving the second order homogeneous (if $f(x) = 0$) and non-homogeneous ($f(x) \neq 0$) ordinary differential equations. To determine the adjustable parameters $a_r = a_0, a_1, a_2, \cdots, a_n$, $n-2$ matrix of algebraic equations is generated from equation (3.13) for x_i such that $i = 1, 2, \cdots, n-2$ and the other two algebraic equations are generated using the boundary conditions for an nth degree basis of Chebyshev polynomial equation (3.6), given as:

$$y(x) = \sum_{r=0}^{n} a_r T_r(x)$$

The approximate solution to the ODE using the CFDM Scheme is then obtained by

$$\bar{y}(x) = \sum_{r=0}^{n} a_r^* T_r(x) \tag{3.14}$$

Where a_r^* is the solution of the adjustable parameters, that is $a_r^* = a_0, a_1, a_2, a_3, \cdots, a_n$.
Remark. If the boundary conditions of the problem to solve is within the interval $[0, 1]$, this require a shift in the interval of Chebyshev polynomials from $[-1, 1]$ to $[0, 1]$. Hence, the shifted Chebyshev polynomials is the choice for $T_r(x)$ rather than the first or second kind Chebyshev polynomial.

3.1.2 Perturbed Chebyshev-Finite Difference Method (PCFDM)

In this section, the Perturbed version of the Chebyshev Finite Difference Method (PCFDM) is presented to improve the solution to the boundary value problem. Let $P_n(x)$ represent the perturbation term with nth degree of Chebyshev polynomial and τ_r denote some perturbation constant. Hence,the following working steps are implemented:

$$y(x) = \sum_{r=0}^{n-2} a_r T_r(x) \tag{3.15}$$

$$P_n(x) = \left[\sum_{r=1}^{n} \tau_r T_{(n-(r-1))}(x)\right] \; r = 1, 2. \tag{3.16}$$

From Equation (3.4), the following are obtained:

$$\begin{gathered} \frac{P(x)}{h^2}\left(\left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i+1}\right) - 2\left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i}\right) + \left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i-1}\right)\right) + \\ \frac{Q(x)}{2h}\left(\left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i+1}\right) + \left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i-1}\right)\right) + R(x)\left[\sum_{r=0}^{n} a_r T_r(x)\right]_i \\ = f(x) + P_n(x) \end{gathered} \tag{3.17}$$

$$\begin{gathered} \frac{P(x)}{h^2}\left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i+1} - 2\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i} + \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right) + \\ \frac{Q(x)}{2h}\left(\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_{i+1} + \left[\sum_{r=0}^{n} a_r T_r(x)\right]_{i-1}\right) + R(x)\left[\sum_{r=0}^{n-2} a_r T_r(x)\right]_i \\ = f(x) + \left[\sum_{r=1}^{n} \tau_r T_{(n-(r-1))}(x)\right] \end{gathered} \tag{3.18}$$

Suppose the nth degree of the Chebyshev basis functions is 6, equation (3.18) can be expanded as

$$\begin{gathered} P(x)([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_{i+1}} - 2[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_i} + \cdots \\ [a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_{i-1}}) + \cdots \\ \frac{h}{2}Q(x)\left([a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_{i+1}} - [a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_{i-1}}\right) + \cdots \\ h^2R(x)[a_0T_0 + a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4]_{x_i} = h^2f(x_i) + \tau_1 * T_6(x_i) + \tau_2 * T_5(x_i) \end{gathered} \tag{3.19}$$

The matrix form of equation (3.19) for an nth degree is obtained using the following steps.
Let

$$\mathbf{T}_i = \begin{pmatrix} T_0(x_i) \\ T_1(x_i) \\ T_2(x_i) \\ T_3(x_i) \\ T_4(x_i) \\ \vdots \\ T_n(x_i) \end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_n \end{pmatrix}$$

Thus, the expression

$$a_0T_0(x_i) + a_1T_1(x_i) + a_2T_2(x_i) + a_3T_3(x_i) + a_4T_4(x_i) + \cdots + a_nT_n(x_i)$$

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can be written as

$$\mathbf{a}^T\mathbf{T}_i,$$

such that equation (3.19) becomes

$$\begin{aligned}&P(x_i)\left(\mathbf{a}^T\mathbf{T}_{i+1}-2\mathbf{a}^T\mathbf{T}_i+\mathbf{a}^T\mathbf{T}_{i-1}\right)+\cdots\\&+\frac{h}{2}Q(x_i)\left(\mathbf{a}^T\mathbf{T}_{i+1}-\mathbf{a}^T\mathbf{T}_{i-1}\right)+\cdots\\&+h^2R(x_i)\mathbf{a}^T\mathbf{T}_i=h^2f(x_i)+\tau_1T_n(x_i)+\tau_2T_{n-1}(x_i)\end{aligned}\tag{3.20}$$

The following matrices and vectors are defined to form a compact representation of equation (3.20):

- $\mathbf{P}$ is a diagonal matrix with $P(x_i)$ on the diagonal.
- $\mathbf{Q}$ is a diagonal matrix with $Q(x_i)$ on the diagonal.
- $\mathbf{R}$ is a diagonal matrix with $R(x_i)$ on the diagonal.
- $\mathbf{f}$ is a vector with $f(x_i)$ as its entries.
- $\mathbf{T}_n$ and $\mathbf{T}_{n-1}$ are vectors with $T_n(x_i)$ and $T_{n-1}(x_i)$ as their entries, respectively.

The second derivative finite difference operator matrix $\mathbf{D}_2$ and the first derivative finite difference operator matrix $\mathbf{D}_1$ are defined as follows

$$\mathbf{D}_1=\begin{pmatrix}0&1&0&\cdots&0\\-1&0&1&\cdots&0\\0&-1&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\end{pmatrix}$$

$$\mathbf{D}_2=\begin{pmatrix}-2&1&0&\cdots&0\\1&-2&1&\cdots&0\\0&1&-2&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&-2\end{pmatrix}$$

Combining these terms, equation (3.20) yields

$$\mathbf{P}(\mathbf{D}_2(\mathbf{AT}))+\frac{h}{2}\mathbf{Q}(\mathbf{D}_1(\mathbf{AT}))+h^2\mathbf{R}(\mathbf{AT})=h^2\mathbf{f}+\tau_1\mathbf{T}_n+\tau_2\mathbf{T}_{n-1},\tag{3.21}$$

where $\mathbf{AT}$ represents the dot product of the vector $\mathbf{a}$ with the matrix $\mathbf{T}$ (where $\mathbf{T}$ has the Chebyshev polynomials evaluated at different points as its rows).

The derived polynomial matrix equation (3.21) is the Perturbed Chebyshev-Finite Difference Method (PCFDM) for solving the second order homogeneous (if $f(x)=0$) and non-homogeneous ($f(x)\neq 0$) ordinary differential equations. To determine the adjustable parameters $a_r = a_0, a_1, a_2, \cdots, a_n$ and the perturbation parameters τ_1 τ_2, $n-2$ matrix of algebraic equations is generated from equation (3.21) for x_i such that $i = 1, 2, \cdots, n-2$ and the other two algebraic equations matrix is generated using the boundary conditions for an nth degree basis of Chebyshev polynomial equation (3.15), given by

$$y(x)=\sum_{i=0}^{n-2}a_rT_r(x)$$

The approximate solution is obtains by the equation

$$\bar{y}(x)=\sum_{r=0}^{n}a_r^*T_r(x)\tag{3.22}$$

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4 Numerical Results and Discussion

In this section, MATLAB software is used to simulate the proposed methods in equation (3.13) and equation (3.21) to solve some boundary value problems of second order ordinary differential equations and compare the results of the two methods to the traditional finite difference method.

4.1 Examples of Second Order Boundary Value Problem of ODEs

Example 1:
Consider the Dirichlet boundary value problem of the form

$$y''(x) = 4y(x),\ [0,1],\ y(0) = 0,\ y(1) = e\ (2.7183). \tag{4.1}$$

with the exact solution given by

$$y(x) = \frac{(-1 + e^{(4x)})e^{(3-2x)}}{-1 + e^{(4)}}$$

Example 2:
Consider the boundary value problem given below

$$y'' - y = x,\ \ 0 < x < 1\ \ y(0) = 0,\ y(1) = 0 \tag{4.2}$$

with the exact solution given by

$$y(x) = \frac{sinh(x)}{sinh1} - x$$

Example 3:
Given the boundary value problem defined as

$$-y'' = \pi^2 cos(\pi x),\ 0 < x < 1\quad y(0) = 1,\ y(1) = -1 \tag{4.3}$$

with the exact solution given by

$$y(x) = cos(\pi x)$$

4.2 Numerical approximation of Boundary Value Problems

In this section, MATLAB is used to implement the proposed methods in (3.13 and 3.21) to obtained the numerical solutions to the class of homogeneous and non-homogeneous Examples presented in section 4.1. The results obtained with the polynomials of degree 8 and the absolute errors calculated are presented in Table 1, Table 2 and Table 3.

In Table 1, the results show the computational errors obtained for solving example 1 using the traditional finite difference method (FDM), enhance Chebyshev finite difference method (EFDM) and perturbed Chebyshev finite difference method (PCFDM). The results indicate that the PCFDM outperform the FDM and CFDM as the computed errors for PCFDM was minimal compared to FDM and CFDM.

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Table 1: Comparison of Example 1 (Homogeneous) Result for $n = 8$

x_i	Exact	FDM Error ($*10^{-4}$)	CFDM Error ($*10^{-4}$)	PCFDM Error ($*10^{-4}$)
0	0	0	0	0
0.125000000000000	0.189329445638137	5.17535775642	5.17535775642	3.32459726049
0.250000000000000	0.390553740848224	10.05658317696	10.05658317696	6.46053403853
0.375000000000000	0.616315043397523	14.29235968406	14.29235968406	9.18368805273
0.500000000000000	0.880797077977883	17.41099048428	17.41099048428	11.19104207216
0.625000000000000	1.200616245726553	18.74465024283	18.74465024283	12.05481560683
0.750000000000000	1.595865569563315	17.33344333974	17.33344333973	11.15981188842
0.875000000000000	2.091377062462207	11.79986700115	11.79986700115	7.62151618917
1.000000000000000	2.718281828459046	1.8171540955e-01	1.8171540955e-01	1.8171540955e-01

Table 2: Comparison of Example 2 (Non-homogeneous) Result for $n = 8$

x_i	Exact	FDM Error1($*10^{-5}$)	CFDM Error ($*10^{-5}$)	PCFDM Error ($*10^{-5}$)
0	1.000000000000000	0	0	0
0.125000000000000	-0.018358025911688	2.1330852864	2.1330852864	1.3661304924
0.250000000000000	-0.035047600211395	4.0824233453	4.0824233453	2.6145772135
0.375000000000000	-0.048374167768282	5.6579996568	5.6579996568	3.6236618005
0.500000000000000	-0.056590558014963	6.6571140542	6.6571140542	4.2635526230
0.625000000000000	-0.057869649842589	6.8576570549	6.8576570549	4.3920030255
0.750000000000000	-0.050275785641288	6.0109201753	6.0109201753	3.8497272379
0.875000000000000	-0.031734487248432	3.8337697035	3.8337697035	2.4553672890
1.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000

The result in Table 2, shows the computational errors of the traditional finite difference method (FDM), enhance Chebyshev finite difference method (EFDM) and perturbed Chebyshev finite difference method (PCFDM). The results indicate that the PCFDM outperform the FDM and CFDM as the computed errors for PCFDM was minimal compared to FDM and CFDM.

Table 3: Comparison of Example 3 (Non-homogeneous) Result for $n = 8$

x_i	Exact	FDM Error1($*10^{-3}$)	CFDM Error ($*10^{-3}$)	PCFDM Error ($*10^{-3}$)
0	1.000000000000000	0	0	0
0.125000000000000	0.923879532511287	2.251869785672	2.251869785672	1.436449866736
0.250000000000000	0.707106781186548	2.682187467531	2.682187467531	1.712253255357
0.375000000000000	0.382683432365090	1.718349526750	1.718349526750	1.096055316236
0.500000000000000	0	0	0	0
0.625000000000000	-0.382683432365090	1.718349526750	1.718349526750	1.096055316236
0.750000000000000	-0.707106781186548	2.682187467531	2.682187467531	1.712253255357
0.875000000000000	-0.923879532511287	2.251869785672	2.251869785672	1.436449866736
1.000000000000000	-1.000000000000000	0	0	0

The result in Table 3, shows the computational errors obtained for solving example 3 using the traditional finite difference method (FDM), enhance Chebyshev finite difference method (CFDM) and perturbed Chebyshev finite difference method (PCFDM). The results indicate that the PCFDM outperform the FDM and CFDM as the computed errors for PCFDM were minimal compared to FDM and CFDM.

4.3 Discussion and Conclusion

In this study, two advance numerical techniques for solving second-order boundary value problems of ordinary differential equations (ODEs) are presented. The Chebyshev-Finite Difference Method (CFDM) and the Perturbed Chebyshev-Finite Difference Method (PCFDM). These methods were applied to solve both homogeneous and non-homogeneous boundary value problems of ODEs, and their results were compared to those obtained using the traditional finite difference method. The numerical results demonstrated that the perturbed Chebyshev finite difference method outperformed both the traditional finite difference method and the unperturbed CFDM. The PCFDM has proven to be an efficient and powerful mathematical tool for addressing these class of boundary value problems of ODEs, offering superior accuracy and reliability in solving both homogeneous and non-homogeneous cases.

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